INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 15 NO. 2 PAGE 253–265 (2022)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1033345

Proper Semi-Slant Pseudo-Riemannian Submersions in Para-Kaehler Geometry

Esra Başarır Noyan and Yılmaz Gündüzalp *

(Communicated by Ramesh Sharma)

ABSTRACT

In this paper, we examine the proper semi-slant pseudo-Riemannian submersions in para-Kaehler geometry and prove some fundamental results on such submersions. In particular we obtain curvature relations in para-Kaehler space forms. Moreover, we provide examples of proper semi-slant pseudo-Riemannian submersions.

Keywords: *Para-Kaehler manifold, para-Hermitian manifold, invariant distribution, pseudo-Riemannian submersion, proper slant distribution, proper semi-slant submersion.*

AMS Subject Classification (2020): *Primary: 53C43; Secondary: 53B20; 53C40.*

1. Introduction

A $C^{\infty}-$submersion ψ can be defined according to the following conditions. A pseudo-Riemannian submersion ([4],[15],[21],[22],[34]), an almost Hermitian submersion ([37],[32],[10]), a slant submersion ([19],[9],[25],[31]), a para quaternionic submersion ([5],[16]), a Clairaut Submersion ([12]), an anti-invariant submersion ([11],[13],[30],[8]), anti-invariant Riemnnian submersion from cosymplectic manifolds ([30],[14]), a quasi-bi-slant Submersion ([27]), a pointwise slant submersion([20],[28]), a hemi-slant submersion ([35],[29]), a semi-invariant submersion ([23],[33]), a semi-slant $\xi^{\perp}$- Riemannian submersions ([1],[26]), etc. As we know, Riemannian submersions were severally introduced by B. O'Neill ([22]) and A. Gray ([15]) in 1960s. In particular, by using the concept of almost Hermitian submersions, B. Watson ([37]) gave some differential geometric properties among fibers, base manifolds, and total manifolds. Some interesting results concerning para-Kaehler-like statistical submersions were obtained by G.E. Vîlcu ([36]).

In this paper, we examine some geometric properties of three types of proper semi-slant pseudo-Riemannian submersions. Let's list the section of our work. In section 2, we gather some concepts, which are needed in the following parts. In section 3, we study some geometric properties of three types of proper semi-slant pseudo-Riemannian submersions from almost para-Hermitian manifolds onto pseudo-Riemannian manifolds.

We present examples, study the geometry of leaves of distributions. We also obtain necessary and sufficient conditions for a proper semi- slant pseudo-Riemannian submersions to be totally geodesic map. In section 4, we research proper semi-slant pseudo-Riemannian submersions with umbilical fibers and obtain characterization of such maps. In the final section, we obtain curvature properties of distribution for proper semi-slant pseudo-Riemannian submersions from para-Kaehler space forms.

Received : 07–11–2021, Accepted : 12–10–2022
** Corresponding author*

2. Preliminaries

By a para-Hermitian manifold we mean a triple $(\mathcal{B}, \mathcal{P}, g_{\mathcal{B}})$,where $\mathcal{B}$ is a connected $2n$- dimensional differentiable manifold, $\mathcal{P}$ is a tensor field of type (1,1) and a pseudo-Riemannian metric $g_{\mathcal{B}}$ on $\mathcal{B}$, satisfying

$$\mathcal{P}^2 U = U, \quad g_{\mathcal{B}}(\mathcal{P}U, \mathcal{P}V) = -g_{\mathcal{B}}(U, V) \tag{2.1}$$

where U, V are vector fields on $\mathcal{B}$. An almost para-Hermitian manifold $\mathcal{B}$ is said to be a para-Kaehler manifold if

$$\nabla \mathcal{P} = 0; \tag{2.2}$$

where ∇ denotes the Riemannian connection on $\mathcal{B}$ ([18]).

Let $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be two pseudo-Riemannian manifolds. A pseudo-Riemannian submersion is a smooth map $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ which is onto and provides some conditions such that
(i) the fibres $\psi^{-1}(q)$, $q \in \tilde{\mathcal{B}}$, are $r-$ dimensional pseudo-Riemannian submanifolds of $\mathcal{B}$, where $r = dim(\mathcal{B}) - dim(\tilde{\mathcal{B}})$.
(ii) ψ_* preserves scalar products of vectors normal to fibres.
The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. We indicate by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. A horizontal vector field U on $\mathcal{B}$ is said to be *fundamental* if U is $\psi-$related to a vector field U_* on $\tilde{\mathcal{B}}$.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by:

$$\mathcal{T}_U \mathcal{W} = h\nabla_{vU} v\mathcal{W} + v\nabla_{vU} h\mathcal{W} \tag{2.3}$$

and

$$\mathcal{A}_U \mathcal{W} = v\nabla_{hU} h\mathcal{W} + h\nabla_{hU} v\mathcal{W} \tag{2.4}$$

for every $U, \mathcal{W} \in \chi(\mathcal{B})$, on $\mathcal{B}$ where ∇ is the Levi-Civita connection of $g_{\mathcal{B}}$.
It is easy to see that a semi-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. Also, if $\mathcal{A}$ vanishes then the horizontal distribution is integrable. (see [4],[7]). Using (2.3) and (2.4), we get

$$\nabla_U W = \mathcal{T}_U W + \hat{\nabla}_U W; \tag{2.5}$$

$$\nabla_U \zeta = \mathcal{T}_U \zeta + h\nabla_U \zeta; \tag{2.6}$$

$$\nabla_\zeta U = \mathcal{A}_\zeta U + v\nabla_\zeta U; \tag{2.7}$$

$$\nabla_\zeta \eta = \mathcal{A}_\zeta \eta + h\nabla_\zeta \eta, \tag{2.8}$$

for any $\zeta, \eta \in \Gamma((ker\psi_*)^\perp)$, $U, W \in \Gamma(ker\psi_*)$. Also, if ζ is basic then $h\nabla_U \zeta = h\nabla_\zeta U = \mathcal{A}_\zeta U$.

It is easily seen that $\mathcal{T}$ is symmetric on the vertical distribution and $\mathcal{A}$ is alternating on the horizontal distribution such that

$$\mathcal{T}_\mathcal{W} U = \mathcal{T}_U \mathcal{W}, \quad \mathcal{W}, U \in \Gamma(ker\psi_*); \tag{2.9}$$

$$\mathcal{A}_Y V = -\mathcal{A}_V Y = \frac{1}{2} v[Y, V], \quad Y, V \in \Gamma((ker\psi_*)^\perp). \tag{2.10}$$

Let $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be pseudo-Riemannian manifolds and $\pi : \mathcal{B} \to \tilde{\mathcal{B}}$ is a differentiable map. Then the second fundamental form of ψ is given by

$$(\nabla\psi_*)(X, V) = \nabla^\psi_X \psi_* V - \psi_*(\nabla_X V) \tag{2.11}$$

for $X, V \in \Gamma(\mathcal{B})$, here we indicate conveniently by ∇ the Riemannian connections of the metrics $g_{\mathcal{B}}$ and $g_{\tilde{\mathcal{B}}}$. Recall that ψ is said to be *harmonic* if $trace(\nabla\psi_*) = 0$ and ψ is called a *totally geodesic* map if $(\nabla\psi_*)(X, V) = 0$ for $X, V \in \Gamma(T\mathcal{B})$ ([17]). Where ∇^ψ is the pullback connection.

For every vertical vector fields $U, W, \mathcal{X}, \mathcal{Y}$ we have

$$R(U, W, \mathcal{X}, \mathcal{Y}) = \tilde{R}(U, W, \mathcal{X}, \mathcal{Y}) - g(\mathcal{T}_U \mathcal{X}, \mathcal{T}_W \mathcal{Y}) + g(\mathcal{T}_W \mathcal{X}, \mathcal{T}_U \mathcal{Y}) \tag{2.12}$$

where R and $\tilde{R}$ is pseudo-Riemannian curvature tensor of $\mathcal{B}$ and $\psi^{-1}(q)$.

Moreover, if $\{U, W\}$ is orthonormal basis of vertical 2-plane, then from (12) we obtain

$$K(U,W) = \tilde{K}(U,W) + ||\mathcal{T}_U W||^2 - g(\mathcal{T}_U U, \mathcal{T}_W W) \tag{2.13}$$

where K and $\tilde{K}$ are sectional curvature of $\mathcal{B}$ and $\psi^{-1}(q)$ ([4]).

3. Semi-slant submersions

Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$.
For any non-null vector field $W \in (ker\psi_*)$, we get

$$W = QW + FW, \tag{3.1}$$

where $QW \in \Gamma(D_1)$ and $FW \in \Gamma(D_2)$ and put

$$\mathcal{P}W = tW + nW, \tag{3.2}$$

where tW and nW are vertical and horizontal parts of $\mathcal{P}V$.
Also, for non-null vector field $\zeta \in (ker\psi_*)^\perp$, we have

$$\mathcal{P}\zeta = B\zeta + C\zeta, \tag{3.3}$$

where $B\zeta \in ker\psi_*$ and $C\zeta \in (ker\psi_*)^\perp$.
In adittion, $(ker\psi_*)^\perp$ is decomposed as

$$(ker\psi_*)^\perp = nD_2 \oplus \mu \tag{3.4}$$

where μ is the orthogonal complementary distribution of nD_2. We say that μ is invariant distribution of $(ker\psi_*)^\perp$ with respect to $\mathcal{P}$.

Definition 3.1. ([12]) Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper slant submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. We have
type $\sim$1 if for every space-like (time-like) vector field $W \in \Gamma(ker\psi_*)$, tW is time-like (space-like), and $\frac{\|tW\|}{\|\mathcal{P}W\|} > 1$,
type $\sim$ 2 if for every space-like (time-like) vector field $W \in \Gamma(ker\psi_*)$, tW is time-like (space-like), and $\frac{\|tW\|}{\|\mathcal{P}W\|} < 1$,
type $\sim$ 3 if for every space-like (time-like) vector field $W \in \Gamma(ker\psi_*)$, tW is space-like (time-like).

A differentiable distribution D on $(ker\psi_*)$ is called a slant distribution if for all non-null $\mathcal{X} \in D$, the quotient $g_{\mathcal{B}}(t\mathcal{X}, t\mathcal{X})/g_{\mathcal{B}}(\mathcal{P}\mathcal{X}, \mathcal{P}\mathcal{X})$ is constant. A distribution is called invariant if it is a slant with slant angle zero, that is if $g_{\mathcal{B}}(t\mathcal{X}, t\mathcal{X})/g_{\mathcal{B}}(\mathcal{P}\mathcal{X}, \mathcal{P}\mathcal{X}) = 1$ for every non-null $\mathcal{X} \in D$. It is called anti-invariant if $t\mathcal{X} = 0$ for all $\mathcal{X} \in D$. In other cases, ıt is called a proper slant distribution.

Now, we can give our definition.

Definition 3.2. Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is said to be a semi-slant submersion if there is a distribution $D_1 \in ker\psi_*$ such that

$$ker\psi_* = D_1 \oplus D_2, \quad \mathcal{P}(D_1) = D_1 \tag{3.5}$$

and the angle $\varphi = \varphi(W)$ between $\mathcal{P}W$ and space $(D_2)_q$ is constant for non-null vector field $W \in (D_2)_q$ and $q \in \mathcal{B}$, we can say that φ is a semi-slant angle of the submersion where D_2 is the orthogonal complement of D_1 in $ker\psi_*$.
Hence, using (3.2), (3.3) and (3.4), we have:

Lemma 3.1. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a semi-slant submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. Then, we obtain the following equations.*
$(i)\, tD_1 = D_1$ $\quad (ii)\, nD_1 = 0$
$(iii)\, tD_2 \subset D_2$ $\quad (iv)\, B((ker\pi_*)^{\perp}) = D_2$.

Then, we can easily see that $\mathcal{P}^2 = I$ and from (3.2) and (3.3) we get:

Lemma 3.2. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a semi-slant submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. Then, we obtain the following equations.*
$(i)\ t^2 + Bn = I$ $\quad (ii)\ C^2 + nB = I$
$(iii) tB + BC = \{0\}$ $\quad (iv)\ nt + Cn = \{0\}$.

The proof of the following Theorems is similar to the proof of ([2],[3]). Therefore we skip its proof.

Theorem 3.1. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, ψ is proper semi-slant submersion of type $\sim$1 if and only if for any space-like(time-like) vector field $W \in \Gamma(ker\psi_*)$, tW time-like(space-like), there exist a constant $\gamma \in (1, \infty)$ and a distribution of type $\sim$1 D on $ker\psi_*$ as follows*
(i) $D = \{W \in \Gamma(ker\psi_*)\}|t^2W = \gamma W\}$,
(ii) for all non-null vector field $W \in \Gamma(ker\psi_*)$ orthogonal to D, we get $tW = 0$.
Also, $\gamma = \cosh^2\varphi$, with $\varphi > 0$.

From here, for any non-null vector field $W \in \Gamma(D_2)$, we obtain

$$t^2W = \cosh^2\varphi W. \tag{3.6}$$

Moreover, for any non-null vector fields $U, W \in \Gamma(D_2)$, from (2.1), (3.2) and (3.6), we have

$$g_{\mathcal{B}}(tU, tW) = -\cosh^2\varphi g_{\mathcal{B}}(U, W). \tag{3.7}$$

Besides, using (2.1), (3.1), (3.6) and Lemma 3.2 (i), we arrive at

$$g_{\mathcal{B}}(nU, nW) = \sinh^2\varphi g_{\mathcal{B}}(U, W). \tag{3.8}$$

Theorem 3.2. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, ψ is proper semi-slant submersion of type $\sim$2 if and only if for any space-like(time-like) vector field $W \in \Gamma(ker\psi_*)$, tW time-like(space-like), there exist a constant $\gamma \in (0, 1)$ and a distribution of type $\sim$2 D on $ker\psi_*$ as follows*
(i) $D = \{W \in \Gamma(ker\psi_*)\}|t^2W = \gamma W\}$,
(ii) for all non-null vector field $W \in \Gamma(ker\psi_*)$ orthogonal to D, we get $tW = 0$.
Also, $\gamma = \cos^2\varphi$, with $0 < \varphi < 2\pi$.

From here, for any non-null vector field $V \in \Gamma(D_2)$, we obtain

$$t^2W = \cos^2\varphi W. \tag{3.9}$$

Moreover, for every non-null vector fields $U, W \in \Gamma(D_2)$, from (2.1), (3.1) and (3.9), we have

$$g_{\mathcal{B}}(tU, tW) = -\cos^2\varphi g_{\mathcal{B}}(U, W). \tag{3.10}$$

Besides, using (2.1), (3.2), (3.9) and Lemma 3.2 (i), we arrive at

$$g_{\mathcal{B}}(nU, nW) = -\sin^2\varphi g_{\mathcal{B}}(U, W). \tag{3.11}$$

Theorem 3.3. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, ψ is proper semi-slant submersion of type $\sim$3 if and only if for any space-like(time-like) vector field $W \in \Gamma(ker\psi_*)$, tW time-like(space-like), there exist a constant $\gamma \in (-\infty, 0)$ and a distribution of type $\sim$3 D on $ker\psi_*$ as follows*

(i) $D = \{W \in \Gamma(ker\psi_*)\}|t^2W = \gamma W\}$,
(ii) for all non-null vector field $W \in \Gamma(ker\psi_*)$ *orthogonal to* D, *we get* $tW = 0$.
Also, $\gamma = -\sinh^2\varphi$, *with* $\varphi > 0$.

From here, for any non-null vector field $W \in \Gamma(D_2)$, *we obtain*

$$t^2W = -\sinh^2\varphi W. \tag{3.12}$$

Moreover, for any non-null vector fields $U, W \in \Gamma(D_2)$, *from (2.1), (3.2) and (3.12), we have*

$$g_{\mathcal{B}}(tU, tW) = \sinh^2\varphi g_{\mathcal{B}}(U, W). \tag{3.13}$$

Besides, using (2.1), (3.1), (3.12) and Lemma 3.2 (i), we arrive at

$$g_{\mathcal{B}}(nU, nW) = -\cosh^2\varphi g_{\mathcal{B}}(U, W). \tag{3.14}$$

Let's consider para-Kaehler structure on R^{2n}_n :

$$\mathcal{P}(\frac{\partial}{\partial y_{2i}}) = \frac{\partial}{\partial y_{2i-1}},\ \mathcal{P}(\frac{\partial}{\partial y_{2i-1}}) = \frac{\partial}{\partial y_{2i}},\ g = (dy^1)^2 - (dy^2)^2 + (dy^3)^2 - ... - (dy^{2n})^2$$

where $i \in \{1, ..., n\}$. Also, $(y_1, y_2, ..., y_{2n})$ denotes the cartesian coordinates over R^{2n}_n.

We can easily present non-trivial examples of proper semi-slant pseudo-Riemannian submersions of type∼1, 2 and 3.

Example 3.1. Let's determine map $\psi : R^8_4 \to R^4_2$

$$\psi(y_1, ..., y_8) = (y_4, y_2\sinh\beta + y_3\cosh\beta, y_7, y_8),$$

here $\beta \neq 0$. So, ψ is a proper semi-slant pseudo-Riemannian submersion of type ~ 1. By direct calculations

$D_1 =< \frac{\partial}{\partial y_5}, \frac{\partial}{\partial y_6} >$ and $D_2 =< \frac{\partial}{\partial y_1}, \cosh\beta\frac{\partial}{\partial y_2} - \sinh\beta\frac{\partial}{\partial y_3}, >$

with the semi-slant angle φ with $\cosh\varphi = cosh\beta$.

Let R^8_4 be a pseud-Euclidean space of signature (-,+,-,+,-,+,-,+) with respect to the canonical basis $(\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_8})$ and R^4_2 be a pseud-Euclidean space of signature (+,-,-,+) with respect to the canonical basis $(\frac{\partial}{\partial \bar{y}_1}, ..., \frac{\partial}{\partial \bar{y}_4})$.

Example 3.2. Let's determine map $\psi : R^8_4 \to R^4_2$

$$\psi(y_1, ..., y_8) = (y_1\sin\omega + y_3\cos\omega, y_2\sin\theta + y_4\cos\theta, y_5, y_6)$$

here $\omega \neq \theta \neq 0$. So, ψ is a proper semi-slant pseudo-Riemannian submersion of type ~ 2. By direct calculations

$D_1 =< \frac{\partial}{\partial y_7}, \frac{\partial}{\partial y_8} >$ and $D_2 =< \cos\omega\frac{\partial}{\partial y_1} - \sin\omega\frac{\partial}{\partial y_3}, -\cos\theta\frac{\partial}{\partial y_2} + \sin\theta\frac{\partial}{\partial y_4} >$

with the semi-slant angle φ with $\varphi = (\omega - \theta)$.

Let R^8_4 be a pseud-Euclidean space of signature (+,-,+,-,+,-,+,-) with respect to the canonical basis $(\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_8})$ and R^4_2 be a pseud-Euclidean space of signature (+,-,+,-) with respect to the canonical basis $(\frac{\partial}{\partial \bar{y}_1}, ..., \frac{\partial}{\partial \bar{y}_4})$.

Example 3.3. Let's determine map $\psi : R^8_4 \to R^4_3$

$$\psi(y_1, ..., y_8) = (y_2\cosh\theta + y_3\sinh\theta, y_4, y_5, y_6),$$

for any $\theta \neq 0$. So, ψ is a proper semi-slant pseudo-Riemannian submersion of type ~ 3. By direct calculations

$D_1 =< \frac{\partial}{\partial y_7}, \frac{\partial}{\partial y_8} >$ and $D_2 =< \sin\theta\frac{\partial}{\partial y_2} - \cos\theta\frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_1} >$

with the semi-slant angle φ with $t^2 = -\sinh^2\theta I$.

Let R^8_4 be a pseud-Euclidean space of signature (+,-,+,-,+,-,+,-) with respect to the canonical basis $(\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_8})$ and R^4_3 be a pseud-Euclidean space of signature (-,-,+,-) with respect to the canonical basis $(\frac{\partial}{\partial \bar{y}_1}, ..., \frac{\partial}{\partial \bar{y}_4})$.

Using equations (2.1), (2.5)∼(2.8) and (3.2)∼(3.3), we get:

Lemma 3.3. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion of type ∼1, 2, 3 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. So, we obtain the following equations.*

$$\hat{\nabla}_U tW + \mathcal{T}_U nW = t\hat{\nabla}_U W + \mathcal{B}\mathcal{T}_U W \tag{3.15}$$

$$\mathcal{T}_U tW + \mathcal{H}\nabla_U nW = n\hat{\nabla}_U W + \mathcal{C}\mathcal{T}_U W \tag{3.16}$$

$$\mathcal{V}\nabla_{\mathcal{X}}\mathcal{Y} + \mathcal{A}_{\mathcal{X}}\mathcal{C}\mathcal{Y} = t\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{B}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} \tag{3.17}$$

$$\mathcal{A}_{\mathcal{X}}\mathcal{B}\mathcal{Y} + \mathcal{H}\nabla_{\mathcal{X}}\mathcal{C}\mathcal{Y} = n\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{C}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} \tag{3.18}$$

$$\hat{\nabla}_U \mathcal{B}\mathcal{X} + \mathcal{T}_U \mathcal{C}\mathcal{X} = t\mathcal{T}_U\mathcal{X} + \mathcal{B}\mathcal{H}\nabla_U\mathcal{X} \tag{3.19}$$

$$\mathcal{T}_U \mathcal{B}\mathcal{X} + \mathcal{H}\nabla_U \mathcal{C}\mathcal{X} = n\mathcal{T}_U\mathcal{X} + \mathcal{C}\mathcal{H}\nabla_U\mathcal{X}, \tag{3.20}$$

for any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(ker\psi_*)^{\perp}$.

Now we can show

$$(\nabla_U t)W = \hat{\nabla}_U tW - t\hat{\nabla}_U W \tag{3.21}$$

and

$$(\nabla_U n)W = \mathcal{H}\nabla_U nW - n\hat{\nabla}_U W, \tag{3.22}$$

for any non-null vector fields $U, W \in ker\psi_*$. Then from (3.15) and (3.16), we get

$$\nabla t \equiv 0 \iff \mathcal{T}_U nW = \mathcal{B}\mathcal{T}_U W \tag{3.23}$$

and

$$\nabla n \equiv 0 \iff \mathcal{T}_U tW = \mathcal{C}\mathcal{T}_U W, \tag{3.24}$$

for any non-null vector fields $U, W \in \Gamma(ker\psi_*)$.

Theorem 3.4. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. The complex distribution D_1 is integrable if and only if*

$$n(\hat{\nabla}_U W - \hat{\nabla}_W U) = C(\mathcal{T}_W U - \mathcal{T}_U W) \tag{3.25}$$

for any non-null vector fields $U, W \in \Gamma(D_1)$.

Proof. For any non-null vector fields $U, W \in \Gamma(D_1)$ and $\mathcal{X} \in \Gamma((ker\psi_*))^{\perp}$ since $[U, W] \in \Gamma(ker\psi_*)$, we get:

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{P}[U, W], \mathcal{X}) &= g_{\mathcal{B}}(\mathcal{P}(\nabla_U W - \nabla_W U), \mathcal{X}) \\ &= g_{\mathcal{B}}(t\hat{\nabla}_U W + n\hat{\nabla}_U W + B\mathcal{T}_U W + C\mathcal{T}_U W - t\hat{\nabla}_W U - n\hat{\nabla}_W U \\ &- B\mathcal{T}_W U - C\mathcal{T}_W U, \mathcal{X}) \\ &= g_{\mathcal{B}}(n\hat{\nabla}_U W + C\mathcal{T}_U W - n\hat{\nabla}_W U - C\mathcal{T}_W U, \mathcal{X}) \end{aligned} \tag{3.26}$$

so, the proof is complete. Similarly, the following conclusion is obtained. □

Theorem 3.5. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. The proper semi- slant distribution D_2 is integrable if and only if*

$$Q(t(\hat{\nabla}_U W - \hat{\nabla}_W U)) + B(\mathcal{T}_U W - \mathcal{T}_W U) = 0 \tag{3.27}$$

for any non-null vector fields $U, W \in \Gamma(D_2)$.

Lemma 3.4. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. The proper semi- slant distribution D_2 is integrable if and only if*

$$\mathcal{P}(\hat{\nabla}_U tW - \hat{\nabla}_W tU + \mathcal{T}_U nW - \mathcal{T}_W nU) = 0 \tag{3.28}$$

for any non-null vector fields $U, W \in \Gamma(D_2)$.

Proof. For any non-null vector fields $U, W \in \Gamma(D_2)$ and for every non-lightlike $\mathcal{X} \in \Gamma(D_1)$ since $[U, W] \in \Gamma(ker\psi_*)$, we get:

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{P}[U, W], \mathcal{X}) &= g_{\mathcal{B}}(\nabla_U \mathcal{P}W - \nabla_W \mathcal{P}U, \mathcal{X}) \\ &= g_{\mathcal{B}}(\hat{\nabla}_U tW + \mathcal{T}_U tW + \mathcal{T}_U nW + \mathcal{H}\nabla_U nW - \hat{\nabla}_W tU \\ &- \mathcal{T}_W tU - \mathcal{T}_W nU - \mathcal{H}\nabla_W nU, \mathcal{X}) \\ &= g_{\mathcal{B}}(\hat{\nabla}_U tW + \mathcal{T}_U nW - \hat{\nabla}_W tU - \mathcal{T}_W nU, \mathcal{X}). \end{aligned} \tag{3.29}$$

So, the proof is complete.

□

Now, let's investigate the cases where the vertical and horizontal distribution are totally geodesic.

Proposition 3.1. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, the vertical distribution describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$n(\hat{\nabla}_U tW + \mathcal{T}_U nW) + C(\mathcal{T}_U tW + \mathcal{H}\nabla_U nW) = 0 \tag{3.30}$$

for any non-null vector fields $U, W \in \Gamma(ker\psi_*)$.

Proof. For any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ we get:

$$\begin{aligned} \nabla_U W &= \mathcal{P}\nabla_U \mathcal{P}W \\ &= \mathcal{P}(\hat{\nabla}_U tW + \mathcal{T}_U tW + \mathcal{T}_U nW + \mathcal{H}\nabla_U nW) \\ &= (t\hat{\nabla}_U tW + n\hat{\nabla}_U tW + B\mathcal{T}_U tW + C\mathcal{T}_U tW + t\mathcal{T}_U nW + n\mathcal{T}_U nW \\ &+ B\mathcal{H}\nabla_U nW + C\mathcal{H}\nabla_U nW). \end{aligned} \tag{3.31}$$

Thus,

$$\nabla_U W \in \Gamma(ker\psi_*) \iff n(\hat{\nabla}_U tW + \mathcal{T}_U nW) + C(\mathcal{T}_U tW + \mathcal{H}\nabla_U nW) = 0. \tag{3.32}$$

□

Similarly, the following conclusion is obtained.

Proposition 3.2. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, the horizontal distribution describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$t(\mathcal{V}\nabla_U BW + \mathcal{A}_U CW) + B(\mathcal{A}_U BW + \mathcal{H}\nabla_U CW) = 0 \tag{3.33}$$

for any non-null vector fields $U, W \in \Gamma(ker\psi_*)^\perp$.

Proposition 3.3. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, the distribution D_1 describes a totally geodesic foliation if and only if*

$$\nabla_U W \in \Gamma(D_1) \iff F(t\hat{\nabla}_U tW + B\mathcal{T}_U tW) = 0$$

and

$$n\hat{\nabla}_U tW + C\mathcal{T}_U tW = 0$$

for any non-null vector fields $U, W \in \Gamma(D_1)$.

Proof. For any non-null vector fields $U, W \in \Gamma(D_1)$ we get:

$$\begin{aligned} \nabla_U W &= \mathcal{P}\nabla_U \mathcal{P}W \\ &= \mathcal{P}(\hat{\nabla}_U tW + \mathcal{T}_U tW) \\ &= (t\hat{\nabla}_U tW + n\hat{\nabla}_U tW + B\mathcal{T}_U tW + C\mathcal{T}_U tW) \end{aligned}$$

Hence,

$$\nabla_U W \in \Gamma(D_1) \iff F(t\hat{\nabla}_U tW + B\mathcal{T}_U tW) = 0$$

and

$$n\hat{\nabla}_U tW + C\mathcal{T}_U tW = 0$$

□

Similarly, the following conclusions are obtained.

Proposition 3.4. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a proper semi-slant pseudo-Riemannian submersion of type* ∼*1 or 2 from a para-Kaehler manifold* $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ *onto a pseudo-Riemannian manifold* $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. *In this case, the distribution* $\mathcal{D}_2$ *describes a totally geodesic foliation if and only if*

$$Q(t(\hat{\nabla}_U tW + \mathcal{T}_U nW) + B(\mathcal{T}_U tW + \mathcal{H}\nabla_U nW)) = 0$$

$$n(\hat{\nabla}_U tW + \mathcal{T}_U nW) + C(\mathcal{T}_U tW + \mathcal{H}\nabla_U nW) = 0$$

for any non-null vector fields $U, W \in \Gamma(\mathcal{D}_2)$.

Theorem 3.6. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a proper semi-slant pseudo-Riemannian submersion of type* ∼*1 or 2 from a para-Kaehler manifold* $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ *onto a pseudo-Riemannian manifold* $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. *In this case,* ψ *is a totally geodesic*

$$n(\hat{\nabla}_U tW + \mathcal{T}_U nW) + C(\mathcal{T}_U tW + \mathcal{H}\nabla_U nW) = 0$$

$$n(\hat{\nabla}_U B\mathcal{X} + \mathcal{T}_U C\mathcal{X}) + C(\mathcal{T}_U B\mathcal{X} + \mathcal{H}\nabla_U C\mathcal{X}) = 0$$

for non-null vector fields $U, W \in \Gamma(ker\psi_*)$ *and* $\mathcal{X} \in \Gamma((ker\psi_*)^{\perp})$.

4. Proper semi-slant pseudo-Riemannian submersions with Totally Umbilical Fibers

A pseudo-Riemannian submersion ψ is said to be totally umbilical if

$$\mathcal{T}_U W = g(U, W)H \tag{4.1}$$

here H is the mean curvature vector field of the fibre in $\mathcal{B}$ for all non-null vector fields $U, W \in \Gamma(ker\psi_*)$.

Then we obtain:

Lemma 4.1. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a proper semi-slant pseudo-Riemannian submersion with totally umbilical fibers from a para-Kaehler manifold* $(B, g_{\mathcal{B}}, \mathcal{P})$ *onto a pseudo-Riemannian manifold* $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. *Then we obtain:*

$$H \in \Gamma(nD_2). \tag{4.2}$$

Proof. For $U, W \in \Gamma(D_1)$ and $\mathcal{X} \in \Gamma(\mu)$ we obtain

$$\begin{aligned} \mathcal{T}_U \mathcal{P}W + h\nabla_U \mathcal{P}W = \nabla_U \mathcal{P}W &= \mathcal{P}\nabla_U W \\ &= t\hat{\nabla}_U W + n\hat{\nabla}_U W + \mathcal{B}\mathcal{T}_U W + \mathcal{C}\mathcal{T}_U W. \end{aligned}$$

So,

$$g_{\mathcal{B}}(\mathcal{T}_U\mathcal{P}W,\mathcal{X}) = g_{\mathcal{B}}(C\mathcal{T}_U W,\mathcal{X}). \tag{4.3}$$

From (4.1), with a simple calculation we get

$$g_{\mathcal{B}}(U,\mathcal{P}W)g_{\mathcal{B}}(H,\mathcal{X}) = g_{\mathcal{B}}(U,W)g_{\mathcal{B}}(H,\mathcal{P}\mathcal{X})$$

Interchanging the role of U and W,we obtain:

$$g_{\mathcal{B}}(W,\mathcal{P}U)g_{\mathcal{B}}(H,\mathcal{X}) = g_{\mathcal{B}}(W,U)g_{\mathcal{B}}(H,\mathcal{P}\mathcal{X})$$

so that adding the above two equations, we obtain:

$$g_{\mathcal{B}}(U,W)g_{\mathcal{B}}(H,\mathcal{P}\mathcal{X}) = 0 \tag{4.4}$$

that means $H \in \Gamma(nD_2)$. □

5. Proper semi-slant pseudo-Riemannian submersions in para-Kaehler space forms

Let's define a plane Q is invariant by $\mathcal{P}$ in $T_qB, q \in B$. Moreover $\{U,\mathcal{P}U\}$ is an orthonormal plane basis of Q. Indicate by $K(Q)$, $K_*(Q)$ and $\hat{K}(Q)$ the para-sectional curvatures of the plane in B, $\tilde{\mathcal{B}}$ and the fiber $\psi^{-1}(\psi(q))$ where $K_*(Q)$ indicate the para-sectional curvature of the plane $Q_* = \langle \psi_* U, \psi_* \mathcal{P}U\rangle$ in $\tilde{\mathcal{B}}$. Using both Corollary 1 of [21, p.465] and (1.27) of [6, p.12] ,we have the following:
i) If $Q \subset (D_1)_q$,we get the following results with calculations.

$$K(Q) = \hat{K}(Q) + \epsilon_U\epsilon_{\mathcal{P}U}\{-||\mathcal{T}_{\mathcal{P}U}U||^2 + ||\mathcal{T}_U U||^2 + g_{\mathcal{B}}(\mathcal{P}[\mathcal{P}U,U],\mathcal{T}_U U)\}$$

ii)If $Q \subset (\mu)_q$, then we get

$$K(Q) = K_*(Q) + 3||\mathcal{A}_U\mathcal{P}U||^2,$$

where $\epsilon_U = g_{\mathcal{B}}(U,U) \in \{\pm 1\}, \epsilon_{\mathcal{P}U} = g_{\mathcal{B}}(\mathcal{P}U,\mathcal{P}U) \in \{\pm 1\}$.

For a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$, (2.1) implies that

$$g_{\mathcal{B}}(\mathcal{P}U,W) + g_{\mathcal{B}}(U,\mathcal{P}W) = 0,$$

for any non-null vector fields $U,W \in \Gamma(\mathcal{T}_q\mathcal{B})$, $q \in \mathcal{B}$. Hence we have $g_{\mathcal{B}}(\mathcal{P}U,U) = 0$.
If $\{U,\mathcal{P}U\}$ spans a non-degenerate plane section, the sectional curvature $H_{\mathcal{B}}(U) = K_{\mathcal{B}}(U\wedge\mathcal{P}U)$ of $span\{U,\mathcal{P}U\}$ is called a para-sectional curvature. A para-Kaehler space form, by definition, is a para-Kaehler manifold of constant para-sectional curvature ([6]). The Riemannian-Christoffel curvature tensor of a para-Kaehler space form $\mathcal{B}^{2n}_n(\nu)$ of constant para-holomorphic sectional curvature ν satisfies

$$\begin{aligned} R_{\mathcal{B}}(U,W,\mathcal{X},\mathcal{Y}) &= \frac{\nu}{4}\{g_{\mathcal{B}}(U,\mathcal{X})g_{\mathcal{B}}(W,\mathcal{Y}) - g_{\mathcal{B}}(U,\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{X}) \\ &- g_{\mathcal{B}}(U,\mathcal{P}\mathcal{X})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{Y}) + g_{\mathcal{B}}(U,\mathcal{P}\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{X}) \\ &- 2g_{\mathcal{B}}(U,\mathcal{P}W)g_{\mathcal{B}}(\mathcal{X},\mathcal{P}\mathcal{Y})\} \end{aligned} \tag{5.1}$$

for all non-null vector fields $U,W,\mathcal{X},\mathcal{Y} \in \Gamma(\mathcal{T}\mathcal{B})$([24]).

Theorem 5.1. *Let $\psi : (\mathcal{B}(\nu), g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from a para-Kaehler space form $(\mathcal{B}(\nu), g_{\mathcal{B}})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. Then we get*

$$\begin{aligned}
\tilde{R}(U,W,\mathcal{X},\mathcal{Y}) &= \frac{\nu}{4}\{g_{\mathcal{B}}(U,\mathcal{X})g_{\mathcal{B}}(W,\mathcal{Y}) - g_{\mathcal{B}}(U,\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{X}) \\
&- g_{\mathcal{B}}(U,\mathcal{P}\mathcal{X})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{Y}) + g_{\mathcal{B}}(U,\mathcal{P}\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{X}) \\
&- 2g_{\mathcal{B}}(U,\mathcal{P}W)g_{\mathcal{B}}(\mathcal{X},\mathcal{P}\mathcal{Y})\} \\
&+ g_{\mathcal{B}}(\mathcal{T}_U\mathcal{X},\mathcal{T}_W\mathcal{Y}) - g_{\mathcal{B}}(\mathcal{T}_W\mathcal{X},\mathcal{T}_U\mathcal{Y}) \qquad (5.2)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{K}(U,W) &= \frac{\nu}{4}\{||U||^2||W||^2 - g^2_{\mathcal{B}}(U,W) - 3g^2_{\mathcal{B}}(U,\mathcal{P}W)\} \\
&+ g_{\mathcal{B}}(\mathcal{T}_U U,\mathcal{T}_W W) + ||\mathcal{T}_U W||^2 \qquad (5.3)
\end{aligned}$$

for all non-null vector fields $U,W,\mathcal{X},\mathcal{Y} \in \Gamma(D_1)$.

Proof. For all non-null vector fields $U,W,\mathcal{X},\mathcal{Y} \in \Gamma(D_1)$, by using (5.1) and (2.12) we obtain

$$\begin{aligned}
\tilde{R}(U,W,\mathcal{X},\mathcal{Y}) &= \frac{\nu}{4}\{g_{\mathcal{B}}(U,\mathcal{X})g_{\mathcal{B}}(W,\mathcal{Y}) - g_{\mathcal{B}}(U,\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{X}) \\
&- g_{\mathcal{B}}(U,\mathcal{P}\mathcal{X})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{Y}) + g_{\mathcal{B}}(U,\mathcal{P}\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{X}) \\
&- 2g_{\mathcal{B}}(U,\mathcal{P}W)g_{\mathcal{B}}(\mathcal{X},\mathcal{P}\mathcal{Y})\} \\
&+ g_{\mathcal{B}}(\mathcal{T}_U\mathcal{X},\mathcal{T}_W\mathcal{Y}) - g_{\mathcal{B}}(\mathcal{T}_W\mathcal{X},\mathcal{T}_U\mathcal{Y})
\end{aligned}$$

which gives (5.2). If $\mathcal{X} = W$ and $\mathcal{Y} = U$ is taken in (5.2), we obtain (5.3). □

Theorem 5.2. *Let $\psi : (\mathcal{B}(\nu), g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 or 2 from a para-Kaehler space form $(\mathcal{B}(\nu), g_{\mathcal{B}})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. If $\mathcal{D}_1$ is totally geodesic, we get*

$$\tilde{\sigma}_{D_1} = -\frac{\nu}{4}(p+2)p$$

where $\tilde{\sigma}_{D_1}$ is the scaler curvature of fibres with $dim(D_1) = p$.

Proof. Let's remember that the trace of scalar curvature is Ricci curvature. Then, we get

$$\tilde{S}_{D_1}(U,W) = \sum_{i=1}^{p} \tilde{R}(E_i,U,W,E_i)$$

for all non-null vector fields $U,W \in \Gamma(D_1)$ and $\{E_1,...E_p\}$ is time-like(space-like) orthonormal basis on $\Gamma(D_1)$. Then, if $\mathcal{D}_1$ is totally geodesic from (5.1), we obtain

$$\tilde{S}_{D_1}(U,W) = \sum_{i=1}^{p}\{\frac{\nu}{4}\{g_{\mathcal{B}}(U,W) - pg_{\mathcal{B}}(U,W) + 3g_{\mathcal{B}}(E_i,\mathcal{P}W)g_{\mathcal{B}}(\mathcal{P}U,E_i)\}.$$

From above equation, we obtain

$$\tilde{S}_{D_1}(U,W) = -\frac{\nu}{4}(p+2)pg_{\mathcal{B}}(U,W). \qquad (5.4)$$

If $U = W = E_i, i = 1,...,p$ is accepted and taking into account (5.4), we obtain the proof.
From (5.4), we can obtain the following conclusion. □

Corollary 5.1. *Let $\psi : (\mathcal{B}(\nu), g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 and 2 from a para-Kaehler space form $(\mathcal{B}(\nu), g_{\mathcal{B}})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ with a semi-slant angle φ. We deduce that if D_1 is totally geodesic distribution, D_1 is Einstein.*

Theorem 5.3. *Let $\psi : (\mathcal{B}(\nu), g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 from a para-Kaehler space form $(\mathcal{B}(\nu), g_{\mathcal{B}})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. Then , we get*

$$\begin{aligned}
\tilde{R}(U,W,\mathcal{X},\mathcal{Y}) &= \frac{\nu}{4}\{g_{\mathcal{B}}(U,\mathcal{X})g_{\mathcal{B}}(W,\mathcal{Y}) - g_{\mathcal{B}}(U,\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{X}) \\
&- g_{\mathcal{B}}(U,\mathcal{P}\mathcal{X})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{Y}) + g_{\mathcal{B}}(U,\mathcal{P}\mathcal{Y})g_{\mathcal{B}}(W,\mathcal{P}\mathcal{X}) \\
&- 2g_{\mathcal{B}}(U,\mathcal{P}W)g_{\mathcal{B}}(\mathcal{X},\mathcal{P}\mathcal{Y})\} \\
&+ g_{\mathcal{B}}(\mathcal{T}_U\mathcal{X},\mathcal{T}_W\mathcal{Y}) - g_{\mathcal{B}}(\mathcal{T}_W\mathcal{X},\mathcal{T}_U\mathcal{Y}) \qquad (5.5)
\end{aligned}$$

and

$$\tilde{K}(U,W) = \frac{\nu}{4}\{||U||^2||W||^2 - g^2_{\mathcal{B}}(U,W) - 3g^2_{\mathcal{B}}(U,\mathcal{P}W)\} + g_{\mathcal{B}}(\mathcal{T}_U U,\mathcal{T}_W W) + ||\mathcal{T}_U W||^2 \tag{5.6}$$

for all non-null vector fields $U, W, \mathcal{X}, \mathcal{Y} \in \Gamma(D_2)$.

Proof. For all non-null vector fields $U, W, \mathcal{X}, \mathcal{Y} \in \Gamma(D_2)$, by using (5.1) we obtain (5.5). Then, here choosing $\mathcal{X} = W$ and $\mathcal{Y} = U$ by direct calculations, we obtain (5.6). □

Theorem 5.4. *Let $\psi : (\mathcal{B}(\nu), g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 from a para-Kaehler space form $(\mathcal{B}(\nu), g_{\mathcal{B}})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ with a semi-slant angle φ. If $\mathcal{D}_2$ is totally geodesic, then we obtain*

$$\tilde{\sigma}_{D_2} = \frac{\nu}{2}(-1 - 2q + 3\tanh^2\varphi\sinh^2\varphi), \tag{5.7}$$

where $dim(D_2) = 2q$.

Proof. Suppose that $E_1, E_2, ..., E_q$ are space-like and $E_{q+1} = sech\varphi n E_1, E_{q+2} = sech\varphi n E_2, ..., E_{2q} = sech\varphi n E_q$ are time-like orthonormal basis on $\Gamma(D_2)$. For any non-null vector fields $U, W \in \Gamma(D_2)$, we get

$$\tilde{S}_{D_2}(U,W) = \sum_{i=1}^{q}\tilde{R}(E_i,U,W,E_i) + \sum_{j=q+1}^{2q}\tilde{R}(sech\varphi n E_i,U,W,sech\varphi n E_i).$$

Using (5.5), we have

$$\tilde{S}_{D_2}(U,W) = \frac{\nu}{4}(-1 - 2q + 3\tanh^2\varphi\sinh^2\varphi)g_{\mathcal{B}}(U,W). \tag{5.8}$$

If we take $U = W = E_i, i = 1, 2, ..., 2q$ in (58), we complete the proof. From (5.8), we obtain: □

Corollary 5.2. *Let $\psi : (\mathcal{B}(\nu), g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper semi-slant pseudo-Riemannian submersion of type ∼1 from a para-Kaehler space form $(\mathcal{B}(\nu), g_{\mathcal{B}})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ with a semi-slant angle φ. We deduce that if D_2 is totally geodesic distribution, D_2 is Einstein.*

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Akyol, M.A., Sarı, R.: *On semi-slant $\xi^{\perp}$ -Riemannian submersions,* Preprint arxiv:1704.01412 (2017).
[2] Alegre, P., Carriazo, A.: *Slant submanifolds of para-Hermitian manifolds.* Mediterr. J. Math. **14** (5), 1-14 (2017).
[3] Alegre, P., Carriazo, A.: *Bi-slant submanifolds of para-Hermitian manifolds.* Mathematics. **7** (7), 618 (2019)
[4] Baditoiu, G., Ianus, S.: *Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces.* Diff. Geom. and appl. **16** 1, 79-84 (2002).
[5] Caldarella, A.V.: *On para-quaternionic submersions between para-quaternionic Kähler manifolds.* Acta Applicandae Mathematicae. **112** 1, 1-14 (2010).
[6] Chen, B. Y.: *Classification of flat Lagrangian H-umbilical submanifolds in para-Kähler n-plane.* International Electronic Journal of Geometry. **4** 1, 1-14 (2011).
[7] Falcitelli, M., Ianus, S., Pastore, A. M.: *Riemannian Submersions and Related Topics.* World Scientific. (2004).
[8] Gilkey, P., Itoh, M., Park, J.H.: *Anti-invariant Riemannian submersions: A Lietheoretical approach.* Taiwanese J. Math. **20** 4, 787-800, (2016).
[9] Gündüzalp, Y.: *Slant submersions in paracontact geometry.* Hacet. J. Math. Stat. **49** 2 , 822-834 (2020).
[10] Gündüzalp, Y.: *Almost para-Hermitian submersions.* Matematicki Vesnik. **68** 4, 241-253 (2016).
[11] Gündüzalp, Y.: *Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds.* Journal of Function Spaces and Applications. 2013 (2013) .
[12] Gündüzalp, Y.: *Anti-invariant Pseudo-Riemannian Submersions and Clairaut Submersions from Paracosymplectic Manifolds.* Mediterr. J. Math. **16** 4, 1-18 (2019).
[13] Gündüzalp, Y.: *Neutral slant submersions in paracomplex geometry,* Afr. Mat. **32** 5, 1095-1110 (2021).
[14] Gündüzalp, Y., Akyol, M.A.:*Conformal slant submersions from cosymplectic manifolds.* Turk. J. Math. **42** 5, 2672–2689 (2018).
[15] Gray, A.: *Pseudo-Riemannian almost product manifolds and submersions.* Journal of Mathematics and Mechanics. **16** 7, 715-737 (1967).
[16] Ianus, S., Mazzocco, R., Vilcu, G. E.: *Riemannian submersions from quaternionic manifolds.* Acta Applicandae Mathematicae. **104** 1, 83-89 (2008).
[17] Ianus, S., Vilcu, G.E., Voicu, R.C.: *Harmonic maps and Riemannian submersions between manifolds endowed with special structures.* Banach Center Publications.**93** 277-288, (2011).
[18] Ivanov, S., Zamkovoy, S.: *Para-Hermitian and para-quaternionic manifolds.* Differential Geometry and its Applications. **23** 2, 205-234 (2005).
[19] Erken, I.K., Murathan, C.: *On slant Riemannian submersions for cosymplectic manifolds.* Bull. Korean Math. Soc. **51** 6 (2014).
[20] Lee, J. W., Ṣahin, B.: *Pointwise slant submersions.* Bull. Korean Math. Soc. **51** 4, 1115–1126 (2014).
[21] Lee, C. W., Lee, J. W., Ṣahin B., Vîlcu, G.E.: *Optimal inequalities for Riemannian maps and Riemannian submersions involving Casorati curvatures.* Annali di Matematica Pura ed Applicata.**200** 3 1277-1295 (2021).
[22] O'Neill, B.: *The fundamental equations of a submersion.* Michigan Mathematical Journal. **13** 4, 459-469 (1966).
[23] Özdemir, F., Sayar, C., Taṣtan, H.M.: *Semi-invariant submersions whose total manifolds are locally product Riemannian.* Quaestiones Mathematicae. **40** 7, 909-926 (2017).
[24] Prvanovicć, M.: *Holomorphically projective transformations in a locally product space.* Math. Balkanica **1** , 195-213 (1971).
[25] Park, K.S.: *H-slant submersions.* Bull. Korean Math. Soc. **49** 2, 329-338 (2012).
[26] Park, K.S., Prasad, R.: *Semi-slant submersions.* Bull. Korean Math. Soc. **50** 3, 951-962 (2013).
[27] Prasad, R., Shukla, S. S., Kumar, S.: *On Quasi-bi-slant Submersions.* Mediterr. J. Math. **16** 6, 1-18 (2019).
[28] Sepet, S. A., Ergut, M.: *Pointwise slant submersions from cosymplectic manifolds.* Turkish J.Math.**40** 3, 582-593 (2016).
[29] Sarı R., Akyol M.A.: *Hemi-slant $\xi^{\perp}$ -Riemannian submersions in contact geometry.* Filomat. **34** 11, 3747–3758 (2020).
[30] Ṣahin, B.: *Anti-invariant Riemannian submersions from almost Hermitian manifolds.* Central European J.Math. **8** 3, 437-447 (2010).
[31] Ṣahin, B.: *Slant submersions from almost Hermitian manifolds.* Bull. Math. Soc.Sci. Math. Roumanie Tome. **54** 102, 93-105 (2011).
[32] Ṣahin, B.: *Riemannian submersions from almost Hermitian manifolds.* Taiwanese J. Math. **17** 2, 629-659 (2013).
[33] Ṣahin, B.: *Semi-invariant Submersions from Almost Hermitian Manifold.* Canadian Mathematical Bulletin. **56** 1, 173-183 (2013).
[34] Ṣahin, B.: *Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and their Applications.* Academic Press, (2017).
[35] Tastan, H. M., Ṣahin B., Yanan, Ṣ.: *Hemi-slant submersions.* Mediterr. J. Math. **13** 4, 2171–2184 (2016).
[36] Vîlcu, G.E.:*Almost product structures on statistical manifolds and para-Kähler-like statistical submersions.* Bulletin des Sciences Mathématiques. **171**, 103018 (2021).
[37] Watson, B.: *Almost Hermitian submersions.* Journal of Differential Geometry. **11** 1, 147-165 (1976).

Affiliations

ESRA BAŞARIR NOYAN
ADDRESS: Department of Mathematics, Dicle University, 21280, Sur, Diyarbakır, Türkiye.
E-MAIL: bsrrnoyan@gmail.com
ORCID ID: https:0000-0001-6535-7498

YILMAZ GÜNDÜZALP
ADDRESS: Department of Mathematics, Dicle University, 21280, Sur, Diyarbakır, Türkiye.
E-MAIL: ygunduzalp@dicle.edu.tr
ORCID ID: 0000-0002-0932-949X