Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1033708
(**2022**) **3**(1) 57 – 74

Conformal Hemi-Slant Riemannian Maps

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Received: 07 December 2021 **Accepted:** 20 January 2022

Abstract: In this study, we define conformal hemi-slant Riemannian maps from an almost Hermitian manifold to a Riemannian manifold as a generalization of conformal anti-invariant Riemannian maps, conformal semi-invariant Riemannian maps and conformal slant Riemannian maps. Then, we obtain integrability conditions for certain distributions which are included in the notion of hemi-slant Riemannian maps and investigate their leaves. Also, we get totally geodesic conditions for this type maps. Lastly, we introduce some geometric properties under the notion of pluri-harmonic map.

Keywords: Riemannian submersion, Riemannian map, conformal Riemannian map, conformal hemi-slant Riemannian map.

1. Introduction

Particularly, the concept of Riemannian submersions [6] and isometric immersions [5] were studied by Falcitelli and Chen. Then, Riemannian submersions were studied in various types as an anti-invariant, a semi-invariant, a slant and a hemi-slant [16]. Then, this concept generalized to the notion of Riemannian map by Fischer [7]. Riemannian maps between Riemannian manifolds are generalization of isometric immersions and Riemannian submersions. Let $\Phi : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rank\Phi < min\{dim(M_1), dim(M_2)\}$. Then, the tangent bundle TM_1 of M_1 has the following decomposition:

$$TM_1 = ker\Phi_* \oplus (ker\Phi_*)^\perp.$$

Since $rank\Phi < min\{dim(M_1), dim(M_2)\}$, always we have $(range\Phi_*)^\perp$. In this way, tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (range\Phi_*) \oplus (range\Phi_*)^\perp.$$

A smooth map $\Phi : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $\Phi^h_{*p_1} : (ker\Phi_{*p_1})^\perp \longrightarrow (range\Phi_*)$ is a linear isometry. Hence, a Riemannian map

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2020 *AMS Mathematics Subject Classification*: 53C15, 53C55

satisfies the equation

$$g_1(X,Y) = g_2(\Phi_*(X), \Phi_*(Y)) \tag{1}$$

for $X, Y \in \Gamma((ker\Phi_*)^{\perp})$. So that isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $ker\Phi_* = \{0\}$ and $(range\Phi_*)^{\perp} = \{0\}$ [6]. An important application field of Riemannian maps is the eikonal equation. It acts as a bridge between geometric optics and physical optics. Also, Riemannian maps and their applications studied by Garcia-Rio and Kupeli in semi-Riemannian geometry [8].

Moreover, Şahin introduced any other types of Riemannian maps [13–16]. In further studies, in particular Akyol, Şahin and Yanan searched this type submersions [1–3] and Riemannian maps [18–21] under conformality case, see also [9]. We say that $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rank\Phi_{*p} \le min\{m,n\}$ and Φ_{*p} maps the horizontal space $(ker(\Phi_{*p})^{\perp})$ conformally onto $range(\Phi_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(\Phi_{*p}(X), \Phi_{*p}(Y)) = \lambda^2(p) g_M(X,Y) \tag{2}$$

for $X, Y \in \Gamma((ker(\Phi_{*p})^{\perp})$. Also, Φ is called conformal Riemannian if Φ is conformal Riemannian at each $p \in M$ [17].

An even-dimensional Riemannian manifold (M, g_M, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1,1)$ on M such that $J^2 = -I$ where I denotes the identity transformation of TM and

$$g_M(X,Y) = g_M(JX, JY), \forall X, Y \in \Gamma(TM). \tag{3}$$

Let (M, g_M, J) is an almost Hermitian manifold and its Levi-Civita connection is ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.,

$$(\nabla_X J)Y = 0, \tag{4}$$

we say M is a Kähler manifold [22].

Therefore, in Section 2; we present background concepts to be used in this paper. In Section 3; we study conformal hemi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds as a generalization of conformal semi-invariant Riemannian maps and conformal slant Riemannian maps. In Section 4; we use the concept of pluriharmonicity to introduce geometric properties.

2. Preliminaries

In this section, we give several definitions and results to be used throughout the study for conformal hemi-slant Riemannian maps. Let $\Phi : (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds. The second fundamental form of Φ is defined by

$$(\nabla\Phi_*)(X,Y) = \overset{N}{\nabla}{}^{\Phi}_{X}\Phi_*(Y) - \Phi_*(\overset{M}{\nabla}_X Y) \tag{5}$$

for $X,Y \in \Gamma(TM)$. The second fundamental form $\nabla\Phi_*$ is symmetric [10].

Then, we define O'Neill's tensor fields $\mathcal{T}$ and $\mathcal{A}$ for Riemannian submersions as

$$\mathcal{A}_X Y = h\overset{M}{\nabla}_{hX} vY + v\overset{M}{\nabla}_{hX} hY, \tag{6}$$

$$\mathcal{T}_X Y = h\overset{M}{\nabla}_{vX} vY + v\overset{M}{\nabla}_{vX} hY \tag{7}$$

for $X,Y \in \Gamma(TM)$ with the Levi-Civita connection $\overset{M}{\nabla}$ of g_M [12]. As usual, we denote by v and h the projections on the vertical distribution $ker\Phi_*$ and the horizontal distribution $(ker\,\Phi_*)^\perp$, respectively. For any $X \in \Gamma(TM)$, $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. Also, $\mathcal{T}$ is vertical, $\mathcal{T}_X = \mathcal{T}_{vX}$, and $\mathcal{A}$ is horizontal, $\mathcal{A}_X = \mathcal{A}_{hX}$. Note that the tensor field $\mathcal{T}$ is symmetric on the vertical distribution [12]. Additionally, from (6) and (7) we have

$$\overset{M}{\nabla}_U V = \mathcal{T}_U V + \hat{\nabla}_U V, \tag{8}$$

$$\overset{M}{\nabla}_U X = h\overset{M}{\nabla}_U X + \mathcal{T}_U X, \tag{9}$$

$$\overset{M}{\nabla}_X V = \mathcal{A}_X V + v\overset{M}{\nabla}_X V, \tag{10}$$

$$\overset{M}{\nabla}_X Y = h\overset{M}{\nabla}_X Y + \mathcal{A}_X Y \tag{11}$$

for $X,Y \in \Gamma((ker\,\Phi_*)^\perp)$ and $U,V \in \Gamma(ker\Phi_*)$, where $\hat{\nabla}_U V = v\overset{M}{\nabla}_U V$ [6].

If a vector field X on M is related to a vector field X' on N, we say X is a projectable vector field. If X is both a horizontal and a projectable vector field, we say X is a basic vector field on M. From now on, when we mention a horizontal vector field, we always consider a basic vector field [4].

On the other hand, let $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ be a conformal Riemannian map between Riemannian manifolds. Then, we have

$$\begin{aligned}(\nabla\Phi_*)(X,Y)\,|_{range\Phi_*} &= X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X)\\ &- g_M(X,Y)\Phi_*(grad(\ln\lambda)),\end{aligned} \tag{12}$$

where $X, Y \in \Gamma((ker\Phi_*)^\perp)$. Hence from (12), we obtain $\overset{N}{\nabla}{}^{\Phi}_{X}\Phi_*(Y)$ as

$$\begin{aligned} \overset{N}{\nabla}{}^{\Phi}_{X}\Phi_*(Y) &= \Phi_*(h\overset{M}{\nabla}_X Y) + X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) \\ &- g_M(X,Y)\Phi_*(grad(\ln\lambda)) + (\nabla\Phi_*)^\perp(X,Y), \end{aligned} \tag{13}$$

where $(\nabla\Phi_*)^\perp(X,Y)$ is the component of $(\nabla\Phi_*)(X,Y)$ on $(range\Phi_*)^\perp$ for $X, Y \in \Gamma((ker\Phi_*)^\perp)$ [18, 19].

Lastly, a map Φ from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if Φ satisfies the following equation

$$(\nabla\Phi_*)(X,Y) + (\nabla\Phi_*)(JX,JY) = 0 \tag{14}$$

for $X, Y \in \Gamma(TM)$ [11].

3. Conformal Hemi-slant Riemannian Maps

We define conformal hemi-slant Riemannian maps from almost Hermitian manifolds and give some examples. We examine integrability and totally geodesicity conditions.

Definition 3.1 *A conformal Riemannian map $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ is called a conformal hemi-slant Riemannian map if the vertical distribution $ker\Phi_*$ of Φ admits two orthogonal complementary distributions $\mathcal{D}_\theta$ and $\mathcal{D}_\perp$ such that $\mathcal{D}_\theta$ is slant and $\mathcal{D}_\perp$ is anti-invariant, i.e., we have*

$$ker\Phi_* = \mathcal{D}_\theta \oplus \mathcal{D}_\perp. \tag{15}$$

Hence, the angel θ is called the hemi-slant angle of the conformal Riemannian map.

Here, if we denote the dimension of $\mathcal{D}_\theta$ and $\mathcal{D}_\perp$ by m_θ and $m_\perp$, respectively, then we get:

i) If $m_\theta = 0$, then Φ is a conformal anti-invariant Riemannian map [18].

ii) If $m_\perp = 0$ and $\theta = 0$, then Φ is a conformal invariant Riemannian map.

iii) If $m_\perp = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then Φ is a proper conformal slant Riemannian map [21].

iv) If $\theta = \frac{\pi}{2}$, then Φ is a conformal anti-invariant Riemannian map.

Now, we give some examples for conformal hemi-slant Riemannian maps.

Example 3.2 *Every conformal slant submersion [3] from an almost Hermitian manifold to a Riemannian manifold is a conformal hemi-slant Riemannian map with $\mathcal{D}_\perp = \{0\}$ and $(range\Phi_*)^\perp = \{0\}$.*

Example 3.3 *Every conformal hemi-slant submersion [9] from an almost Hermitian manifold to a Riemannian manifold is a conformal hemi-slant Riemannian map with* $(range\Phi_*)^{\perp} = \{0\}$.

Example 3.4 *Every conformal slant Riemannian map [21] from an almost Hermitian manifold to a Riemannian manifold is a conformal hemi-slant Riemannian map with* $\mathcal{D}_{\perp} = \{0\}$.

Example 3.5 *Every conformal semi-invariant submersion [2] from an almost Hermitian manifold to a Riemannian manifold is a conformal hemi-slant Riemannian map with* $\theta = \frac{\pi}{2}$ *and* $(range\Phi_*)^{\perp} = \{0\}$.

Example 3.6 *Every conformal semi-invariant Riemannian map [19] from an almost Hermitian manifold to a Riemannian manifold is a conformal hemi-slant Riemannian map with* $\theta = \frac{\pi}{2}$.

If $\mathcal{D}_{\perp} \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$,then we say Φ is a proper conformal hemi-slant Riemannian map. Hence, we give an explicit example to proper case.

Example 3.7 *Define a map* $\Phi : \mathbb{R}^8 \longrightarrow \mathbb{R}^5$ *by*

$$\Phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = e(x_2, x_3, \frac{x_6 + x_7}{\sqrt{2}}, x_8, 0)$$

with $\theta \in (0, \frac{\pi}{2})$. *We obtain the horizontal distribution*

$$(ker\Phi_*)^{\perp} = \{Z_1 = e\frac{\partial}{\partial x_2}, Z_2 = e\frac{\partial}{\partial x_3}, Z_3 = \frac{e}{\sqrt{2}}(\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}), Z_4 = e\frac{\partial}{\partial x_8}\}$$

and the vertical distribution

$$ker\Phi_* = \{W_1 = \frac{\partial}{\partial x_1}, W_2 = \frac{\partial}{\partial x_4}, W_3 = \frac{\partial}{\partial x_5}, W_4 = \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7}\},$$

respectively. If the complex structure of $\mathbb{R}^8$ *is* $J = (-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7)$, *we have*

$$JW_1 = \frac{1}{e}Z_1, \quad JW_2 = -\frac{1}{e}Z_2, \quad JW_3 = \frac{\sqrt{2}}{2e}Z_3 + \frac{1}{2}W_4, \quad JW_4 = -\frac{1}{e}Z_4 - W_3.$$

Hence, we obtain $\mathcal{D}_{\perp} = span\{W_1, W_2\}$ *and* $\mathcal{D}_{\theta} = span\{W_3, W_4\}$. *So,* Φ *is a proper conformal hemi-slant Riemannian map with slant angle* $\theta = \frac{\pi}{4}$, $\lambda = e$ *and* $rank\Phi = 4$.

For any $W \in \Gamma(ker\Phi_*)$, we get

$$W = \tilde{P}W + \tilde{Q}W, \tag{16}$$

where $\tilde{P}W \in \Gamma(\mathcal{D}_\theta)$ and $\tilde{Q}W \in \Gamma(\mathcal{D}_\perp)$, and have

$$JW = \phi W + \psi W, \tag{17}$$

where $\phi W \in \Gamma(ker\Phi_*)$ and $\psi W \in \Gamma((ker\Phi_*)^\perp)$. Lastly, for $Z \in \Gamma((ker\Phi_*)^\perp)$, we have

$$JZ = BZ + CZ, \tag{18}$$

where $BZ \in \Gamma(ker\Phi_*)$ and $CZ \in \Gamma((ker\Phi_*)^\perp)$. Hence, we obtain decomposition of $(ker\Phi_*)^\perp$ as

$$(ker\Phi_*)^\perp = \psi\mathcal{D}_\theta \oplus J\mathcal{D}_\perp \oplus \mu, \tag{19}$$

where μ is the orthogonal complement of $\psi\mathcal{D}_\theta \oplus J\mathcal{D}_\perp$ and it is invariant under J. From equations (16)-(19), we obtain followings:

$$\phi\mathcal{D}_\theta = \mathcal{D}_\theta, \quad \phi\mathcal{D}_\perp = \{0\}, \quad B\psi\mathcal{D}_\theta = \mathcal{D}_\theta, \quad BJ\mathcal{D}_\perp = \mathcal{D}_\perp \tag{20}$$

and

$$\phi^2 + B\psi = -I, \quad \psi\phi + C\psi = \{0\}, \quad \phi B + BC = \{0\}, \quad \psi B + C^2 = -I. \tag{21}$$

The proof of the next theorem is exactly same with hemi-slant submanifolds like hemi-slant Riemannian maps; see Theorem 3.6 of [15].

Theorem 3.8 *Let Φ be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, Φ is a conformal hemi-slant Riemannian map if and only if there exists a constant $\lambda \in [0,1]$ and a distribution $\mathcal{D}$ on $ker\Phi_*$ such that*

i) $\mathcal{D} = \{W \in \Gamma(ker\Phi_*) | \phi^2 W = \lambda W\}$,

ii) we have $\phi W = 0$, for any $W \in \Gamma(ker\Phi_*)$ orthogonal to $\mathcal{D}$.

Further, we have $\lambda = -\cos^2\theta$ where θ is the slant angle of Φ.

The next expressions are easy to see their validity

$$g_M(\phi U_1, \phi U_2) = \cos^2\theta g_M(U_1, U_2), \tag{22}$$

$$g_M(\psi U_1, \psi U_2) = \sin^2\theta g_M(U_1, U_2) \tag{23}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_\theta)$.

Now, we give some integrability conditions for leaf of the distributions.

Theorem 3.9 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, the slant distribution $\mathcal{D}_\theta$ is integrable if and only if*

$$\begin{aligned}\lambda^2\{g_M(\mathcal{T}_{U_1}JV, \phi U_2) - g_M(\mathcal{T}_{U_2}JV, \phi U_1)\} &= g_N(\overset{N}{\nabla}{}^{\Phi}_{U_2}\Phi_*(JV) + \Phi_*(\mathcal{A}_{JV}U_2), \Phi_*(\psi U_1)) \\ &- g_N(\overset{N}{\nabla}{}^{\Phi}_{U_1}\Phi_*(JV) + \Phi_*(\mathcal{A}_{JV}U_1), \Phi_*(\psi U_2))\end{aligned}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D}_\perp)$.

Proof Since g_M is the Kähler metric from (9) and (17), we get

$$g_M(\overset{M}{\nabla}_{U_1}U_2, V) = -g_M(\mathcal{T}_{U_1}JV, \phi U_2) - g_M(h\overset{M}{\nabla}_{U_1}JV, \psi U_2) \tag{24}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D}_\perp)$. Now, using (5) and symmetry condition of $\nabla\Phi_*$, we get

$$\Phi_*(h\overset{M}{\nabla}_{U_1}JV) = \overset{N}{\nabla}{}^{\Phi}_{U_1}\Phi_*(JV) + \Phi_*(\mathcal{A}_{JV}U_1). \tag{25}$$

Putting (25) in (24), we have

$$\begin{aligned}g_M(\overset{M}{\nabla}_{U_1}U_2, V) &= -g_M(\mathcal{T}_{U_1}JV, \phi U_2) \\ &- \frac{1}{\lambda^2}g_N(\overset{N}{\nabla}{}^{\Phi}_{U_1}\Phi_*(JV) + \Phi_*(\mathcal{A}_{JV}U_1), \Phi_*(\psi U_2)).\end{aligned} \tag{26}$$

Lastly, changing the roles of U_1 and U_2 in (26) we obtain the proof. □

The integrability condition of $\mathcal{D}_\perp$ is the same with Theorem 3.8 in [15]. Note that, always the distribution $ker\Phi_*$ is integrable. Then, we have the following.

Theorem 3.10 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, the horizontal distribution $(ker\Phi_*)^\perp$ is integrable if and only if*

i)

$$\begin{aligned}&g_N((\nabla\Phi_*)(Z_2, BZ_1) - (\nabla\Phi_*)(Z_1, BZ_2) + \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(CZ_2) - \overset{N}{\nabla}{}^{\Phi}_{Z_2}\Phi_*(CZ_1), \Phi_*(\psi U)) \\ &= \lambda^2\{g_M(v\overset{M}{\nabla}_{Z_1}BZ_2 + \mathcal{A}_{Z_1}CZ_2 - v\overset{M}{\nabla}_{Z_2}BZ_1 - \mathcal{A}_{Z_2}CZ_1, \phi U) \\ &-Z_1(\ln\lambda)g_M(CZ_2, \psi U) - CZ_2(\ln\lambda)g_M(Z_1, \psi U) + Z_2(\ln\lambda)g_M(CZ_1, \psi U) \\ &+CZ_1(\ln\lambda)g_M(Z_2, \psi U) + \psi U(\ln\lambda)(g_M(Z_1, CZ_2) - g_M(Z_2, CZ_1))\},\end{aligned}$$

ii) $\tilde{Q}\{B\{\mathcal{A}_{Z_1}BZ_2 + h\overset{M}{\nabla}_{Z_1}CZ_2 - \mathcal{A}_{Z_2}BZ_1 - h\overset{M}{\nabla}_{Z_2}CZ_1\}\} = 0$

are provided for any $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$, $U \in \Gamma(\mathcal{D}_\theta)$ *and* $V \in \Gamma(\mathcal{D}_\perp)$.

Proof We search $g_M([Z_1, Z_2], U) = 0$ and $g_M([Z_1, Z_2], V) = 0$ for any $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$, $U \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D}_\perp)$. Firstly, using (10), (11) and (17), we get

$$\begin{aligned} g_M([Z_1, Z_2], U) &= g_M(\mathcal{A}_{Z_1} BZ_2 + h\overset{M}{\nabla}_{Z_1} CZ_2 - \mathcal{A}_{Z_2} BZ_1 - h\overset{M}{\nabla}_{Z_2} CZ_1, \psi U) \\ &+ g_M(v\overset{M}{\nabla}_{Z_1} BZ_2 + \mathcal{A}_{Z_1} CZ_2 - v\overset{M}{\nabla}_{Z_2} BZ_1 - \mathcal{A}_{Z_2} CZ_1, \phi U). \end{aligned} \tag{27}$$

We have $(\nabla\Phi_*)(Z_1, BZ_2) = -\Phi_*(\mathcal{A}_{Z_1} BZ_2)$ from (5) and equality of $\Phi_*(h\overset{M}{\nabla}_{Z_1} CZ_2)$ from (13). In (27), we obtain

$$\begin{aligned} g_M([Z_1, Z_2], U) &= \frac{1}{\lambda^2} g_N((\nabla\Phi_*)(Z_2, BZ_1) - (\nabla\Phi_*)(Z_1, BZ_2), \Phi_*(\psi U)) \\ &+ \frac{1}{\lambda^2} g_N(\overset{N}{\nabla}{}^{\Phi}_{Z_1} \Phi_*(CZ_2) - \overset{N}{\nabla}{}^{\Phi}_{Z_2} \Phi_*(CZ_1), \Phi_*(\psi U)) \\ &- Z_1(\ln\lambda) g_M(CZ_2, \psi U) - CZ_2(\ln\lambda) g_M(Z_1, \psi U) \\ &+ \psi U(\ln\lambda) g_M(Z_1, CZ_2) + Z_2(\ln\lambda) g_M(CZ_1, \psi U) \\ &+ CZ_1(\ln\lambda) g_M(Z_2, \psi U) - \psi U(\ln\lambda) g_M(Z_2, CZ_1) \\ &+ g_M(v\overset{M}{\nabla}_{Z_1} BZ_2 + \mathcal{A}_{Z_1} CZ_2 - v\overset{M}{\nabla}_{Z_2} BZ_1 - \mathcal{A}_{Z_2} CZ_1, \phi U). \end{aligned} \tag{28}$$

We get (i) from (28). Now, for (10) and (11) we obtain

$$\begin{aligned} g_M([Z_1, Z_2], V) &= -g_M(B\{\mathcal{A}_{Z_1} BZ_2 + h\overset{M}{\nabla}_{Z_1} CZ_2\}, V) \\ &+ g_M(B\{\mathcal{A}_{Z_2} BZ_1 + h\overset{M}{\nabla}_{Z_2} CZ_1\}, V). \end{aligned} \tag{29}$$

From (16) and (29), we get (ii). □

In the rest of the section, we investigate totally geodesicity conditions on total manifold. Recall that Φ is said to be horizontally homothetic map if $h(grad(\ln\lambda)) = 0$ [4] and Φ is said to be totally geodesic map if $(\nabla\Phi_*)(E, F) = 0$ for all $E, F \in \Gamma(TM)$ [16].

Theorem 3.11 *Let* Φ *be a conformal hemi-slant Riemannian map from a Kähler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N). *Then, any two conditions below imply the third condition;*

i) $ker\Phi_*$ *defines a totally geodesic foliation on* M,

ii) Φ *is a horizontally homothetic map,* *iii)*

$$\overset{N}{\nabla}{}^{\Phi}_{JW_1}\Phi_*(\psi W_2) = \Phi_*(J[JW_1,W_2]) + (\nabla\Phi_*)^{\perp}(\psi W_1,\psi W_2) + \Phi_*(\mathcal{T}_{\phi W_1}\phi W_2 + \mathcal{A}_{\psi W_2}\phi W_1 + \mathcal{A}_{\psi W_1}\phi W_2)$$

for any $W_1, W_2 \in \Gamma(ker\Phi_*)$.

Proof Using equations (5) and (13), we get

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{JW_1}JW_2) &= \overset{N}{\nabla}{}^{\Phi}_{JW_1}\Phi_*(JW_2) - (\nabla\Phi_*)(JW_1,JW_2)\\ &= \overset{N}{\nabla}{}^{\Phi}_{JW_1}\Phi_*(\psi W_2) - \Phi_*(\mathcal{T}_{\phi W_1}\phi W_2 + \mathcal{A}_{\psi W_2}\phi W_1 + \mathcal{A}_{\psi W_1}\phi W_2)\\ &- \psi W_1(\ln\lambda)\Phi_*(\psi W_2) - \psi W_2(\ln\lambda)\Phi_*(\psi W_1)\\ &+ g_M(\psi W_1,\psi W_2)\Phi_*(grad(\ln\lambda)) - (\nabla\Phi_*)^{\perp}(\psi W_1,\psi W_2)\end{aligned} \tag{30}$$

for any $W_1, W_2 \in \Gamma(ker\Phi_*)$. On the other hand, we get

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{JW_1}JW_2) &= \Phi_*(J[JW_1,W_2]) + J\overset{M}{\nabla}_{W_1}JW_2)\\ &= \Phi_*(J[JW_1,W_2]) - \Phi_*(\overset{M}{\nabla}_{W_1}W_2).\end{aligned} \tag{31}$$

Putting (31) in (30), we obtain

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{W_1}W_2) &= \Phi_*(J[JW_1,W_2]) - \overset{N}{\nabla}{}^{\Phi}_{JW_1}\Phi_*(\psi W_2)\\ &+ \Phi_*(\mathcal{T}_{\phi W_1}\phi W_2 + \mathcal{A}_{\psi W_2}\phi W_1 + \mathcal{A}_{\psi W_1}\phi W_2)\\ &+ \psi W_1(\ln\lambda)\Phi_*(\psi W_2) + \psi W_2(\ln\lambda)\Phi_*(\psi W_1)\\ &- g_M(\psi W_1,\psi W_2)\Phi_*(grad(\ln\lambda)) + (\nabla\Phi_*)^{\perp}(\psi W_1,\psi W_2).\end{aligned} \tag{32}$$

Suppose that (i) and (ii) are provided in (32). Then, we have

$$\Phi_*(\overset{M}{\nabla}_{W_1}W_2) = 0$$

and

$$\psi W_1(\ln\lambda)\Phi_*(\psi W_2) + \psi W_2(\ln\lambda)\Phi_*(\psi W_1) - g_M(\psi W_1,\psi W_2)\Phi_*(grad(\ln\lambda)) = 0.$$

Hence, we obtain

$$\begin{aligned}0 &= \Phi_*(J[JW_1,W_2]) - \overset{N}{\nabla}{}^{\Phi}_{JW_1}\Phi_*(\psi W_2) + (\nabla\Phi_*)^{\perp}(\psi W_1,\psi W_2)\\ &+ \Phi_*(\mathcal{T}_{\phi W_1}\phi W_2 + \mathcal{A}_{\psi W_2}\phi W_1 + \mathcal{A}_{\psi W_1}\phi W_2).\end{aligned} \tag{33}$$

We get (iii) from (33). One can easily see that if (ii) and (iii) are provided in (32) we obtain $\Phi_*(\overset{M}{\nabla}_{W_1} W_2) = 0$. So, (i) is satisfied. Lastly, we proof (ii). Suppose that (i) and (iii) are provided in (32). Then, we obtain

$$\begin{aligned} 0 &= \psi W_1(\ln\lambda)\Phi_*(\psi W_2) + \psi W_2(\ln\lambda)\Phi_*(\psi W_1) \\ &- g_M(\psi W_1, \psi W_2)\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{34}$$

For $\psi W_1 \in \Gamma((ker\Phi_*)^\perp)$ in (34), we have

$$0 = \lambda^2 \psi W_2(\ln\lambda) g_M(\psi W_1, \psi W_1).$$

So, we obtain $\psi W_2(\ln\lambda) = 0$. It means λ is a constant on $(ker\Phi_*)^\perp$. Therefore, Φ is a horizontally homothetic map. The proof is complete. □

In a similar way, we have the following.

Theorem 3.12 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, any two conditions below imply the third condition;*

i) $(ker\Phi_*)^\perp$ defines a totally geodesic foliation on M,

ii) Φ is a horizontally homothetic map,

iii)

$$\begin{aligned} \overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_2) &= \Phi_*(J[Z_1, JZ_2]) - (\nabla\Phi_*)^\perp(CZ_1, CZ_2) \\ &+ \Phi_*(\mathcal{T}_{BZ_1}BZ_2 + \mathcal{A}_{CZ_1}BZ_2 + \mathcal{A}_{CZ_2}BZ_1) \end{aligned}$$

for any $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$.

Theorem 3.13 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, the slant distribution $\mathcal{D}_\theta$ defines a totally geodesic foliation on M if and only if*

$$\cos^2\theta \mathcal{T}_{U_1}U_2 = \mathcal{T}_{U_1}B\psi U_2$$

is provided for any $U_1, U_2 \in \Gamma(\mathcal{D}_\theta)$.

Proof From definition of $\nabla\Phi_*$ and (17), we get

$$\begin{aligned}(\nabla\Phi_*)(U_1,U_2) &= \Phi_*(J\overset{M}{\nabla}_{U_1}JU_2)\\ &= \Phi_*(\overset{M}{\nabla}_{U_1}J\phi U_2)+\Phi_*(\overset{M}{\nabla}_{U_1}J\psi U_2)\\ &= \Phi_*(\overset{M}{\nabla}_{U_1}\phi^2U_2+\overset{M}{\nabla}_{U_1}\psi\phi U_2)+\Phi_*(\overset{M}{\nabla}_{U_1}B\psi U_2+\overset{M}{\nabla}_{U_1}C\psi U_2) \qquad (35)\end{aligned}$$

for any $U_1,U_2\in\Gamma(\mathcal{D}_\theta)$. Now, from Theorem 3.8 and by using (20) in (34), we obtain

$$(\nabla\Phi_*)(U_1,U_2) = -\cos^2\theta\Phi_*(\mathcal{T}_{U_1}U_2)+\Phi_*(\mathcal{T}_{U_1}B\psi U_2). \qquad (36)$$

The proof is complete. □

Theorem 3.14 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M,g_M,J) to a Riemannian manifold (N,g_N). Then, any two conditions below imply the third condition;*

i) $\mathcal{D}_\perp$ defines a totally geodesic foliation on M,

ii) λ is a constant on $J(\mathcal{D}_\perp)$,

iii) $\overset{N}{\nabla}{}^{\Phi}_{JV_1}\Phi_*(JV_2) = (\nabla\Phi_*)^\perp(JV_1,JV_2)-\Phi_*(J[V_2,JV_1])$

for any $V_1,V_2\in\Gamma(\mathcal{D}_\perp)$.

Proof From the definition of $\nabla\Phi_*$, we have

$$\begin{aligned}(\nabla\Phi_*)(JV_1,JV_2) &= \overset{N}{\nabla}{}^{\Phi}_{JV_1}\Phi_*(JV_2)-\Phi_*(\overset{M}{\nabla}_{JV_1}JV_2)\\ &= \overset{N}{\nabla}{}^{\Phi}_{JV_1}\Phi_*(JV_2)+\Phi_*(J[V_2,JV_1]-J\overset{M}{\nabla}_{V_2}JV_1)\\ &= \overset{N}{\nabla}{}^{\Phi}_{JV_1}\Phi_*(JV_2)+\Phi_*(J[V_2,JV_1])+\Phi_*(\overset{M}{\nabla}_{V_2}V_1) \qquad (37)\end{aligned}$$

for any $V_1,V_2\in\Gamma(\mathcal{D}_\perp)$. Using (13) in (37), we obtain

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{V_2}V_1) &= -\overset{N}{\nabla}{}^{\Phi}_{JV_1}\Phi_*(JV_2)-\Phi_*(J[V_2,JV_1])\\ &+ JV_1(\ln\lambda)\Phi_*(JV_2)+JV_2(\ln\lambda)\Phi_*(JV_1)\\ &- g_M(JV_1,JV_2)\Phi_*(grad(\ln\lambda))+(\nabla\Phi_*)^\perp(JV_1,JV_2). \qquad (38)\end{aligned}$$

Suppose that (i) and (iii) are satisfies in (38). So, we have

$$0 = \Phi_*(\overset{M}{\nabla}_{V_2}V_1)$$

and

$$0 = -\overset{N}{\nabla}{}^{\Phi}_{JV_1}\Phi_*(JV_2) - \Phi_*(J[V_2, JV_1]) + (\nabla\Phi_*)^{\perp}(JV_1, JV_2).$$

Therefore, we obtain from (38)

$$\begin{aligned} 0 &= JV_1(\ln\lambda)\Phi_*(JV_2) + JV_2(\ln\lambda)\Phi_*(JV_1) \\ &- g_M(JV_1, JV_2)\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{39}$$

Now, we obtain from (39)

$$0 = \lambda^2 JV_2(\ln\lambda)g_M(JV_1, JV_1) \tag{40}$$

for any $V_1 \in \Gamma(\mathcal{D}_{\perp})$. So, we obtain $JV_2(\ln\lambda) = 0$. It means λ is a constant on $J(\mathcal{D}_{\perp})$. The proofs of (i) and (iii) are easy to see from (38). □

Lastly, we present totally geodesicity of the map Φ.

Theorem 3.15 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, the map Φ defines a totally geodesic foliation on M if and only if*

i) Φ is a horizontally homothetic map,

ii)

$$\begin{aligned} \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) &= (\nabla\Phi_*)^{\perp}(Z_1, Z_2) - \Phi_*(C\{\mathcal{A}_{Z_1}BZ_2 + h\overset{M}{\nabla}_{Z_1}CZ_2\}) \\ &- \Phi_*(\psi\{v\overset{M}{\nabla}_{Z_1}BZ_2 + \mathcal{A}_{Z_1}CZ_2\}), \end{aligned}$$

iii)

$$\begin{aligned} \cos^2\theta\mathcal{T}_{W_1}\tilde{P}W_2 &= h\overset{M}{\nabla}_{W_1}\psi\phi\tilde{P}W_2 + \psi\mathcal{T}_{W_1}(\psi\tilde{P}W_2 + J\tilde{Q}W_2) \\ &+ Ch\overset{M}{\nabla}_{W_1}(\psi\tilde{P}W_2 + J\tilde{Q}W_2) \end{aligned}$$

are provided for any $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$ and $W_1, W_2 \in \Gamma(ker\Phi_*)$.

Proof Because of rank condition of the conformal hemi-slant Riemannian map Φ, we have $(\nabla\Phi_*)(Z_1, Z_2) = (\nabla\Phi_*)^{\perp}(Z_1, Z_2) + (\nabla\Phi_*)^{\top}(Z_1, Z_2)$ for any $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$. We know that $(\nabla\Phi_*)^{\perp}(Z_1, Z_2) \in \Gamma((range\Phi_*)^{\perp})$ and $(\nabla\Phi_*)^{\top}(Z_1, Z_2) \in \Gamma(range\Phi_*)$, see (12) and (13). Using these equations, we obtain

$$(\nabla\Phi_*)(Z_1, Z_2) = \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) - \Phi_*(\overset{M}{\nabla}_{Z_1}Z_2). \tag{41}$$

Since $(\nabla\Phi_*)(Z_1,Z_2)=0$,

$$\begin{aligned} 0 &= \overset{N}{\nabla}{}^{\Phi}{}_{Z_1}\Phi_*(Z_2)-(\nabla\Phi_*)^{\perp}(Z_1,Z_2) \\ &+ \Phi_*(C\mathcal{A}_{Z_1}BZ_2+\psi v\overset{M}{\nabla}_{Z_1}BZ_2) \\ &+ \Phi_*(\psi\mathcal{A}_{Z_1}CZ_2+Ch\overset{M}{\nabla}_{Z_1}CZ_2) \\ &- Z_1(\ln\lambda)\Phi_*(Z_2)-Z_2(\ln\lambda)\Phi_*(Z_1) \\ &+ g_M(Z_1,Z_2)\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{42}$$

From (42), we have

$$\begin{aligned} \overset{N}{\nabla}{}^{\Phi}{}_{Z_1}\Phi_*(Z_2) &= (\nabla\Phi_*)^{\perp}(Z_1,Z_2)-\Phi_*(C\{\mathcal{A}_{Z_1}BZ_2+h\overset{M}{\nabla}_{Z_1}CZ_2\}) \\ &- \Phi_*(\psi\{v\overset{M}{\nabla}_{Z_1}BZ_2+\mathcal{A}_{Z_1}CZ_2\}) \end{aligned} \tag{43}$$

and

$$\begin{aligned} 0 &= Z_1(\ln\lambda)\Phi_*(Z_2)+Z_2(\ln\lambda)\Phi_*(Z_1) \\ &- g_M(Z_1,Z_2)\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{44}$$

In (44), for any $Z_1\in\Gamma((ker\Phi_*)^{\perp})$ we get

$$\begin{aligned} 0 &= \lambda^2 Z_1(\ln\lambda)g_M(Z_2,Z_1)+\lambda^2 Z_2(\ln\lambda)g_M(Z_1,Z_1) \\ &- \lambda^2 g_M(Z_1,Z_2)Z_1(\ln\lambda) \\ &= \lambda^2 Z_2(\ln\lambda)g_M(Z_1,Z_1). \end{aligned} \tag{45}$$

So, from (45) we obtain $Z_2(\ln\lambda)=0$. It means Φ is a horizontally homothetic map. We obtain (ii) and (i) from (43) and (45), respectively. In a similar way, we get

$$\begin{aligned} (\nabla\Phi_*)(W_1,W_2) &= \Phi_*(J\overset{M}{\nabla}_{W_1}J\tilde{P}W_2+J\tilde{Q}W_2) \\ &= \Phi_*(\overset{M}{\nabla}_{W_1}\phi^2\tilde{P}W_2+\overset{M}{\nabla}_{W_1}\psi\phi\tilde{P}W_2) \\ &+ \Phi_*(\psi\mathcal{T}_{W_1}\psi\tilde{P}W_2+Ch\overset{M}{\nabla}_{W_1}\psi\tilde{P}W_2) \\ &+ \Phi_*(\psi\mathcal{T}_{W_1}J\tilde{Q}W_2+Ch\overset{M}{\nabla}_{W_1}J\tilde{Q}W_2) \\ &= -\cos^2\theta\Phi_*(\mathcal{T}_{W_1}\tilde{P}W_2)+\Phi_*(h\overset{M}{\nabla}_{W_1}\psi\phi\tilde{P}W_2) \\ &+ \Phi_*(\psi\mathcal{T}_{W_1}(\psi\tilde{P}W_2+J\tilde{Q}W_2)) \\ &+ \Phi_*(Ch\overset{M}{\nabla}_{W_1}(\psi\tilde{P}W_2+J\tilde{Q}W_2)) \end{aligned} \tag{46}$$

for any $W_1, W_2 \in \Gamma(ker\Phi_*)$. We obtain (iii) from (46). The proof is complete. □

4. Pluriharmonic Conformal Hemi-slant Riemannian Maps

In this section, we use the notion of pluriharmonic map on the distributions of a conformal hemi-slant Riemannian map to introduce their geometric properties. Φ is said to be $\mathcal{D}_\theta$-pluriharmonic map ($\mathcal{D}_\perp$, $ker\Phi_*$, $(ker\Phi_*)^\perp$ or *mixed*-pluriharmonic, respectively) if

$$(\nabla\Phi_*)(E,F) + (\nabla\Phi_*)(JE,JF) = 0$$

for $E, F \in \Gamma(\mathcal{D}_\theta)$ ($\mathcal{D}_\perp$, $ker\Phi_*$, $(ker\Phi_*)^\perp$ or $(ker\Phi_*)^\perp \times ker\Phi_*$, respectively) [19].

Theorem 4.1 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, any two conditions below imply the third condition;*

i) Φ *is a* $\mathcal{D}_\theta$*-pluriharmonic map,*

ii) λ *is a constant on* $\psi(\mathcal{D}_\theta)$ *and* $(\nabla\Phi_*)^\perp(\psi U_1, \psi U_2) = 0$,

iii) $\sin^2\theta \mathcal{T}_{U_1}U_2 + \mathcal{A}_{\psi U_2}\phi U_1 + \mathcal{A}_{\psi U_1}\phi U_2 = 0$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_\theta)$.

Proof Using definition of $\mathcal{D}_\theta$-pluriharmonic map and symmetry condition of $\nabla\Phi_*$, we get

$$\begin{aligned} 0 &= (\nabla\Phi_*)(U_1,U_2) + (\nabla\Phi_*)(\phi U_1, \phi U_2) + (\nabla\Phi_*)(\psi U_2, \phi U_1) \\ &+ (\nabla\Phi_*)(\psi U_1, \phi U_2) + (\nabla\Phi_*)(\psi U_1, \psi U_2) \end{aligned} \tag{47}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_\theta)$. From Theorem 3.8 and (12), we obtain

$$\begin{aligned} 0 &= -\sin^2\theta\Phi_*(\mathcal{T}_{U_1}U_2) - \Phi_*(\mathcal{A}_{\psi U_2}\phi U_1 + \mathcal{A}_{\psi U_1}\phi U_2) \\ &+ \psi U_1(\ln\lambda)\Phi_*(\psi U_2) + \psi U_2(\ln\lambda)\Phi_*(\psi U_1) \\ &- g_M(\psi U_1, \psi U_2)\Phi_*(grad(\ln\lambda)) - (\nabla\Phi_*)^\perp(\psi U_1, \psi U_2). \end{aligned} \tag{48}$$

Now, suppose that (i) and (ii) are provided in (48). So, we have

$$(\nabla\Phi_*)(U_1,U_2) + (\nabla\Phi_*)(JU_1, JU_2) = 0,$$

$$\psi U_1(\ln\lambda)\Phi_*(\psi U_2) + \psi U_2(\ln\lambda)\Phi_*(\psi U_1) - g_M(\psi U_1, \psi U_2)\Phi_*(grad(\ln\lambda)) = 0$$

and

$$(\nabla\Phi_*)^\perp(\psi U_1, \psi U_2) = 0,$$

respectively. Hence, we easily obtain (iii) from (48). If we suppose that (ii) and (iii) are provided in (48), we obtain (i) from (47) such that Φ is a $\mathcal{D}_\theta$-pluriharmonic map. Lastly, we suppose that (i) and (iii) are provided in (48), we get

$$\begin{aligned} 0 &= \psi U_1(\ln\lambda)\Phi_*(\psi U_2) + \psi U_2(\ln\lambda)\Phi_*(\psi U_1) \\ &- g_M(\psi U_1, \psi U_2)\Phi_*(grad(\ln\lambda)) - (\nabla\Phi_*)^\perp(\psi U_1, \psi U_2). \end{aligned} \tag{49}$$

We obtain $(\nabla\Phi_*)^\perp(\psi U_1, \psi U_2) = 0$ from (49). For any $\psi U_1 \in \Gamma(\psi(\mathcal{D}_\theta))$, we obtain

$$0 = \lambda^2 \psi U_2(\ln\lambda) g_M(\psi U_1, \psi U_1). \tag{50}$$

So, from (50) we get $\psi U_2(\ln\lambda) = 0$. It means λ is a constant on $\psi(\mathcal{D}_\theta)$. (ii) is provided. The proof is all. □

Similarly, we have the following theorems.

Theorem 4.2 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, any two conditions below imply the third condition;*

i) $\mathcal{D}_\perp$ defines a totally geodesic foliation on M,

ii) Φ is a $\mathcal{D}_\perp$-pluriharmonic map,

iii) λ is a constant on $J(\mathcal{D}_\perp)$ and $(\nabla\Phi_*)^\perp(JV_1, JV_2) = 0$

for any $V_1, V_2 \in \Gamma(\mathcal{D}_\perp)$.

Note that $\mathcal{D}_\perp$-pluriharmonic map and $J(\mathcal{D}_\perp)$-pluriharmonic map give same results for a conformal hemi-slant Riemannian map. Since $\mathcal{D}_\perp$ is an anti-invariant distribution, we obtain the result from the definition of pluriharmonic map.

Theorem 4.3 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, any two conditions below imply the third condition;*

i) Φ is a $ker\Phi_*$-pluriharmonic map,

ii) Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(\psi W_1, \psi W_2) = 0$,

iii) $\sin^2\theta \mathcal{T}_{W_1} W_2 = \mathcal{A}_{\psi W_2}\phi W_1 + \mathcal{A}_{\psi W_1}\phi W_2$

for any $W_1, W_2 \in \Gamma(ker\Phi_*)$.

Theorem 4.4 *Let Φ be a conformal hemi-slant Riemannian map from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, any three conditions below imply the fourth condition;*

i) $(ker\Phi_*)^{\perp}$ *defines a totally geodesic foliation on* M*,*

ii) Φ *is a* $(ker\Phi_*)^{\perp}$*-pluriharmonic map,*

iii) Φ *is a horizontally homothetic map,*

iv) $\overset{N}{\nabla}{}^{\Phi}{}_{Z_1}\Phi_*(Z_2) = \Phi_*(\mathcal{T}_{BZ_1}BZ_2 + \mathcal{A}_{CZ_1}BZ_2 + \mathcal{A}_{CZ_2}BZ_1) + (\nabla\Phi_*)^{\perp}(CZ_1, CZ_2)$

for any $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$*.*

Theorem 4.5 *Let* Φ *be a conformal hemi-slant Riemannian map from a Kähler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then, any two conditions below imply the third condition;*

i) Φ *is a mixed-pluriharmonic map,*

ii) Φ *is a horizontally homothetic map and* $(\nabla\Phi_*)^{\perp}(CZ, \psi W) = 0$,

iii) $\mathcal{A}_Z W + \mathcal{T}_{BZ}\phi W + \mathcal{A}_{\psi W}BZ + \mathcal{A}_{CZ}\phi W = 0$

for any $Z \in \Gamma((ker\Phi_*)^{\perp})$ *and* $W \in \Gamma(ker\Phi_*)$*.*

Proof From definition of *mixed*-pluriharmonic map, we obtain

$$\begin{aligned} 0 &= -\Phi_*(\mathcal{A}_Z W) + (\nabla\Phi_*)^{\perp}(CZ, \psi W) \\ &- \Phi_*(\mathcal{T}_{BZ}\phi W + \mathcal{A}_{\psi W}BZ + \mathcal{A}_{CZ}\phi W) \\ &+ CZ(\ln\lambda)\Phi_*(\psi W) + \psi W(\ln\lambda)\Phi_*(CZ) \\ &- g_M(CZ, \psi W)\Phi_*(grad(\ln\lambda)) \end{aligned} \tag{51}$$

for any $Z \in \Gamma((ker\Phi_*)^{\perp})$ and $W \in \Gamma(ker\Phi_*)$. Now, we only proof (ii). Suppose that (i) and (iii) are provided in (51). We obtain easily $(\nabla\Phi_*)^{\perp}(CZ, \psi W) = 0$ and get

$$0 = \lambda^2\psi W(\ln\lambda)g_M(CZ, CZ) \tag{52}$$

for $CZ \in \Gamma((ker\Phi_*)^{\perp})$ and

$$0 = \lambda^2 CZ(\ln\lambda)g_M(\psi W, \psi W) \tag{53}$$

for $\psi W \in \Gamma((ker\Phi_*)^{\perp})$. So, we have $\psi W(\ln\lambda) = 0$ and $CZ(\ln\lambda) = 0$ from (52) and (53), respectively. They means λ is a constant on horizontal distribution. Hence, Φ is a horizontally homothetic map. (ii) is provided. □

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

References

[1] Akyol M.A., Şahin B., *Conformal anti-invariant submersions from almost Hermitian manifolds*, Turkish Journal of Mathematics, 40, 43-70, 2016.

[2] Akyol M.A., Şahin B., *Conformal semi-invariant submersions*, Communications in Contemporary Mathematics, 19(2), 1650011, 2017.

[3] Akyol M.A., Şahin B., *Conformal slant submersions*, Hacettepe Journal of Mathematics and Statistics, 48, 28-44, 2019.

[4] Baird P., Wood J.C., Harmonic Morphisms between Riemannian Manifolds, Clarendon Press, 2003.

[5] Chen B.Y., Riemannian Submanifolds: Handbook of Differential Geometry, Vol. I, 187-418, 2000.

[6] Falcitelli M., Ianus S., Pastore A.M., Riemannian Submersions and Related Topics, World Scientific, 2004.

[7] Fischer A.E., *Riemannian maps between Riemannian manifolds*, Contemporary Mathematics, 132, 331-366, 1992.

[8] Garcia-Rio E., Kupeli D.N., Semi-Riemannian Maps and Their Applications, Kluwer Academic, 1999.

[9] Kumar S., Kumar S., Pandey S., Prasad R., *Conformal hemi-slant submersions from almost Hermitian manifolds*, Communications of the Korean Mathematical Society, 35(3), 999-1018, 2020.

[10] Nore T., *Second fundamental form of a map*, Annali di Matematica Pura ed Applicata, 146, 281-310, 1986.

[11] Ohnita Y., *On pluriharmonicity of stable harmonic maps*, Journal of the London Mathematical Society, 2(35), 563-587, 1987.

[12] O'Neill B., *The fundamental equations of a submersion*, Michigan Mathematical Journal, 13(4), 458-469, 1966.

[13] Şahin B., *Semi-invariant Riemannian maps from almost Hermitian manifolds*, Indagationes Mathematicae, 23, 80-94, 2012.

[14] Şahin B., *Slant Riemannian maps from almost Hermitian manifolds*, Quaestiones Mathematicae, 36(3), 449-461, 2013.

[15] Şahin B., *Hemi-slant Riemannian maps*, Mediterranean Journal of Mathematics, 14, 2017.

[16] Şahin B., Riemannian Submersions, Riemannian Maps in Hermitian Geometry and Their Applications, Elsevier, 2017.

[17] Şahin B., *Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems*, Acta Applicandae Mathematicae, 109(3), 829-847, 2010.

[18] Şahin B., Yanan Ş., *Conformal Riemannian maps from almost Hermitian manifolds*, Turkish Journal of Mathematics, 42(5), 2436-2451, 2018.

[19] Şahin B., Yanan Ş., *Conformal semi-invariant Riemannian maps from almost Hermitian manifolds*, Filomat, 33(4), 1125-1134, 2019.

[20] Yanan Ş., *Conformal generic Riemannian maps from almost Hermitian manifolds*, Turkish Journal of Science, 6(2), 76-88, 2021.

[21] Yanan Ş., Şahin B., *Conformal slant Riemannian maps*, International Journal of Maps in Mathematics, 2022 (Accepted).

[22] Yano K., Kon M., Structures on Manifolds: Series in Pure Mathematics, World Scientific, 1984.