International Electronic Journal of Geometry
Volume 15 No. 1 Page 83–95 (2022)
DOI: https://doi.org/10.36890/iejg.1033998

Notes About a Harmonicity on the Tangent Bundle with Vertical Rescaled Metric

Abderrahim Zagane* and Nour Elhouda Djaa

(Communicated by Cihan Özgür)

Abstract

In this article, we present some results concerning the harmonicity on the tangent bundle equipped with the vertical rescaled metric. We establish necessary and sufficient conditions under which a vector field is harmonic with respect to the vertical rescaled metric and we construct some examples of harmonic vector fields. We also study the harmonicity of a vector field along with a map between Riemannian manifolds, the target manifold is equipped with a vertical rescaled metric on its tangent bundle. Next we also discuss the harmonicity of the composition of the projection map of the tangent bundle of a Riemannian manifold with a map from this manifold into another Riemannian manifold, the source manifold being whose tangent bundle is endowed with a vertical rescaled metric. Finally, we study the harmonicity of the tangent map also the harmonicity of the identity map of the tangent bundle.

Keywords: *Tangent bundle, horizontal lift, vertical lift, vertical rescaled metric, harmonic maps.*

AMS Subject Classification (2020): *Primary: 53C20; 58E20; Secondary: 53C43.*

1. Introduction

The tangent bundle equipped with the Sasaki metric has been studied by many authors among them are: Sasaki, S. [20], Crasmareanu, M. [4], Dombrowski, P. [6], Salimov, A., Gezer, A., Cengiz, N. [3, 10, 18] etc... The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on the tangent bundle. Musso, E., Tricerri, F. [16] has introced the notion of Cheeger-Gromoll metric, this metric has been studied also by many authors (see [1, 11, 21, 19]). There are other metrics on the tangent bundle that is studied and published in articles (see for example [2, 22].

The main idea in this note consists in study of the harmonicity with respect to the vertical rescaled metric on the tangent bundle [5]. We establish necessary and sufficient conditions under which a vector field is harmonic (Theorem 3.3, Theorem 3.5 and Theorem 3.6 (case the real euclidean space)). We also construct some examples of harmonic vector fields (Example 3.1, Example 3.2 and Example 3.3). After that we study the harmonicity of the map $\sigma : (M, g) \longrightarrow)(TN, H^f)$, $x \to (Y \circ \varphi)(x)$ where Y be a vector field on N (Theorem 3.7 and Theorem 3.8) and the map $\phi : (TM, G^f) \longrightarrow (N, h)$, $(x, u) \to \varphi(x)$ (Theorem 3.9 and Theorem 3.10), $\varphi : (M, g) \longrightarrow (N, h)$ is a smooth map and (TN, H^f) (resp (TM, G^f)) is a tangent bundle equipped with the vertical rescaled metric on N (resp on M). Finally, we study the harmonicity of the tangent map (Theorem 3.11 and Theorem 3.12), also the harmonicity of the identity map of the tangent bundle (Proposition 3.2 and Theorem 3.13).

2. Preliminaries

Let M^m be an m-dimensional Riemannian manifold with a Riemannian metric g and TM be its tangent bundle and the natural projection $\pi : TM \to M$. A system of local coordinates (U, x^i) in M induces on TM

Received : 07–12–2021, Accepted : 06–04–2022
** Corresponding author*

a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i\right)$, $\bar{i} = m + i = m+1, ..., 2m$, where (y^i) is the cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\{\frac{\partial}{\partial x^i}|_P\}$, P being an arbitrary point in U whose coordinates are (x^i). Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im_s^r(M)$ be the module over $C^\infty(M)$ of C^∞ tensor fields of type (r,s). In particular, $\Im_0^1(M)$ denote the module over $C^\infty(M)$ of C^∞ vector fields on M.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM. \tag{2.1}$$

of the tangent bundle to TM at any $(x,u) \in TM$ into vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \partial_{\bar{i}}|_{(x,u)},\ \xi^i \in \mathbb{R}\}, \tag{2.2}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{\xi^i \partial_i|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \partial_{\bar{k}}|_{(x,u)},\ \xi^i \in \mathbb{R}\}. \tag{2.3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$.

Let $X = X^i \partial_i$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$^V X = X^i \partial_{\bar{i}}, \tag{2.4}$$

$$^H X = = X^i \{\partial_i - y^j \Gamma_{ij}^k \partial_{\bar{k}}\}. \tag{2.5}$$

For consequences, $({}^H\partial_i, {}^V\partial_i)_{i=\overline{1,m}}$ is a local adapted frame on TTM.

Lemma 2.1. *[6] Let (M,g) be a Riemannian manifold. The Lie bracket of vertical and horizontal vector fields is given by*

1. $[{}^H X, Y^H] = {}^H[X,Y] - {}^V(R(X,Y)u)$,
2. $[{}^H X, Y^V] = {}^V(\nabla_X Y)$,
3. $[X^V, Y^V] = 0$,

for any vector fields X, Y on M , where R its tensor curvature of (M,g).

Consider a smooth map $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of φ is defined by

$$(\nabla d\varphi)(X,Y) = \nabla_X^\varphi d\varphi(Y) - d\varphi(\nabla_X Y). \tag{2.6}$$

Here ∇ is the Riemannian connection on M and ∇^φ is the pull-back connection on the pull-back bundle $\varphi^{-1}TN$. If $\nabla d\varphi = 0$ then φ is called totally geodesic.

The tension field of φ is defined by

$$\tau(\varphi) = trace_g \nabla d\varphi. \tag{2.7}$$

The energy functional of φ is defined by

$$E(\varphi) = \int_K e(\varphi) v_g, \tag{2.8}$$

such that K is any compact of M, where v_g is the canonical measure on M induced by g and

$$e(\varphi) = \frac{1}{2} trace_g h(d\varphi, d\varphi) \tag{2.9}$$

is the energy density of φ.

A map is called harmonic if it is a critical point of the energy functional. For any smooth variation $\{\varphi_t\}_{t\in I}$ of φ with $\varphi_0 = \varphi$ and $V = \frac{d}{dt}\varphi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\varphi_t)\Big|_{t=0} = -\int_K h(\tau(\varphi), V) v_g. \tag{2.10}$$

Then φ is harmonic if and only if $\tau(\varphi) = 0$. One can refer to [8, 9, 12, 14, 17] for background on harmonic maps.

3. Vertical rescaled metric and Harmonicity.

3.1. Vertical rescaled metric

Definition 3.1. [5] Let (M^m, g) be a Riemannian manifold and f be a strictly positive smooth function on M. We define the vertical rescaled metric G^f on the tangent bundle TM by

$$\begin{aligned} i)\, G^f({}^HX, {}^HY) &= g(X,Y), \\ ii)\, G^f({}^HX, {}^VY) &= 0, \\ iii)\, G^f({}^VX, {}^VY) &= fg(X,Y), \end{aligned}$$

for any vector fields X, Y on M^m.

Theorem 3.1. *[5] Let (M^m, g) be a m-dimensional Riemannian manifold and $\widetilde{\nabla}$ be a Levi-Civita connection of (TM, G^f). Then, we have*

$$\begin{aligned} i)\, \widetilde{\nabla}_{{}^HX}{}^HY &= {}^H(\nabla_X Y) - \frac{1}{2}{}^V(R(X,Y)u), \\ ii)\, \widetilde{\nabla}_{{}^HX}{}^VY &= \frac{f}{2}{}^H(R(u,Y)X) + {}^V(\nabla_X Y) + \frac{X(f)}{2f}{}^VY, \\ iii)\, \widetilde{\nabla}_{{}^VX}{}^HY &= \frac{f}{2}{}^H(R(u,X)Y) + \frac{Y(f)}{2f}{}^VX, \\ iv)\, \widetilde{\nabla}_{{}^VX}{}^VY &= \frac{-1}{2}g(X,Y){}^H(grad\, f), \end{aligned}$$

for any vector fields X, Y on M^m, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^m, g).

3.2. Harmonicity of a vector field $\xi : (M, g) \longrightarrow (TM, G^f)$

Lemma 3.1. *[14, 15] Let (M^m, g) be a Riemannian manifold. If X, Y are vector fields on M^m and $(x, u) \in TM$ such that $Y_x = u$, then we have:*

$$d_xY(X_x) = {}^HX_{(x,u)} + {}^V(\nabla_X Y)_{(x,u)}.$$

Lemma 3.2. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled metric. If ξ is a vector field on M^m, then the energy density associated to ξ is given by:*

$$e(\xi) = \frac{m}{2} + \frac{f}{2}trace_g\, g(\nabla\xi, \nabla\xi). \tag{3.1}$$

Proof. Let $(x, u) \in TM$, ξ be a vector field on M^m, $\xi_x = u$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M at x, then:

$$\begin{aligned} e(\xi)_x &= \frac{1}{2}trace_g\, G^f(d\xi, d\xi)_{(x,u)} \\ &= \frac{1}{2}\sum_{i=1}^m G^f(d\xi(E_i), d\xi(E_i))_{(x,u)} \end{aligned}$$

Using Lemma 3.1, we obtain:

$$\begin{aligned} e(\xi) &= \frac{1}{2}\sum_{i=1}^m G^f({}^HE_i + {}^V(\nabla_{E_i}\xi), {}^HE_i + {}^V(\nabla_{E_i}\xi)) \\ &= \frac{1}{2}\sum_{i=1}^m \big[G^f({}^HE_i, {}^HE_i) + G^f({}^V(\nabla_{E_i}\xi), {}^V(\nabla_{E_i}\xi))\big] \\ &= \frac{1}{2}\sum_{i=1}^m \big[g(E_i, E_i) + fg(\nabla_{E_i}\xi, \nabla_{E_i}\xi)\big] \\ &= \frac{m}{2} + \frac{f}{2}trace_g\, g(\nabla\xi, \nabla\xi). \end{aligned}$$

□

Theorem 3.2. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled metric. If ξ is a vector field on M^m, then the tension field associated to ξ is given by:*

$$\tau(\xi) = {}^H\big(trace_g\big(R(\xi, \nabla\xi) * -\frac{1}{2}g(\nabla\xi, \nabla\xi)grad\, f\big)\big) + {}^V\big(trace_g\big(\nabla^2\xi + d(\ln f)(*)\nabla\xi\big)\big). \tag{3.2}$$

where $\nabla^2\xi = \nabla\nabla\xi - \nabla_\nabla\xi$.

Proof. Let $(x, u) \in TM$, ξ be a vector field on M^m, $\xi_x = u$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M^m, then

$$\begin{aligned}\tau(\xi)_x &= trace_g\,(\nabla d\xi)_x\\ &= \sum_{i=1}^m \big(\nabla^\xi_{E_i} d\xi(E_i) - d\xi(\nabla_{E_i}E_i)\big)_x\end{aligned}$$

where ∇^ξ is the pull-back connection.

$$\begin{aligned}\tau(\xi)_x &= \sum_{i=1}^m \big(\widetilde{\nabla}_{d\xi(E_i)} d\xi(E_i)\big)_{(x,u)} - d_x\xi(\nabla_{E_i}E_i)_x\\ &= \sum_{i=1}^m \big(\widetilde{\nabla}_{({}^HE_i + {}^V(\nabla_{E_i}\xi))}({}^HE_i + {}^V(\nabla_{E_i}\xi)) - {}^H(\nabla_{E_i}E_i) - {}^V(\nabla_{(\nabla_{E_i}E_i)}\xi)\big)_{(x,u)}\\ &= \sum_{i=1}^m \big(\widetilde{\nabla}_{{}^HE_i}{}^HE_i + \widetilde{\nabla}_{{}^HE_i}{}^V(\nabla_{E_i}\xi) + \widetilde{\nabla}_{{}^V(\nabla_{E_i}\xi)}(E_i)^H + \widetilde{\nabla}_{{}^V(\nabla_{E_i}\xi)}{}^V(\nabla_{E_i}\xi) - {}^H(\nabla_{E_i}E_i) - {}^V(\nabla_{(\nabla_{E_i}E_i)}\xi)\big)_{(x,u)}.\end{aligned}$$

Using Theorem 3.1, we obtain

$$\begin{aligned}\tau(\xi) &= \sum_{i=1}^m \big({}^H(\nabla_{E_i}E_i) - \frac{1}{2}{}^V(R(E_i, E_i)\xi) + \frac{f}{2}{}^H(R(\xi, \nabla_{E_i}\xi)E_i) + {}^V(\nabla_{E_i}\nabla_{E_i}\xi) + \frac{1}{2f}E_i(f){}^V(\nabla_{E_i}\xi)\\ &\quad + \frac{f}{2}{}^H(R(\xi, \nabla_{E_i}\xi)E_i) + \frac{1}{2f}E_i(f){}^V(\nabla_{E_i}\xi) - \frac{1}{2}g(\nabla_{E_i}\xi, \nabla_{E_i}\xi){}^H(grad\, f)\big) - {}^H(\nabla_{E_i}E_i) - {}^V(\nabla_{(\nabla_{E_i}E_i)}\xi)\\ &= \sum_{i=1}^m {}^H\big(f(R(\xi, \nabla_{E_i}\xi)E_i) - \frac{1}{2}g(\nabla_{E_i}\xi, \nabla_{E_i}\xi)grad\, f\big) + {}^V\big(\nabla_{E_i}\nabla_{E_i}\xi - \nabla_{(\nabla_{E_i}E_i)}\xi + \frac{1}{f}E_i(f)\nabla_{E_i}\xi\big)\\ &= {}^H\big(trace_g\big(R(\xi, \nabla\xi) * -\frac{1}{2}g(\nabla\xi, \nabla\xi)grad\, f\big)\big) + {}^V\big(trace_g\big(\nabla^2\xi + \frac{1}{f}df(*)\nabla\xi\big)\big).\end{aligned}$$

□

Theorem 3.3. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled metric. If ξ is a vector field on M^m, then ξ is harmonic vector field if and only the following conditions are verified*

$$trace_g\big(R(\xi, \nabla\xi) * -\frac{1}{2}g(\nabla\xi, \nabla\xi)grad\, f\big) = 0, \tag{3.3}$$

$$trace_g\big(\nabla^2\xi + d(\ln f)(*)\nabla\xi\big) = 0. \tag{3.4}$$

Proof. The statement is a direct consequence of Theorem 3.2. □

Corollary 3.1. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled metric. Then any parallel vector field on M^m is harmonic.*

Example 3.1. Let $\mathbb{R}^2$ be endowed with the Riemannian metric

$$g = e^{2x}dx^2 + e^{2y}dy^2.$$

The vector field $\xi = e^{-x}\partial_x + e^{-y}\partial_y$ is harmnic. Indeed, It is enough to set $u = e^x$ and $v = e^y$ to get the euclidean metric $g = du^2 + dv^2$ and $\xi = \partial_u + \partial_v$ which is trivially parallel.

Example 3.2. Let $\mathbb{R}^3$ be endowed with the cylindrical Riemannian metric given by:

$$g = dr^2 + r^2 d\theta^2 + dz^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\ \Gamma^1_{22} = -r,$$

then we have,

$$\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{1}{r}\partial_\theta,\ \nabla_{\partial_\theta}\partial_\theta = -r\partial_r,\ \nabla_{\partial_r}\partial_r = \nabla_{\partial_r}\partial_z = \nabla_{\partial_z}\partial_r = \nabla_{\partial_\theta}\partial_z = \nabla_{\partial_z}\partial_\theta = \nabla_{\partial_z}\partial_z = 0,$$

the vector field $\xi = \sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta + \partial_z$ is harmnic because ξ is parallel, indeed,

$$\nabla_{\partial_r}\xi = \sin\theta\nabla_{\partial_r}\partial_r - \frac{1}{r^2}\cos\theta\partial_\theta + \frac{1}{r}\cos\theta\nabla_{\partial_r}\partial_\theta + \nabla_{\partial_r}\partial_z = 0,$$

$$\nabla_{\partial_\theta}\xi = \cos\theta\partial_r + \sin\theta\nabla_{\partial_\theta}\partial_r - \frac{1}{r}\sin\theta\partial_\theta + \frac{1}{r}\cos\theta\nabla_{\partial_\theta}\partial_\theta + \nabla_{\partial_\theta}\partial_z = 0,$$

$$\nabla_{\partial_z}\xi = \sin\theta\nabla_{\partial_z}\partial_r + \frac{1}{r}\cos\theta\nabla_{\partial_z}\partial_\theta + \nabla_{\partial_z}\partial_z = 0,$$

i.e $\nabla\xi = 0$, then ξ is harmonic.

Proposition 3.1. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) its tangent bundle equipped with the vertical rescaled metric and $\xi : (M^m, g) \to (TM, G^f)$ is an isometric immersion if and only if $\nabla\xi = 0$.*

Proof. Let X, Y be vector fields. From Lemma 3.1 we have

$$\begin{aligned} G^f(d\xi(X), d\xi(Y)) &= G^f({}^HX + {}^V(\nabla_X\xi), {}^HY + {}^V(\nabla_Y\xi)) \\ &= G^f({}^HX, {}^HY) + G^f({}^V(\nabla_X\xi), {}^V(\nabla_Y\xi)) \\ &= g(X, Y) + fg(\nabla_X\xi, \nabla_Y\xi), \end{aligned}$$

from which it follows that

$$G^f(d\xi(X), d\xi(Y)) = g(X, Y).$$

Therefore, ξ is an isometric immersion if and only if

$$fg(\nabla_X\xi, \nabla_Y\xi) = 0,$$

which is equivalent to $\nabla\xi = 0$. □

As a direct consequence of Theorem 3.3 and Proposition 3.1, we obtain the following theorem.

Theorem 3.4. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) its tangent bundle equipped with the vertical rescaled metric. If $\xi : (M, F, g) \to (TM, G^f)$ is isometric immersion, then ξ is harmonic.*

Theorem 3.5. *Let (M^m, g) be a Riemannian compact manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled metric. If ξ is a vector field on M^m, then ξ is harmonic vector field if and only if ξ is parallel.*

Proof. If ξ is parallel, from Corollary 3.1, we deduce that ξ is harmonic vector field.
Conversely, let φ_t be a variation of ξ defined by:

$$\begin{aligned} \mathbb{R} \times M &\longrightarrow T_xM \\ (t, x) &\longmapsto \varphi_t(x) = (1+t)\xi_x \end{aligned}$$

From lemma 3.2 we have:

$$e(\varphi_t) = \frac{m}{2} + \frac{(1+t)^2 f}{2} trace_g\, g(\nabla\xi, \nabla\xi)$$

$$E(\varphi_t) = \frac{m}{2}Vol(M) + \frac{(1+t)^2}{2}\int_M f\, trace_g\, g(\nabla\xi,\nabla\xi)v_g$$

If ξ is a critical point of the energy functional, then we have :

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big(\frac{m}{2}Vol(M) + \frac{(1+t)^2}{2}\int_M f\, trace_g\, g(\nabla\xi,\nabla\xi)v_g\Big)_{t=0} \\ &= \int_M f\, trace_g\, g(\nabla\xi,\nabla\xi)v_g \end{aligned}$$

which gives

$$g(\nabla\xi,\nabla\xi) = 0,$$

Hence, it follows that $\nabla\xi = 0$. □

Example 3.3. The torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ (Riemannian compact 2-dimensional manifold) equipped with the product of canonical metric:

$$g = \frac{4}{(1+x^2)^2}dx^2 + \frac{4}{(1+y^2)^2}dy^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^1_{11} = \frac{-2x}{1+x^2},\ \Gamma^2_{22} = \frac{-2y}{1+y^2}$$

then we have,

$$\nabla_{\partial_x}\partial_x = \frac{-2x}{1+x^2}\partial_x,\ \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 0,\ \nabla_{\partial_y}\partial_y = \frac{-2y}{1+y^2}\partial_y.$$

The vector field $\xi = (1+x^2)\partial_x + (1+y^2)\partial_y$ is harmonic if and only if ξ is parallel, indeed

$$\begin{aligned} \nabla_{\partial_x}\xi &= 2x\partial_x + (1+x^2)\nabla_{\partial_x}\partial_x + (1+y^2)\nabla_{\partial_x}\partial_y = 0, \\ \nabla_{\partial_y}\xi &= (1+x^2)\nabla_{\partial_y}\partial_x + 2y\partial_y + (1+y^2)\nabla_{\partial_y}\partial_y = 0, \end{aligned}$$

i.e $\nabla\xi = 0$, then ξ is harmonic.

Remark 3.1. In general , using Corollary 3.1 and Theorem 3.5, we can construct many examples for harmonic vector fields.

Theorem 3.6. *Let $(\mathbb{R}^m, <,>)$ the real euclidian space and $T\mathbb{R}^m$ its tangent bundle equipped with the vertical rescaled metric. If $\xi = (\xi^1,\cdots,\xi^m)$ is a vector field on $\mathbb{R}^m$, then ξ is harmonic if and only if the following conditions are verified*

$$\xi = constant\ \ or\ \ f = constant, \tag{3.5}$$

$$\sum_{i=1}^m \Big(\frac{\partial^2\xi^k}{\partial(x^i)^2} + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\xi^k}{\partial x^i}\Big) = 0. \tag{3.6}$$

for all $k = \overline{1,m}$.

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,m}}$ be a canonical frame on $\mathbb{R}^m$. Using Theorem 3.3, we have

$$\begin{aligned} (3.3) &\Leftrightarrow trace_g\big(g(\nabla\xi,\nabla\xi)grad\, f\big) = 0 \\ &\Leftrightarrow \sum_{i=1}^m g(\nabla_{\frac{\partial}{\partial x^i}}\xi, \nabla_{\frac{\partial}{\partial x^i}}\xi) = 0\ \ or\ \ grad\, f = 0 \\ &\Leftrightarrow \sum_{i=1}^m\sum_{j=1}^m \big(\frac{\partial\xi^j}{\partial x^i}\big)^2 = 0\ \ or\ \ f = constant \\ &\Leftrightarrow \frac{\partial\xi^j}{\partial x^i} = 0\ ,\ for\ any\ i,j = \overline{1,m}\ \ or\ \ f = constant \\ &\Leftrightarrow \xi = constant\ \ or\ \ f = constant. \end{aligned}$$

$$
\begin{aligned}
(3.4) \quad &\Leftrightarrow \quad trace_g\big(\nabla^2\xi + \frac{1}{f}df(*)\nabla\xi\big) = 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{m}\big(\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^i}}\xi + \frac{1}{f}df(\frac{\partial}{\partial x^i})(\nabla_{\frac{\partial}{\partial x^i}}\xi\big) = 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{m}\Big(\sum_{k=1}^{m}\big(\frac{\partial^2\xi^k}{\partial(x^i)^2}\frac{\partial}{\partial x^k}\big) + \frac{1}{f}\frac{\partial f}{\partial x^i}\sum_{k=1}^{m}\big(\frac{\partial\xi^k}{\partial x^i}\frac{\partial}{\partial x^k}\big)\Big) = 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{m}\Big(\frac{\partial^2\xi^k}{\partial(x^i)^2} + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\xi^k}{\partial x^i}\Big) = 0,
\end{aligned}
$$

for all $k = \overline{1, m}$. □

Corollary 3.2. *Let $(\mathbb{R}^m, <,>)$ the real euclidean space and $T\mathbb{R}^m$ its tangent bundle equipped with the vertical rescaled metric. If f is a constant function, then $\xi = (\xi^1, \cdots, \xi^m)$ is a harmonic vector field on $\mathbb{R}^m$ if and only if for all $k = \overline{1, m}$, ξ^k is a real harmonic function on $\mathbb{R}^m$.*

Corollary 3.3. *Let $(\mathbb{R}^m, <,>)$ the real euclidean space and $T\mathbb{R}^m$ its tangent bundle equipped with the vertical rescaled metric. If f is not a constant function, then $\xi = (\xi^1, \cdots, \xi^m)$ is a harmonic vector field on $\mathbb{R}^m$ if and only if ξ is constant.*

3.3. Harmonicity of the map $\sigma : (M, g) \longrightarrow (TN, H^f)$

Lemma 3.3. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and*

$$
\begin{aligned}
\sigma : M \quad &\longrightarrow \quad TN \\
x \quad &\longmapsto \quad (Y \circ \varphi)(x) = (\varphi(x), Y_{\varphi(x)})
\end{aligned}
$$

a smooth map, such that Y be a vector field on N. Then

$$d\sigma(X) = {}^H(d\varphi(X)) + {}^V(\nabla^\varphi_X\sigma)$$

for any vector field X on M.

Proof. Let $x \in M$ and $v \in T_{\varphi(x)}N$, such that $v = Y_{\varphi(x)}$, for any vector field X on M. Using Lemma 3.3, we obtain

$$
\begin{aligned}
d_x\sigma(X_x) &= d_x(Y \circ \varphi)(X_x) \\
&= d_{\varphi(x)}Y(d_x\varphi(X_x)) \\
&= {}^H(d\varphi(X))_{(\varphi(x),v)} + {}^V(\nabla_{d\varphi(X)}Y)_{(\varphi(x),v)} \\
&= {}^H(d\varphi(X))_{(\varphi(x),v)} + {}^V(\nabla^\varphi_X\sigma)_{(\varphi(x),v)}.
\end{aligned}
$$

□

Theorem 3.7. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, f be a strictly positive smooth function on N and (TN, H^f) the tangent bundle of N equipped with vertical rescaled metric. Let*

$$
\begin{aligned}
\sigma : M \quad &\longrightarrow \quad TN \\
x \quad &\longmapsto \quad (Y \circ \varphi)(x) = (\varphi(x), Y_{\varphi(x)})
\end{aligned}
$$

be a smooth map, such that Y be a vector field on N. The tension field of σ is given by

$$
\begin{aligned}
\tau(\sigma) \quad = \quad &{}^H\Big(\tau(\varphi) + trace_g\big(fR^N(\sigma, \nabla^\varphi\sigma)d\varphi(*) - \frac{1}{2}h(\nabla^\varphi\sigma, \nabla^\varphi\sigma)grad\, f\big)\Big) \\
&+{}^V\Big(trace_g\big((\nabla^\varphi)^2\sigma + \frac{1}{f}d\varphi(*)(f)\nabla^\varphi\sigma\big)\Big), \qquad (3.7)
\end{aligned}
$$

where $(\nabla^\varphi)^2\sigma = \nabla^\varphi\nabla^\varphi\sigma - \nabla^\varphi_\nabla\sigma$.

Proof. Let $x \in M$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M at x, $\sigma(x) = (\varphi(x), v)$ and $v = Y_{\varphi(x)} \in T_{\varphi(x)}N$. Using lemma 3.3, we have

$$\begin{aligned}
\tau(\sigma)_x &= trace_g\,(\nabla d\sigma)_x \\
&= \sum_{i=1}^{m}\big(\nabla^{\sigma}_{E_i} d\sigma(E_i) - d\sigma(\nabla_{E_i}E_i)\big)_x \\
&= \sum_{i=1}^{m}\big(\nabla^{TN}_{d\sigma(E_i)} d\sigma(E_i) - {}^{H}(d\varphi(\nabla_{E_i}E_i)) - {}^{V}(\nabla^{\varphi}_{\nabla_{E_i}E_i}\sigma)\big)_{(\varphi(x),v)} \\
&= \sum_{i=1}^{m}\big(\nabla^{TN}_{({}^{H}(d\varphi(E_i))+{}^{V}(\nabla^{\varphi}_{E_i}\sigma))}({}^{H}(d\varphi(E_i)) + {}^{V}(\nabla^{\varphi}_{E_i}\sigma)) - {}^{H}(d\varphi(\nabla_{E_i}E_i)) - {}^{V}(\nabla^{\varphi}_{\nabla_{E_i}E_i}\sigma)\big)_{(\varphi(x),v)} \\
&= \sum_{i=1}^{m}\big(\nabla^{TN}_{{}^{H}(d\varphi(E_i))}{}^{H}(d\varphi(E_i)) + \nabla^{TN}_{{}^{H}(d\varphi(E_i))}{}^{V}(\nabla^{\varphi}_{E_i}\sigma) + \nabla^{TN}_{{}^{V}(\nabla^{\varphi}_{E_i}\sigma)}{}^{H}(d\varphi(E_i)) + \nabla^{TN}_{{}^{V}(\nabla^{\varphi}_{E_i}\sigma)}{}^{V}(\nabla^{\varphi}_{E_i}\sigma) \\
&\quad - {}^{H}(d\varphi(\nabla_{E_i}E_i)) - {}^{V}(\nabla^{\varphi}_{\nabla_{E_i}E_i}\sigma)\big)_{(\varphi(x),v)}.
\end{aligned}$$

From the theorem 3.1, we obtain

$$\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^{m}\Big({}^{H}(\nabla^{N}_{d\varphi(E_i)}d\varphi(E_i)) - \frac{1}{2}{}^{V}(R^N(d\varphi(E_i), d\varphi(E_i))\sigma) + \frac{f}{2}{}^{H}(R^N(\sigma, \nabla^{\varphi}_{E_i}\sigma)d\varphi(E_i)) + {}^{V}(\nabla^{N}_{d\varphi(E_i)}\nabla^{\varphi}_{E_i}\sigma) \\
&\quad + \frac{1}{2f}d\varphi(E_i)(f)^{V}(\nabla^{\varphi}_{E_i}\sigma) + \frac{f}{2}{}^{H}(R^N(\sigma, \nabla^{\varphi}_{E_i}\sigma)d\varphi(E_i)) + \frac{1}{2f}d\varphi(E_i)(f)^{V}(\nabla^{\varphi}_{E_i}\sigma) \\
&\quad - \frac{1}{2}h(\nabla^{\varphi}_{E_i}\sigma, \nabla^{\varphi}_{E_i}\sigma)^{H}(grad\, f) - {}^{H}(d\varphi(\nabla_{E_i}E_i)) - {}^{V}(\nabla^{\varphi}_{\nabla_{E_i}E_i}\sigma)\Big) \\
&= \sum_{i=1}^{m}\Big({}^{H}(\nabla^{\varphi}_{E_i}d\varphi(E_i)) - {}^{H}(d\varphi(\nabla_{E_i}E_i)) + f^{H}(R^N(\sigma, \nabla^{\varphi}_{E_i}\sigma)d\varphi(E_i)) - \frac{1}{2}h(\nabla^{\varphi}_{E_i}\sigma, \nabla^{\varphi}_{E_i}\sigma)^{H}(grad\, f) \\
&\quad + {}^{V}(\nabla^{\varphi}_{E_i}\nabla^{\varphi}_{E_i}\sigma) - {}^{V}(\nabla^{\varphi}_{\nabla_{E_i}E_i}\sigma) + \frac{1}{f}d\varphi(E_i)(f)^{V}(\nabla^{\varphi}_{E_i}\sigma)\Big) \\
&= {}^{H}\Big(\tau(\varphi) + trace_g\big(fR^N(\sigma, \nabla^{\varphi}\sigma)d\varphi(*) - \frac{1}{2}h(\nabla^{\varphi}\sigma, \nabla^{\varphi}\sigma)grad\, f\big)\Big) \\
&\quad + {}^{V}\Big(trace_g\big((\nabla^{\varphi})^2\sigma + \frac{1}{f}d\varphi(*)(f)\nabla^{\varphi}\sigma\big)\Big).
\end{aligned}$$

□

From Theorem 3.7 we obtain.

Theorem 3.8. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, f be a strictly positive smooth function on N and (TN, H^f) the tangent bundle of N equipped with vertical rescaled metric. Let*

$$\begin{aligned}
\sigma : M &\longrightarrow TN \\
x &\longmapsto (Y \circ \varphi)(x) = (\varphi(x), Y_{\varphi(x)})
\end{aligned}$$

be a smooth map, such that Y be a vector field on N. Then σ is a harmonic if and only if the following conditions are verified

$$\begin{cases} \tau(\varphi) = -trace_g\big(fR^N(\sigma, \nabla^{\varphi}\sigma)d\varphi(*) - \frac{1}{2}h(\nabla^{\varphi}\sigma, \nabla^{\varphi}\sigma)grad\, f\big), \\ trace_g\big((\nabla^{\varphi})^2\sigma + \frac{1}{f}d\varphi(*)(f)\nabla^{\varphi}\sigma\big) = 0. \end{cases} \tag{3.8}$$

3.4. *Harmonicity of the map $\phi : (TM, G^f) \longrightarrow (N, h)$*

Lemma 3.4. *Let (M^m, g) be a Riemannian manifold, f be a strictly positive smooth function on M and (TM, G^f) its tangent bundle equipped with the vertical rescaled metric. The tension field of the canonical projection*

$$\begin{aligned}
\pi : (TM, G^f) &\longrightarrow (M, g) \\
(x, u) &\longmapsto x
\end{aligned}$$

is given by:

$$\tau(\pi) \quad = \quad \frac{m}{2f}(grad\, f)\circ\pi. \tag{3.9}$$

Proof. Let $(x,u)\in TM$ and $(E_i)_{i=\overline{1,m}}$ be local orthonormal frame on M at x, we put $F_i=\frac{1}{\sqrt{f}}E_i$ then $\left({}^HE_i,{}^VF_i\right)_{i=\overline{1,m}}$ is a local orthonormal frame on (TM,G^f) at (x,u).

$$\begin{aligned}
\tau(\pi) &= trace_{G^f}\nabla d\pi \\
&= \sum_{i=1}^m \left(\nabla^\pi_{{}^HE_i}d\pi({}^HE_i)-d\pi(\widetilde{\nabla}_{{}^HE_i}{}^HE_i)+\nabla^\pi_{{}^VF_i}d\pi({}^VF_i)-d\pi(\widetilde{\nabla}_{{}^VF_i}{}^VF_i)\right)\\
&= \sum_{i=1}^m \left(\nabla_{d\pi({}^HE_i)}d\pi({}^HE_i)-d\pi({}^H(\nabla_{E_i}E_i)-\frac{1}{2}{}^V(R(E_i,E_i)u))+\nabla_{d\pi({}^VF_i)}d\pi({}^VF_i)+\frac{1}{2}g(F_i,F_i)d\pi({}^H(grad\, f))\right)
\end{aligned}$$

With $d\pi({}^VX)=0$ and $d\pi({}^HX)=X\circ\pi$, for any vector field X on M, then we find

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^m \left((\nabla_{E_i\circ\pi}E_i\circ\pi)-(\nabla_{E_i}E_i)\circ\pi\right)+\frac{1}{2f}\sum_{i=1}^m g(E_i,E_i)d\pi({}^H(grad\, f))\\
&= \sum_{i=1}^m \left((\nabla_{E_i}E_i)\circ\pi-(\nabla_{E_i}E_i)\circ\pi\right)+\frac{1}{2f}\sum_{i=1}^m g(E_i,E_i)\,(grad\, f)\circ\pi\\
&= \frac{m}{2f}(grad\, f)\circ\pi.
\end{aligned}$$

□

Theorem 3.9. *Let $\varphi:(M^m,g)\to(N^n,h)$ be a smooth map between Riemannian manifolds, f be a strictly positive smooth function on M and (TM,G^f) the tangent bundle of M equipped with vertical rescaled metric. The tension field of the map*

$$\begin{aligned}
\phi:(TM,G^f) &\longrightarrow (N,h)\\
(x,u) &\longmapsto \varphi(x)
\end{aligned}$$

is given by:

$$\tau(\phi) \quad = \quad (\tau(\varphi)+\frac{m}{2f}d\varphi(grad\, f))\circ\pi. \tag{3.10}$$

Proof. Let $(x,u)\in TM$ and $(E_i)_{i=\overline{1,m}}$ be local orthonormal frame on M at x and $\left({}^HE_i,{}^VF_i\right)_{i=\overline{1,m}}$ is a local orthonormal frame on (TM,G^f) at (x,u) where $F_i=\frac{1}{\sqrt{f}}E_i$. Since ϕ is written in the form $\phi=\varphi\circ\pi$, we have:

$$\begin{aligned}
\tau(\phi)_{(x,u)} &= \tau(\varphi\circ\pi)_{(x,u)}\\
&= (d\varphi(\tau(\pi))+trace_{G^f}\nabla d\varphi(d\pi,d\pi))_{(x,u)}.
\end{aligned}$$

$$\begin{aligned}
trace_{G^f}\nabla d\varphi(d\pi,d\pi) &= \sum_{i=1}^m \left(\nabla^\varphi_{d\pi({}^HE_i)}d\varphi(d\pi({}^HE_i))-d\varphi(\nabla^M_{d\pi({}^HE_i)}d\pi({}^HE_i))\right)\\
&\quad+\sum_{i=1}^m \left(\nabla^\varphi_{d\pi({}^VF_i)}d\varphi(d\pi({}^VF_i))-d\varphi(\nabla^M_{d\pi({}^VF_i)}d\pi({}^VF_i))\right)
\end{aligned}$$

With $d\pi({}^VX)=0$ and $d\pi({}^HX)=X\circ\pi$, for any vector field X on M, then we find

$$\begin{aligned}
trace_{G^f}\nabla d\varphi(d\pi,d\pi) &= \sum_{i=1}^m \left(\nabla^\varphi_{(E_i\circ\pi)}d\varphi(E_i\circ\pi)-d\varphi(\nabla^M_{E_i\circ\pi}(E_i\circ\pi))\right)\\
&= \sum_{i=1}^m \left(\nabla^\varphi_{E_i}d\varphi(E_i)-d\varphi(\nabla^M_{E_i}E_i)\right)\circ\pi\\
&= \tau(\varphi)\circ\pi,
\end{aligned}$$

Using Lemma 3.4, we obtain:

$$\tau(\phi) \quad = \quad (\tau(\varphi) + \frac{m}{2f}\, d\varphi(grad\, f)) \circ \pi.$$

□

Theorem 3.10. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, f be a strictly positive smooth function on M and (TM, G^f) the tangent bundle of M equipped with vertical rescaled metric. The map*

$$\begin{array}{rcl} \phi : (TM, G^f) & \longrightarrow & (N, h) \\ (x, u) & \longmapsto & \varphi(x) \end{array}$$

is a harmonic if and only if.

$$\tau(\varphi) = -\frac{m}{2f}\, d\varphi(grad\, f).$$

3.5. Harmonicity of the tangent map

Let (M^m, g) (resp. (N^n, h)) be a Riemannian manifold and (TM, G^{f_1}) (resp. (TN, H^{f_2})) its tangent bundle equipped with the vertical rescaled metric, such that f_1 (resp. f_2) is a strictly positive smooth function on M (resp. N).

Lemma 3.5. *[7] Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. . The map φ induces the tangent map*

$$\begin{array}{rcl} \Phi = d\varphi : TM & \longrightarrow & TN \\ (x, u) & \longmapsto & (\varphi(x), d\varphi(u)) \end{array}$$

and we have

$$\begin{array}{rcl} d\Phi({}^V X) & = & {}^V\big(d\varphi(X)\big), \\ d\Phi({}^H X) & = & {}^H\big(d\varphi(X)\big) + {}^V\big(\nabla d\varphi(u, X)\big). \end{array}$$

for any vector field X on M.

Theorem 3.11. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a isometric smooth map between Riemannian manifolds, then the tension field associated to the tangent map*
$\Phi : (TM, G^{f_1}) \longrightarrow (TN, H^{f_2})$ is given by:

$$\begin{aligned} \tau(\Phi) \quad = \quad & {}^H\big[\tau(\varphi) + \frac{m}{2f_1}\big(d\varphi(grad^M f_1) - (grad^N f_2) \circ \varphi\big) \\ & + trace_g\big((f_2 \circ \varphi) R^N(d\varphi(u), \nabla d\varphi(u, *)) d\varphi(*) \circ \varphi - \frac{1}{2} h(\nabla d\varphi(u, *), \nabla d\varphi(u, *))(grad^N f_2) \circ \varphi\big)\big] \\ & + {}^V\big[\frac{1}{f_2 \circ \varphi} \nabla d\varphi(u, grad^M(f_2 \circ \varphi)) + \frac{m}{f_1} \nabla d\varphi(u, grad^M f_1)\big) + trace_g \nabla^{\varphi}(\nabla d\varphi(u, *))\big]. \end{aligned} \tag{3.11}$$

Proof. Let $(x, u) \in TM$, $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M at x such that $(\nabla^M_{E_i} E_i)_x = 0$ and $\big({}^H E_i, {}^V F_i\big)_{i=\overline{1,m}}$ be a local orthonormal frame on (TM, G^{f_1}) at (x, u) where $F_i = \frac{1}{\sqrt{f_1}} E_i$, then

$$\begin{aligned} \tau(\Phi)_{(x,u)} \quad & = \quad trace_{G^f}(\nabla d\Phi)_{(x,u)} \\ & = \quad \sum_{i=1}^{m} \big(\nabla^{TN}_{d\Phi({}^H E_i)} d\Phi({}^H E_i) - d\Phi(\nabla^{TM}_{{}^H E_i} {}^H E_i) + \nabla^{TN}_{d\Phi({}^V F_i)} d\Phi({}^V F_i) - d\Phi(\nabla^{TM}_{{}^V F_i} {}^V F_i)\big)_{(\varphi(x), d\varphi(u))}. \end{aligned}$$

Using Lemma 3.5, we obtain:

$$\begin{aligned}
\tau(\Phi)_{(x,u)} &= \sum_{i=1}^{m}\Big(\nabla^{TN}_{^H(d\varphi(E_i))}{}^H(d\varphi(E_i)) + \nabla^{TN}_{^H(d\varphi(E_i))}{}^V(\nabla d\varphi(u,E_i)) + \nabla^{TN}_{^V(\nabla d\varphi(u,E_i))}{}^H(d\varphi(E_i))\\
&\quad +\nabla^{TN}_{^V(\nabla d\varphi(u,E_i))}{}^V(\nabla d\varphi(u,E_i)) - d\Phi(\nabla^{TM}_{^HE_i}{}^HE_i) + \nabla^{TN}_{^V(d\varphi(F_i))}{}^V(d\varphi(F_i)) - d\Phi(\nabla^{TM}_{^VF_i}{}^VF_i)\Big)_{(\varphi(x),d\varphi(u))}\\
&= \sum_{i=1}^{m}\Big({}^H(\nabla^N_{d\varphi(E_i)}d\varphi(E_i)) + f_2{}^H(R^N(d\varphi(u),\nabla d\varphi(u,E_i))d\varphi(E_i)) + {}^V(\nabla^N_{d\varphi(E_i)}\nabla d\varphi(u,E_i))\\
&\quad +\frac{1}{f_2}d\varphi(E_i)(f_2){}^V(\nabla d\varphi(u,E_i)) - \frac{1}{2}h(\nabla d\varphi(u,E_i),\nabla d\varphi(u,E_i)){}^H(grad^N f_2)\\
&\quad +\frac{1}{2f_1}g(E_j,E_i){}^H(d\varphi(grad^M f_1)) + \frac{1}{2f_1}g(E_i,E_i){}^V(\nabla d\varphi(u,grad^M f_1))\\
&\quad -\frac{1}{2f_1}h(d\varphi(E_i),d\varphi(E_i)){}^H(grad^N f_2)\Big)_{(\varphi(x),d\varphi(u))}
\end{aligned}$$

From the isometry property of the map φ, we have

$$\begin{aligned}
\tau(\Phi)_{(x,u)} &= \sum_{i=1}^{m}\Big({}^H\big[(\nabla^N_{d\varphi(E_i)}d\varphi(E_i))_{\varphi(x)} + f_2(\varphi(x))R^N_{\varphi(x)}(d\varphi(u),\nabla d\varphi(u,E_i))d\varphi(E_i)\\
&\quad -\frac{1}{2}h_{\varphi(x)}(\nabla d\varphi(u,E_i),\nabla d\varphi(u,E_i))(grad^N f_2)_{\varphi(x)} + \frac{1}{2f_1(x)}g_x(E_j,E_i)d\varphi(grad^M f_1)_x\\
&\quad -\frac{1}{2f_1(x)}g_x(E_i,E_i)(grad^N f_2)_{\varphi(x)}\big]\\
&\quad +{}^V\big[(\nabla^N_{d\varphi(E_i)}\nabla d\varphi(u,E_i))_{\varphi(x)} + \frac{1}{f_2(\varphi(x))}(\nabla d\varphi(u,E_i(f_2\circ\varphi)E_i))_{\varphi(x)}\\
&\quad +\frac{1}{2f_1(x)}g_x(E_i,E_i)(\nabla d\varphi(u,grad^M f_1))_x\big]\Big)\\
&= {}^H\big[\tau(\varphi)_x + \frac{m}{2f_1}\big(d\varphi(grad^M f_1)_x - (grad^N f_2)_{\varphi(x)}\big) + trace_g\big(f_2(\varphi(x))R^N_{\varphi(x)}(d\varphi(u),\nabla d\varphi(u,*))d\varphi(*)\\
&\quad -\frac{1}{2}h_{\varphi(x)}(\nabla d\varphi(u,*),\nabla d\varphi(u,*))(grad^N f_2)_{\varphi(x)}\big)\big]\\
&\quad +{}^V\big[\frac{1}{f_2(\varphi(x))}\nabla d\varphi(u,grad^M(f_2\circ\varphi))_x + \frac{m}{2f_1(x)}\nabla d\varphi(u,grad^M f_1)_x + trace_g\nabla^{\varphi}(\nabla d\varphi(u,*))_x\big].
\end{aligned}$$

□

Theorem 3.12. *Let $\varphi:(M^m,g)\to(N^n,h)$ be a isometric smooth map between Riemannian manifolds, then the tangent map $\Phi:(TM,G^{f_1})\longrightarrow(TN,H^{f_2})$ is harmonic if and only if*

$$\begin{aligned}
\tau(\varphi) &= -\frac{m}{2f_1}\big(d\varphi(grad^M f_1) - (grad^N f_2)\circ\varphi\big) - trace_g\big((f_2\circ\varphi)R^N(d\varphi(u),\nabla d\varphi(u,*))d\varphi(*)\circ\varphi\\
&\quad -\frac{1}{2}h(\nabla d\varphi(u,*),\nabla d\varphi(u,*))(grad^N f_2)\circ\varphi\big), \qquad (3.12)
\end{aligned}$$

$$trace_g\nabla^{\varphi}(\nabla d\varphi(u,*)) = -\frac{1}{f_2\circ\varphi}\nabla d\varphi(u,grad^M(f_2\circ\varphi)) - \frac{m}{f_1}\nabla d\varphi(u,grad^M f_1)\big). \qquad (3.13)$$

Corollary 3.4. *Let $\Phi:(TM,G^{f_1})\longrightarrow(TN,H^{f_2})$ be a the tangent map of isometric smooth map between Riemannian manifolds $\varphi:(M^m,g)\to(N^n,h)$. If φ is totally geodesic, then the tangent map Φ is harmonic if and only if*

$$\tau(\varphi) = -\frac{m}{2f_1}\big(d\varphi(grad^M f_1) - (grad^N f_2)\circ\varphi\big). \qquad (3.14)$$

3.6. Harmonicity of the identity map $I:(TM,G^{f_1})\to(TM,G^{f_2})$

Let (M^m,g) be a Riemannian manifold and (TM,G^{f_1}) (resp. (TM,G^{f_2})) its tangent bundle equipped with the vertical rescaled metric G^{f_1} (resp. G^{f_2}), such that f_1 (resp. f_2) is a strictly positive smooth function on M.

Proposition 3.2. *Let* (M^m, g) *be a Riemannian manifold, then the tension field associated to the identity map* $I : (TM, G^{f_1}) \to (TM, G^{f_2})$ *is given by*

$$\tau(I) = \frac{m}{2f_1}{}^H(grad\,(f_1 - f_2)).$$

Proof. Let $(x, u) \in TM$, $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M at x and $\left({}^HE_i, {}^VF_i\right)_{i=\overline{1,m}}$ be a local orthonormal frame on (TM, G^{f_1}) at (x, u) where $F_i = \frac{1}{\sqrt{f_1}}E_i$. If $\widetilde{\nabla}$ (resp. $\overline{\nabla}$) denote the Levi-Civita connection of (TM, G^{f_1}) (resp. (TM, G^{f_2})), then, we have

$$\begin{aligned}
\tau(I)_{(x,u)} &= trace_{G^{f_1}}(\nabla dI)_{(x,u)} \\
&= \sum_{i=1}^{m}\left(\nabla^I_{{}^HE_i}dI({}^HE_i) - dI(\widetilde{\nabla}_{{}^HE_i}{}^HE_i) + \nabla^I_{{}^VF_i}dI({}^VF_i) - dI(\widetilde{\nabla}_{{}^VF_i}{}^VF_i)\right)_{(x,u)}.
\end{aligned}$$

$$\begin{aligned}
\tau(I) &= \sum_{i=1}^{m}\left(\overline{\nabla}_{dI({}^HE_i)}dI({}^HE_i) - dI(\widetilde{\nabla}_{{}^HE_i}{}^HE_i) + \overline{\nabla}_{dI({}^VF_i)}dI({}^VF_i) - dI(\widetilde{\nabla}_{{}^VF_i}{}^VF_i)\right) \\
&= \sum_{i=1}^{m}\left(\overline{\nabla}_{{}^HE_i}{}^HE_i - \widetilde{\nabla}_{{}^HE_i}{}^HE_i + \overline{\nabla}_{{}^VF_i}{}^VF_i - \widetilde{\nabla}_{{}^VF_i}{}^VF_i\right) \\
&= \sum_{i=1}^{m}\left(\overline{\nabla}_{{}^VF_i}{}^VF_i - \widetilde{\nabla}_{{}^VF_i}{}^VF_i\right) \\
&= \sum_{i=1}^{m}\left(\frac{-1}{2}g(F_i, F_i){}^H(grad\,f_2) + \frac{1}{2}g(F_i, F_i){}^H(grad\,f_1)\right) \\
&= \frac{m}{2f_1}{}^H(grad\,(f_1 - f_2)).
\end{aligned}$$

□

Theorem 3.13. *Let* (M^m, g) *be a Riemannian manifold, then identity map* $I : (TM, G^{f_1}) \to (TM, G^{f_2})$ *is harmonic if and only if* $f_1 = f_2$ *or* f_1 *and* f_2 *are constant functions.*

Acknowledgments

We would like to thank the anonymous referees for their valuable and insightful comments, which helped improve the paper. We also thank Professor M. Djaa advice and helpful suggestions.

References

[1] Abbassi, M. T. K., Sarih, M.: *On Natural Metrics on Tangent Bundles of Riemannian Manifolds.* Arch. Math. (Brno). **41** (1), 71-92 (2005).
[2] Altunbas, M., Simsek, R., Gezer, A.: *A Study Concerning Berger type deformed Sasaki Metric on the Tangent Bundle.* Zh. Mat. Fiz. Anal.Geom. **15** (4), 435-447 (2019) . https://doi.org/10.15407/mag15.04.435
[3] Cengiz, N. , Salimov, A.A.: *Diagonal lift in the tensor bundle and its applications.* Appl. Math. Comput. **142** (2-3), 309-319 (2003). https://doi.org/10.1016/S0096-3003(02)00305-3.
[4] Crasmareanu, M.: *Liouville and geodesic Ricci solitons*, Zbl 1183.53036 C. R., Math., Acad. Sci. Paris **347**, No. 21-22, 1305-1308 (2009).
[5] Dida, H.M., Hathout, F., Azzouz, A.: *On the geometry of the tangent bundle with vertical rescaled metric.* Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (1), 222-235 (2019). https://doi.org/10.31801/cfsuasmas.443735
[6] Dombrowski, P.: *On the Geometry of the Tangent Bundle.* J. Reine Angew. Math. **210** , 73-88 (1962). https://doi.org/10.1515/crll.1962.210.73
[7] El Hendi, H., Belarbi, L.: *Naturally harmonic maps between tangent bundles.* Balkan J. Geom. Appl. **25** (1), 34-46 (2020).
[8] Ells, J., Lemaire, L.: *Another report on harmonic maps.* Bull. London Math. Soc. **20** (5), 385-524 (1988). https://doi.org/10.1112/blms/20.5.385
[9] Ells, J., Sampson, J. H.: *Harmonic mappings of Riemannian manifolds.* Amer.J. Maths. **86**, 109-160 (1964). https://doi.org/10.2307/2373037
[10] Gezer, A.: *On the Tangent Bundle with Deformed Sasaki Metric.* Int. Electron. J. Geom. **6** (2), 19-31 (2013).
[11] Gudmundsson, S., Kappos, E.: *On the geometry of the tangent bundle with the Cheeger-Gromoll metric.* Tokyo J. Math. **25** (1), 75-83 (2002). https://doi.org/10.3836/tjm/1244208938
[12] Ishihara, T.: *Harmonic sections of tangent bundles.* J.Math. Tokushima Univ. **13**, 23-27 (1979).

[13] Kada Ben Otmane, R., Zagane, A., Djaa, M.: *On generalized Cheeger-Gromoll metric and harmonicity.* Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69** (1), 629-645 (2020). https://doi.org/10.31801/cfsuasmas.487296
[14] Konderak, J. J.: *On Harmonic Vector Fields.* Publications Mathematiques. **36**, 217-288 (1992) .
[15] Latti, F., Djaa, M., Zagane, A.: *Mus-Sasaki Metric and Harmonicity.* Math. Sci. Appl. E-Notes. **6** (1), 29-36 (2018). https://doi.org/10.36753/mathenot.421753
[16] Musso, E., Tricerri, F.: *Riemannian Metrics on Tangent Bundles.* Ann. Mat. Pura. Appl. **150** (4), 1-19 (1988).
[17] Opriou, V.: *Harmonic Maps Between tangent bundles.* Rend. Sem. Mat. Univ. Politec. Torino. **47** (1), 47-55 (1989).
[18] Salimov, A. A., Gezer, A.: *On the geometry of the (1, 1)-tensor bundle with Sasaki type metric.* Chin. Ann. Math. Ser. B. **32** (3), 369-386 (2011). DOI: 10.1007/s11401-011-0646-3
[19] Salimov, A. A., Kazimova, S.: *Geodesics of the Cheeger-Gromoll Metric.* Turkish J. Math. **33**, 99-105 (2009). doi:10.3906/mat-0804-24
[20] Sasaki, S.: *On the differential geometry of tangent bundles of Riemannian manifolds II.* Tohoku Math. J. **14**, 146-155 (1962). https://doi.org/10.2748/tmj/1178244169
[21] Sekizawa, M.: *Curvatures of Tangent Bundles with Cheeger-Gromoll Metric.* Tokyo J. Math. **14** (2), 407-417 (1991). DOI: 10.3836/tjm/1270130381
[22] Zagane, A., Djaa, M.: *Geometry of Mus-Sasaki metric.* Commun. Math. **26** 113-126 (2018). https://doi.org/10.2478/cm-2018-0008

Affiliations

Abderrahim Zagane
Address: Relizane University, Faculty of Sciences and Technology, Department of Mathematics, 48000, Relizane-Algeria.
e-mail: Zaganeabr2018@gmail.com
ORCID ID:0000-0001-9339-3787

Nour Elhouda Djaa
Address: Relizane University, Faculty of Sciences and Technology, Department of Mathematics, 48000, Relizane-Algeria.
e-mail: djaanor@hotmail.fr
ORCID ID:0000-0002-0568-0629