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Investigation of Matrices $Q^{n.L}$, $M^{n.L}$ and $R^{n.L}$ and Some Related Identities

İbrahim GÖKCAN*[1], Ali Hikmet DEĞER[2]

Abstract

In this study, it is aimed to use the Lorenz matrix multiplication to find the n^{th} powers of some special matrices and to reach the quadratic equations and characteristic roots of the matrices obtained in this way. In addition, it is aimed to contribute literature to the studies in the field by reaching some identities.

Keywords: Characteristic Roots, Fibonacci and Lucas Numbers, Lorentz Matrix Multiplication, Identity, Quadratic Equations

1. INTRODUCTION

Fibonacci and Lucas numbers are studied extensively in the literature and are associated with many scientific facts. The Fibonacci number sequence is $0,1,1,2,3,\cdots$ known by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Similarly, the Lucas number sequence is $2,1,3,4,\cdots$ known by the recurrence relation $L_n = L_{n-1} + L_{n-2}$. [1, 2] can be examined about Fibonacci and Lucas numbers. In [3], Ruggles obtained some findings related with Fibonacci and Lucas numbers. In recent years, studies realised on the generalization of the Fibonacci and Lucas numbers. In [4], Tasci and Kilic achieved generalization of Lucas numbers by the help of matrices and a relation between generalized order-k Lucas and Fibonacci numbers. In [5], Kilic and Tasci studied on the generalized order-k Fibonacci and Lucas numbers. They generalized Binet's formula for these numbers and obtained some data that also included these numbers. In [6], Kızılateş and Tuglu examined extented concolved (p,q)-Fibonacci and Lucas polynomials and obtained some recurrence relations about these polynomials. In [7], Qi, Kızılateş, and Du achieved a closed formula for Horadam polynomials by using tridiagonal determinat and from here they reproduced closed formulas for other poylnomial sequences. In [8], Kızılateş worked finite operators on Horadam sequences and defined Horadam finite operator sequences. In addition, in [8], Kızılateş investigated recurrence relation, Binet-like formula, summation formula and generating function by using this sequences. Moreover, in [8], Kızılateş obtained a closed formula that given number of

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this sequences by using tridiagonal determinant. In [9], Kızılateş, Du and Qi made a few explicit formulas for (p,q,r)-tribonacci polynomials and generalized tribonacci sequences by using Hessenberg determinants and from here they reproduced explicit formulas for some other sequences. One of the studies in this field is to obtain identities related to Fibonacci, Lucas and other number sequences with the help of n^{th} powers of the matrices consisting of the elements of Fibonacci and Lucas number sequences, to reach the quadratic equations and characteristic roots of the matrices with the help of the determinants of the n^{th} powers of the matrices. Matrices $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, M=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ and $R=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$ were used specially.

In this section, some historical information were given about the matrices Q, M and R in order to make sense of our work. In this regard, [10] was our main reference source. The subject of examining Fibonacci sequences with the method matrix Q and discovering the relations between their elements has been discussed by mathematicians for many years. With initial conditions $F_0=0$ and $F_1=1$, the Fibonacci sequence is obtained with the recurrence relation $F_n=F_{n-1}+F_{n-2}$. The n^{th} power of the matrix Q formed by the elements of the Fibonacci sequence becomes $Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}$. The determinant of this matrix satisfies the equation $F_{n+1}F_{n-1}-F_n^2=(-1)^n$ given by Robert Simson in 1753. This equation is the basis of Lewis Carroll's famous geometric paradox. Basin and Hoggatt [11] refer to Charles King's [12] master's thesis for the first definition of the matrix Q. Earlier, work on using the matrix Q to construct Fibonacci sequences was published by Brenner [13]. J. Sutherland Frame [14] used matrix $\begin{pmatrix}\alpha & 1\\ 1 & 0\end{pmatrix}$ for studying continued fractions [10]. Note that this matrix becomes the matrix Q for $\alpha=1$ and the matrix which produces the Pell sequences for $\alpha=2$. Schwerdtfeger [15] studied Jacobsthal [16]'s Fibonacci polynomial matrix methods and defined the Fibonacci polynomial with the elements of Q^n. White [17] used the matrices $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ to generate $GL\,(2,\mathbb{Z})$. Bicknell [18] found the square root of the matrix Q and used it in fractional powers [10].

1.1. Examining the Matrix Q

Determinant of matrix Q is -1. Under classical matrix multiplication, the powers of the matrix Q are found as $Q^2=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$, $Q^3=\begin{pmatrix}3 & 2\\ 2 & 1\end{pmatrix}$, $Q^4=\begin{pmatrix}5 & 3\\ 3 & 2\end{pmatrix},\cdots$. From here, n^{th} power of matrix Q is found as $Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}$.

Theorem 1. [19] Let $n\geq 1$. Then $Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}$.

Proposition 1. [19] Let $n\geq 1$. Then $F_{n+1}F_{n-1}-F_n^2=(-1)^n$.

Proposition 2. [19] Let $m,n\in\mathbb{N}$. Then,

i) $F_{m+n+1}=F_{m+1}F_{n+1}+F_mF_n$

ii) $F_{m+n}=F_{m+1}F_n+F_mF_{n-1}$

iii) $F_{m+n}=F_mF_{n+1}+F_{m-1}F_n$

iv) $F_{m+n-1}=F_mF_n+F_{m-1}F_{n-1}$

1.2. Examining the Matrix M

Determinant of matrix M is 1. Under classical matrix multiplication, the powers of the matrix M are found as $M^2=\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}$, $M^3=\begin{pmatrix}5 & 8\\ 8 & 13\end{pmatrix}$, $M^4=\begin{pmatrix}13 & 21\\ 21 & 34\end{pmatrix},\cdots$. From here, n^{th} power of matrix M is found as $M^n=\begin{pmatrix}F_{2n-1} & F_{2n}\\ F_{2n} & F_{2n+1}\end{pmatrix}$.

Proposition 3. [19] Let $n\geq 1$. Then $M^n=\begin{pmatrix}F_{2n-1} & F_{2n}\\ F_{2n} & F_{2n+1}\end{pmatrix}$.

Conclusion1. Let $n\geq 1$. Then $F_{2n-1}F_{2n+1}-F_{2n}^2=1$.

Conclusion 2. Let $m, n \in \mathbb{N}$.Then,

i) $F_{2m+2n-1} = F_{2m-1}F_{2n-1} + F_{2m}F_{2n}$

ii) $F_{2(m+n)} = F_{2m-1}F_{2n} + F_{2m}F_{2n+1}$

iii) $F_{2(m+n)} = F_{2m}F_{2n-1} + F_{2m+1}F_{2n}$

iv) $F_{2m+2n+1} = F_{2m}F_{2n} + F_{2m+1}F_{2n+1}$

1.3. Examining the Matrix *R*

Determinant of matrix R is -5. Under classical matrix multiplication, the powers of the matrix R are found as $R^2 = 5I$, $R^3 = 5R$, $R^4 = 5^2I$, $R^5 = 5^2R$, $R^6 = 5^3I$, $R^7 = 5^3R$, $\cdots$. Powers of $(2n)^{th}$ and $(2n+1)^{th}$ of matrix R are obtained as $R^{2n} = 5^nI$ and $R^{2n+1} = 5^nR$, respectively.

Conclusion 3. Let $n \geq 1$. Then $R^{2n} = 5^nI$.

Conclusion 4. Let $n \geq 0$. Then $R^{2n+1} = 5^nR$.

1.4. Quadratic Equations and Characteristic Roots of Matrices Q^n, M^n and R^n

The characteristic equation of Q is $|Q - xI| = 0$, where Q is a 2×2 matrix and I is a 2×2 unit matrix. The roots of the characteristic equation are the characteristic roots of Q. With the help of the equation given above, the characteristic roots of Q^n are found as follows:

$$|Q^n - xI| = \left|\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} F_{n+1} - x & F_n \\ F_n & F_{n-1} - x \end{pmatrix}\right|$$

$$= (F_{n+1} - x)(F_{n-1} - x) - F_n^2$$

$$= x^2 + F_{n+1}F_{n-1} - x(F_{n+1} + F_{n-1}) - F_n^2$$

$$= x^2 - x(F_{n+1} + F_{n-1}) + F_{n+1}F_{n-1} - F_n^2$$

$$= x^2 - xL_n + (-1)^n = 0. \quad (1)$$

Using the quadratic equation, we can obtain the following characteristic roots. From the method $\Delta = b^2 - 4ac$,

$$x_{1,2} = \frac{L_n \pm \sqrt{L_n^2 - 4(-1)^n}}{2} \quad (2)$$

are found. Substituting the identity $L_n^2 - 4(-1)^n = 5F_n^2$ at (2),

$$x_{1,2} = \frac{L_n \pm \sqrt{5F_n^2}}{2} = \frac{L_n \pm \sqrt{5}F_n}{2} \quad (3)$$

roots are found [19].
Let's examine the quadratic equation and characteristic roots of the matrix M^n with a similar method. M^n is a 2×2 matrix and I is a 2×2 unit matrix. The characteristic equation of M^n is $|M^n - xI| = 0$. From here,

$$|M^n - xI| = \left|\begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} F_{2n-1} - x & F_{2n} \\ F_{2n} & F_{2n+1} - x \end{pmatrix}\right|$$

$$= (F_{2n-1} - x)(F_{2n+1} - x) - F_{2n}^2$$

$$= x^2 + F_{2n+1}F_{2n-1} - x(F_{2n+1} + F_{2n-1}) - F_{2n}^2$$

$$= x^2 - x(F_{2n+1} + F_{2n-1}) + F_{2n+1}F_{2n-1} - F_{2n}^2$$

$$= x^2 - xL_{2n} + 1 = 0. \quad (4)$$

From the Δ method,

$$x_{1,2} = \frac{L_{2n} \pm \sqrt{L_{2n}^2 - 4}}{2} \quad (5)$$

roots are found. $L_{2n}^2 - 4 = 5F_{2n}^2$is obtained from $L_n^2 - 4(-1)^n = 5F_n^2$ (please see [19]) for $n \to 2n$. From here, $L_{2n}^2 - 4 = 5F_{2n}^2$ is substituting at (5),

$$x_{1,2} = \frac{L_{2n} \pm \sqrt{5F_{2n}^2}}{2} = \frac{L_{2n} \pm \sqrt{5}F_{2n}}{2} \quad (6)$$

roots are obtained.

Now let's examine the quadratic equation of the matrix R^n and its characteristic roots. R^n is a 2×2 matrix and I is a 2×2 unit matrix. R^n's characteristic equation is $|R^n - xI| = 0$. Since the matrix R^n is found differently for powers $(2n)^{th}$ and $(2n+1)^{th}$, there are different quadratic equations and characteristic roots for both values. Quadratic equation for matrix R^{2n},

$$|R^{2n} - xI| = \left|\begin{pmatrix} 5^n & 0 \\ 0 & 5^n \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 5^n & 0 \\ 0 & 5^n \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 5^n - x & 0 \\ 0 & 5^n - x \end{pmatrix}\right|$$

$$= (5^n - x)(5^n - x)$$

$$= 5^{2n} - 2x5^n + x^2$$

$$= x^2 - 2.5^n x + 5^{2n} = 0. \quad (7)$$

From the Δ method,

$$x_{1,2} = \frac{2.5^n \pm \sqrt{4.5^{2n} - 4.5^{2n}}}{2} = \frac{2.5^n}{2} = 5^n \quad (8)$$

roots are obtained. Quadratic equation for matrix R^{2n+1},

$$|R^{2n+1} - xI| = \left|\begin{pmatrix} 5^n & 2.5^n \\ 2.5^n & -5^n \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 5^n & 2.5^n \\ 2.5^n & -5^n \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 5^n - x & 2.5^n \\ 2.5^n & -5^n - x \end{pmatrix}\right|$$

$$= (5^n - x)(-5^n - x) - 4.5^{2n}$$

$$= (5^n - x)(5^n + x) + 4.5^{2n}$$
$$= -x^2 + 5^{2n+1} = 0. \quad (9)$$

From the Δ method,

$$x_{1,2} = \frac{\pm\sqrt{-4.(-1)5^{2n+1}}}{2} = \pm 5^n\sqrt{5} \quad (10)$$

roots are obtained.

1.5. Lorentz Matrix Multiplication

In [20], the authors defined Lorentz matrix multiplication between two matrices unlike the classical matrix multiplication. Let's R_n^m denote matrices of type $m \times n$ and R_p^n denote matrices of type $n \times p$. Lorentz matrix multiplication

$$\langle a, b\rangle_{\cdot L} = -a_{i1}b_{1k} + \sum_{j=2}^{n} a_{ij}b_{jk} \quad (11)$$

by "$\cdot_L$" is defined between lines of matrices $A = (a_{ij}) \in R_n^m$ and columns of matrices $B = (b_{jk}) \in R_p^n$. $A._L\, B$ is a matrix of type $m \times p$. $(i,j)^{th}$ inner product of $A._L\, B$ is $\langle A_i, B^j\rangle_L$ when A_i is i^{th} line of matrix A and B^j is j^{th} column of matrix B. $A._L\, B$ is defined as follows:

$$A._L\, B = \begin{pmatrix} \langle A_1, B^1\rangle_L & \cdots & \langle A_1, B^j\rangle_L \\ \vdots & \ddots & \vdots \\ \langle A_i, B^1\rangle_L & \cdots & \langle A_i, B^j\rangle_L \end{pmatrix} \quad (12)$$

Determinant of $A._L\, B$ is defined as

$$\det(A._L\, B) = -detA.detB. \quad (13)$$

[20-22] can be examined for Lorentz matrix multiplication and its related properties.

2. OBTAINING IDENTITIES WITH HELP OF MATRICES $Q^{n\cdot_L}$, $M^{n\cdot_L}$ AND $R^{n\cdot_L}$

n^{th} power under Lorentz matrix multiplication of matrices Q, M and R is denoted with $Q^{n\cdot_L}$, $M^{n\cdot_L}$ and $R^{n\cdot_L}$,respectively.

2.1. Matrix $Q^{n\cdot_L}$ and Related Identity

n^{th} power of matrix Q is obtained under Lorentz matrix multiplication as follows for $k \in \mathbb{N}$:

$$Q^{2\cdot_L} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot_L \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_0 & -F_1 \\ -F_1 & -F_1 \end{pmatrix},$$
$$Q^{3\cdot_L} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot_L \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -F_1 & F_0 \\ F_0 & F_1 \end{pmatrix},$$

$$Q^{4\cdot_L}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\cdot_L\begin{pmatrix}-1&0\\0&1\end{pmatrix}=\begin{pmatrix}F_1&F_1\\F_1&F_0\end{pmatrix},$$
$$Q^{5\cdot_L}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\cdot_L\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}F_0&-F_1\\-F_1&-F_1\end{pmatrix},$$
$$Q^{6\cdot_L}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\cdot_L\begin{pmatrix}0&-1\\-1&-1\end{pmatrix}=\begin{pmatrix}-F_1&F_0\\F_0&F_1\end{pmatrix},\cdots$$

$$Q^{n\cdot_L}=\begin{cases}\begin{pmatrix}F_1&F_1\\F_1&F_0\end{pmatrix}, & n=3k+1\\ \begin{pmatrix}F_0&-F_1\\-F_1&-F_1\end{pmatrix}, & n=3k+2\\ \begin{pmatrix}-F_1&F_0\\F_0&F_1\end{pmatrix}, & n=3k+3\end{cases}\tag{14}$$

Let us prove identity (14) by using induction.
It is provided $Q^{1\cdot_L}, Q^{2\cdot_L}$ and $Q^{3\cdot_L}$ for $n=1, n=2$ and $n=3$,respectively. Then,

$$Q^{n\cdot_L}=\begin{cases}\begin{pmatrix}F_1&F_1\\F_1&F_0\end{pmatrix}, & n=1\\ \begin{pmatrix}F_0&-F_1\\-F_1&-F_1\end{pmatrix}, & n=2\\ \begin{pmatrix}-F_1&F_0\\F_0&F_1\end{pmatrix}, & n=3\end{cases}$$

Let assume that $Q^{3k+1\cdot_L}, Q^{3k+2\cdot_L}$ and $Q^{3k+3\cdot_L}$ is provided for $n=3k+1, n=3k+2$ and $n=3k+3$,respectively. Then,

$$Q^{n\cdot_L}=\begin{cases}\begin{pmatrix}F_1&F_1\\F_1&F_0\end{pmatrix}, & n=3k+1\\ \begin{pmatrix}F_0&-F_1\\-F_1&-F_1\end{pmatrix}, & n=3k+2\\ \begin{pmatrix}-F_1&F_0\\F_0&F_1\end{pmatrix}, & n=3k+3\end{cases}$$

Let demonstrate that $Q^{3k+4\cdot_L}, Q^{3k+5\cdot_L}$ and $Q^{3k+6\cdot_L}$ is provided for $n=3k+4, n=3k+5$ and $n=3k+6$, respectively.

$$Q^{n\cdot_L}=\begin{cases}\begin{pmatrix}F_1&F_1\\F_1&F_0\end{pmatrix}, & n=3k+4\\ \begin{pmatrix}F_0&-F_1\\-F_1&-F_1\end{pmatrix}, & n=3k+5\\ \begin{pmatrix}-F_1&F_0\\F_0&F_1\end{pmatrix}, & n=3k+6\end{cases}$$

For $Q^{3k+4\cdot_L}=Q\cdot_L Q^{3k+3\cdot_L}$,
$$Q^{3k+4\cdot_L}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\cdot_L\begin{pmatrix}-1&0\\0&1\end{pmatrix}=\begin{pmatrix}F_1&F_1\\F_1&F_0\end{pmatrix}.$$
For $Q^{3k+5\cdot_L}=Q\cdot_L Q^{3k+4\cdot_L}$,
$$Q^{3k+5\cdot_L}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\cdot_L\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}F_0&-F_1\\-F_1&-F_1\end{pmatrix}.$$
For $Q^{3k+6\cdot_L}=Q\cdot_L Q^{3k+5\cdot_L}$,
$$Q^{3k+6\cdot_L}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\cdot_L\begin{pmatrix}0&-1\\-1&-1\end{pmatrix}=\begin{pmatrix}-F_1&F_0\\F_0&F_1\end{pmatrix}.$$
Determinants of matrices are obtained as follows from (13).
$$detQ^{2\cdot_L}=-det\begin{pmatrix}1&1\\1&0\end{pmatrix}.det\begin{pmatrix}1&1\\1&0\end{pmatrix}=-1$$
$$detQ^{3\cdot_L}=-det\begin{pmatrix}1&1\\1&0\end{pmatrix}.det\begin{pmatrix}0&-1\\-1&-1\end{pmatrix}=-1$$
$$detQ^{4\cdot_L}=-det\begin{pmatrix}1&1\\1&0\end{pmatrix}.det\begin{pmatrix}-1&0\\0&1\end{pmatrix}=-1$$
$$detQ^{5\cdot_L}=-det\begin{pmatrix}1&1\\1&0\end{pmatrix}.det\begin{pmatrix}1&1\\1&0\end{pmatrix}=-1,\cdots$$
Determinants of matrices are $detQ^{n\cdot_L}=-1$ for $n\geqslant 1$.

2.2. Matrix $M^{n\cdot_L}$ and Related Identities

n^{th} power of matrix M is obtained under Lorentz matrix multiplication as follows:
$$M^{2\cdot_L}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\cdot_L\begin{pmatrix}1&1\\1&2\end{pmatrix}=\begin{pmatrix}F_0&F_2\\F_2&F_4\end{pmatrix},$$
$$M^{3\cdot_L}=\begin{pmatrix}0&1\\1&3\end{pmatrix}\cdot_L\begin{pmatrix}1&1\\1&2\end{pmatrix}=\begin{pmatrix}F_1&F_3\\F_3&F_5\end{pmatrix},$$
$$M^{4\cdot_L}=\begin{pmatrix}1&2\\2&5\end{pmatrix}\cdot_L\begin{pmatrix}1&1\\1&2\end{pmatrix}=\begin{pmatrix}F_2&F_4\\F_4&F_6\end{pmatrix},$$
$$M^{5\cdot_L}=\begin{pmatrix}1&3\\3&8\end{pmatrix}\cdot_L\begin{pmatrix}1&1\\1&2\end{pmatrix}=\begin{pmatrix}F_3&F_5\\F_5&F_7\end{pmatrix},\cdots$$

$$M^{n\cdot_L}=\begin{pmatrix}F_{n-2}&F_n\\F_n&F_{n+2}\end{pmatrix}\tag{15}$$
Let us prove identity (15) by using induction.
It is provided $M^{2\cdot_L}$ for $n=2$. Then,
$$M^{2\cdot_L}=\begin{pmatrix}F_0&F_2\\F_2&F_4\end{pmatrix}.$$
Let assume that $M^{k\cdot_L}$ is provided for $n=k$. Then,
$$M^{k\cdot_L}=\begin{pmatrix}F_{k-2}&F_k\\F_k&F_{k+2}\end{pmatrix}.$$
Let demonstrate that $M^{k+1\cdot_L}$ is provided for $n=k+1$.
For $M^{k+1\cdot_L}=M\cdot_L M^{k\cdot_L}$,
$$M^{k+1\cdot_L}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\cdot_L\begin{pmatrix}F_{k-2}&F_k\\F_k&F_{k+2}\end{pmatrix}$$
$$=\begin{pmatrix}-F_{k-2}+F_k&-F_{k-2}+2F_k\\-F_k+F_{k+2}&-F_k+2F_{k+2}\end{pmatrix}$$
$$=\begin{pmatrix}F_{k-1}&F_{k+1}\\F_{k+1}&F_{k+3}\end{pmatrix}.$$
Determinants of matrices are obtained as follows from (13).

$$detM^{2\cdot_L}=-det\begin{pmatrix}1&1\\1&2\end{pmatrix}.det\begin{pmatrix}1&1\\1&2\end{pmatrix}=-1$$

$detM^{3\cdot L} = -det\begin{pmatrix}0 & 1\\1 & 3\end{pmatrix}.det\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = 1$
$detM^{4\cdot L} = -det\begin{pmatrix}1 & 2\\2 & 5\end{pmatrix}.det\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = -1$
$detM^{5\cdot L} = -det\begin{pmatrix}1 & 3\\3 & 8\end{pmatrix}.det\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = 1,\cdots$
If we continue like this, we obtained $detM^{2\cdot L} = detM^{4\cdot L} = \cdots = detM^{2n\cdot L} = -1$ and $detM^{3\cdot L} = detM^{5\cdot L} = \cdots = detM^{2n+1\cdot L} = 1$.
From here, the determinant for the generalized matrix is $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$ for $n \geq 2$. This identity can be proved by inductive method. Among the matrices obtained by Lorentz matrix multiplication, the following operations are provided by Lorentz matrix operations.
$M^{2\cdot L}\cdot_L M^{3\cdot L} = \begin{pmatrix}2 & 5\\5 & 13\end{pmatrix} = M^{5\cdot L}$,
$M^{1\cdot L}\cdot_L M^{5\cdot L} = \begin{pmatrix}3 & 8\\8 & 21\end{pmatrix} = M^{6\cdot L}$,
$M^{2\cdot L}\cdot_L M^{4\cdot L} = \begin{pmatrix}3 & 8\\8 & 21\end{pmatrix} = M^{6\cdot L}$
If the situation is generalized for different m and n numbers, the equation $M^{m\cdot L}\cdot_L M^{n\cdot L} = M^{m+n\cdot L}$ is obtained. Using this equation, identities can be obtained. The identities in the following theorem are reached by both Lorentz and classical matrix multiplication.

Theorem 2.

i) $F_{m+n-2} = -F_{m-2}F_{n-2} + F_mF_n$

ii) $F_{m+n} = -F_{m-2}F_n + F_mF_{n+2}$

iii) $F_{m+n} = -F_mF_{n-2} + F_{m+2}F_n$

iv) $F_{m+n+2} = -F_mF_n + F_{m+2}F_{n+2}$

v) $\frac{F_m}{F_n} = \frac{F_{m+2}+F_{m-2}}{F_{n+2}+F_{n-2}}$

vi) $F_{2n} = L_nF_n$

Proof. Following equation is provided with Lorentz matrix multiplication when m and n are natural numbers for $M^{m\cdot L} = \begin{pmatrix}F_{m-2} & F_m\\F_m & F_{m+2}\end{pmatrix}$ and $M^{n\cdot L} = \begin{pmatrix}F_{n-2} & F_n\\F_n & F_{n+2}\end{pmatrix}$.

$$M^{m\cdot L}\cdot_L M^{n\cdot L} = \begin{pmatrix}F_{m-2} & F_m\\F_m & F_{m+2}\end{pmatrix}\cdot_L\begin{pmatrix}F_{n-2} & F_n\\F_n & F_{n+2}\end{pmatrix}$$
$$= \begin{pmatrix}-F_{m-2}F_{n-2} + F_mF_n & -F_{m-2}F_n + F_mF_{n+2}\\-F_mF_{n-2} + F_{m+2}F_n & -F_mF_n + F_{m+2}F_{n+2}\end{pmatrix}$$
$$= \begin{pmatrix}F_{m+n-2} & F_{m+n}\\F_{m+n} & F_{m+n+2}\end{pmatrix}$$
$$= M^{m+n\cdot L}$$

i) If taken as $r = 2$, $m = n-2$ and $n = m$ in $F_rF_{m+n} = F_{m+r}F_n - (-1)^rF_mF_{n-r}$ (please see [23]), $F_2F_{n-2+m} = F_{n-2+2}F_m - (-1)^2F_{n-2}F_{m-2}$ is obtained. So, $F_{n+m-2} = -F_{n-2}F_{m-2} + F_nF_m$.

ii) If taken as $r = 2, m = n$ and $n = m$ in $F_rF_{m+n} = F_{m+r}F_n - (-1)^rF_mF_{n-r}$ (please see [23]),
$F_2F_{n+m} = F_{n+2}F_m - (-1)^2F_nF_{m-2}$
is obtained.So, $F_{m+n} = -F_nF_{m-2} + F_{n+2}F_m$.

iii) If taken as $r = 2$ in $F_rF_{m+n} = F_{m+r}F_n - (-1)^rF_mF_{n-r}$ (please see [23]),
$F_2F_{m+n} = F_{m+2}F_n - (-1)^2F_mF_{n-2}$ is obtained. So, $F_{m+n} = F_{m+2}F_n - F_mF_{n-2}$.

iv) If taken as $r = 2, n = m+2$ and $m = n$ in $F_rF_{m+n} = F_{m+r}F_n - (-1)^rF_mF_{n-r}$ (please see [23]),
$F_{n+m+2} = -F_nF_m + F_{n+2}F_{m+2}$ is obtained.

v) From (ii) and (iii) identities,
$F_{m+n} = -F_{m-2}F_n + F_mF_{n+2}$ and
$F_{m+n} = -F_mF_{n-2} + F_{m+2}F_n$
are found. From the equality of two identities,
$-F_{m-2}F_n + F_mF_{n+2} = -F_mF_{n-2} + F_{m+2}F_n$
is obtained. So,
$F_mF_{n+2} + F_mF_{n-2} = F_{m+2}F_n + F_{m-2}F_n$.
$$\frac{F_m}{F_n} = \frac{F_{m+2}+F_{m-2}}{F_{n+2}+F_{n-2}}$$
identity is obtained from
$F_m(F_{n+2} + F_{n-2}) = (F_{m+2}+F_{m-2})F_n$.

vi) If taken as $m = n$ in (ii), $F_{2n} = -F_{n-2}F_n + F_nF_{n+2} = (F_{n+2}-F_{n-2})F_n = L_nF_n$
is obtained.

Now let's give the theorem about newly obtained identities according to Lorentz matrix multiplication.

Theorem 3.

i) $3F_{m+n} = F_{m+2}F_{n+2} - F_{m-2}F_{n-2}$

ii) $L_{m+n-1} = -L_{m-1}F_{n-2} + L_{m+1}F_n$

iii) $L_{m+n+1} = -L_{m-1}F_n + L_{m+1}F_{n+2}$

iv) $F_{2n-2} = -F_{n-2}^2 + F_n^2$

v) $F_{2n+2} = -F_n^2 + F_{n+2}^2$

vi) $3F_{2n} = F_{n+2}^2 - F_{n-2}^2$

vii) $L_{2n-1} = -L_{n-1}F_{n-2} + L_{n+1}F_n$

viii) $L_{2n+1} = -L_{n-1}F_n + L_{n+1}F_{n+2}$

ix) $L_{2n-1} + L_{2n+1} = -L_{n-1}^2 + L_{n+1}^2$

x) $L_{2n-1} = -F_{n-2}^2 + F_n^2 + L_nF_n$

Proof.
i) From Theorem 2 (i) and Theorem 2 (iv), we get $F_{m+n-2} + F_{m-2}F_{n-2} = F_mF_n$ and $F_mF_n = F_{m+2}F_{n+2} - F_{m+n+2}$. From the equality of two identities,
$F_{m+n-2} + F_{m-2}F_{n-2} = F_{m+2}F_{n+2} - F_{m+n+2}$
is obtained. So,
$3F_{m+n} = F_{m+2}F_{n+2} - F_{m-2}F_{n-2}$ is obtained from
$F_{m+n-2} + F_{m+n+2} = F_{m+2}F_{n+2} - F_{m-2}F_{n-2}$.

ii) From Theorem 2 (i) and Theorem 2 (iii) , we obtain $F_{m+n-2} = -F_{m-2}F_{n-2} + F_mF_n$ and $F_{m+n} = -F_mF_{n-2} + F_{m+2}F_n$. From the sum of two identities,
$F_{m+n-2} + F_{m+n} =$
$= -F_{m-2}F_{n-2} + F_mF_n - F_mF_{n-2} + F_{m+2}F_n$
$= -(F_{m-2}+F_m)F_{n-2} + (F_m + F_{m+2})F_n$
$= -L_{m-1}F_{n-2} + L_{m+1}F_n$
$F_{m+n-2} + F_{m+n} = -L_{m-1}F_{n-2} + L_{m+1}F_n$
So, $L_{m+n-1} = -L_{m-1}F_{n-2} + L_{m+1}F_n$.

iii) From Theorem 2 (ii) and Theorem 2 (iv), we find that $F_{m+n} = -F_{m-2}F_n + F_mF_{n+2}$ and $F_{m+n+2} = -F_mF_n + F_{m+2}F_{n+2}$. From the sum of two identities,
$F_{m+n} + F_{m+n+2} =$
$= -F_{m-2}F_n + F_mF_{n+2} - F_mF_n + F_{m+2}F_{n+2}$
$= -(F_{m-2} + F_m)F_n + (F_m + F_{m+2})F_{n+2}$
$= -L_{m-1}F_n + L_{m+1}F_{n+2}$
So, $L_{m+n+1} = -L_{m-1}F_n + L_{m+1}F_{n+2}$.

iv) If taken as $m = n$ in Theorem 2 (i),
$F_{2n-2} = -F_{n-2}F_{n-2} + F_nF_n = -F_{n-2}^2 + F_n^2$ is obtained.

v) If taken as $m = n$ in Theorem 2 (iv),
$F_{2n+2} = -F_nF_n + F_{n+2}F_{n+2} = -F_n^2 + F_{n+2}^2$ is obtained.

vi) If taken as $m = n$ in (i),
$3F_{2n} = F_{n+2}F_{n+2} - F_{n-2}F_{n-2} = F_{n+2}^2 - F_{n-2}^2$ is obtained.

vii)If taken as $m = n$ in (ii),
$L_{2n-1} = -L_{n-1}F_{n-2} + L_{n+1}F_n$ is obtained.

viii)If taken as $m = n$ in (iii),
$L_{2n+1} = -L_{n-1}F_n + L_{n+1}F_{n+2}$ is obtained.

ix) From the sum of (vii) and (viii) identities,
$L_{2n-1} + L_{2n+1} =$
$= -L_{n-1}F_{n-2} + L_{n+1}F_n - L_{n-1}F_n + L_{n+1}F_{n+2}$
$= -(F_{n-2} + F_n)L_{n-1} + (F_n + F_{n+2})L_{n+1}$
$= -L_{n-1}L_{n-1} + L_{n+1}L_{n+1}$
$= -L_{n-1}^2 + L_{n+1}^2$
$L_{2n-1} + L_{2n+1} = -L_{n-1}^2 + L_{n+1}^2$ is obtained.

x) From the sum of (iv) and Theorem 2 (vi), $F_{2n-2} + F_{2n} = -F_{n-2}^2 + F_n^2 + L_nF_n$. So,

$L_{2n-1} = -F_{n-2}^2 + F_n^2 + L_nF_n$ is obtained.

2.3. Examining of Matrix $R^{n \cdot L}$

n^{th} power of matrix R is obtained under Lorentz matrix multiplication as follows:

$$R^{2 \cdot L} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot_L \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} L_2 & -L_3 \\ -L_3 & -L_2 \end{pmatrix},$$
$$R^{3 \cdot L} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot_L \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} -L_5 & -F_3 \\ -F_3 & L_5 \end{pmatrix},$$
$$R^{4 \cdot L} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot_L \begin{pmatrix} -11 & -2 \\ -2 & 11 \end{pmatrix} = \begin{pmatrix} L_4 & 24 \\ 24 & -L_4 \end{pmatrix},$$
$$R^{5 \cdot L} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot_L \begin{pmatrix} 7 & 24 \\ 24 & -7 \end{pmatrix} = \begin{pmatrix} 41 & 38 \\ -38 & -41 \end{pmatrix},$$

...

Determinants of matrices are $detR^{2 \cdot L} = -25$, $detR^{3 \cdot L} = -125$, $detR^{4 \cdot L} = -625$ and $detR^{5 \cdot L} = -1325$.
Identities could not be found because matrix $R^{n \cdot L}$ could not be written in terms of the elements of the Fibonacci and Lucas Number sequences.

3. QUADRATIC EQUATIONS AND CHARACTERISTIC ROOTS OF $Q^{n \cdot L}$, $M^{n \cdot L}$ AND $R^{n \cdot L}$

3.1. Quadratic Equations and Characteristic Roots of $Q^{n \cdot L}$

Matrix $Q^{n \cdot L}$ could not be written in terms of the elements of the Fibonacci and Lucas Number sequences. Then, quadratic equation and characteristic roots could not be reached.

3.2. Quadratic Equations and Characteristic Roots of $M^{n \cdot L}$

The determinant of the matrix $M^{n \cdot L}$ found as $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$. The quadratic equation of the matrix $M^{n \cdot L}$ is obtained as follows where $I^{2 \cdot L} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is unit matrix.

$$|M^{n \cdot L} - xI^{2 \cdot L}| = \left| \begin{pmatrix} F_{n-2} & F_n \\ F_n & F_{n+2} \end{pmatrix} - x \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} F_{n-2} & F_n \\ F_n & F_{n+2} \end{pmatrix} - \begin{pmatrix} -x & 0 \\ 0 & x \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} F_{n-2} + x & F_n \\ F_n & F_{n+2} - x \end{pmatrix} \right|$$

$$= (F_{n-2} + x)(F_{n+2} - x) - F_n^2$$

$$= -x^2 + F_{n-2}F_{n+2} - x(F_{n-2} - F_{n+2}) - F_n^2$$

$F_{n+2} - F_{n-2} = L_n$ (please see [19]),
here, $F_{n-2} - F_{n+2} = -L_n$. If the obtained value is substituted in the quadratic equation of the matrix $M^{n \cdot L}$,

$$|M^{n \cdot L} - xI^{2 \cdot L}| = -x^2 + xL_n + (-1)^{n+1} \qquad (16)$$

is obtained. From the method Δ,

$$x_{1,2} = \frac{-L_n \pm \sqrt{L_n^2 - 4(-1)^n}}{-2} \qquad (17)$$

roots are obtained.

3.3. Quadratic Equations and Characteristic Roots of $R^{n \cdot L}$

Matrix $R^{n \cdot L}$ could not be written in terms of the elements of the Fibonacci and Lucas Number sequences. Then, quadratic equation and characteristic roots could not be reached.

4. CONCLUSION

In this study, Lorentz matrix multiplication was used, unlike classical matrix multiplication, to find the n^{th} power of a matrix. Under classical matrix multiplication, previously known identities [Theorem 2] were obtained. However, new identities [Theorem 3] were obtained. In this article, matrices that type of 2×2 are used. In [5], matrices that type of $k \times k$ are worked to obtain generalized order-k Fibonacci and Lucas matrices under classical matrix multiplication. Unlike these matrices, different generalized matrices can be reached under Lorentz multiplication. At the same time, in this study, identities are obtained with classic Fibonacci and Lucas numbers. Differently from this article, in [5], identities are achieved with generalized order-k Fibonacci and Lucas numbers. These identities are generalized of identities obtained with classic Fibonacci and Lucas numbers for different k value. The quadratic equation and characteristic roots of the matrix M, which can be generalized by both classical matrix multiplication and Lorentz matrix multiplication, are obtained.

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The Declaration of Ethics Committee Approval

This study does not require ethics committee permission or any special permission

The Declaration of Research and Publication Ethics

The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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