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CHARACTERIZATIONS MOTIVATED BY THE NEXUS BETWEEN CONVOLUTION AND SIZE BIASING FOR EXPONENTIAL VARIABLES

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Abstract: For a continuous density $f(x)$ with support on the real interval $(0, \infty)$ and finite mean μ, its size biased density is defined to be of the form $(x/\mu)f(x)$. It is well known that for exponential variables, the convolution of two copies of the density yields the size biased form. This is the basis of the so-called inspection paradox. We verify that this agreement between size biasing and convolution actually characterizes the exponential distribution. We next consider the case in which the addition of one more term in a sum of independent identically distributed (i.i.d.) positive random variables also coincides with size biasing. Some related conjectures are also introduced. We then consider the problem of characterizing the class of all pairs of densities that can be called size-bias convolution pairs in the sense that their convolution is just a size biased version of one of them. We then consider discrete analogs to the size bias convolution results. It turns out that matters are more easily dealt with in the case of non-negative integer valued variables. Related geometric and Poisson characterizations are provided. Next, denote the sum of n i.i.d non-negative integer valued random variables $\{X_i\}$, $i = 1, 2, ...$ by S_n. We verify that the ratio of the densities of S_{n_1} and S_{n_2} determines the distribution of the X's. The absolutely continuous version of this result, though judged to be plausible, can only be conjectured at this time.

Key words: Continuous density, Size biased density, Convolution, Non negative integer valued variables, Exponential distribution, Size-bias convolution pairs.

1. Introduction

If we consider the convolution of two identical exponential distributions, the resulting density is just a size biased version of the exponential density involved in the convolution. This observation, discussed below in Section 2, provides a characterization of the exponential density. This characterization is so simply verified that it seems inevitable that it must have been proved in some earlier paper, but we have not been able to find a reference. In fact, in the exponential case, we can observe that the n-fold convolution of the exponential distribution also produces a weighted version of the common density of the convolutants. This too will be shown to be a characteristic property of the exponential density. In Section 3, we investigate the problem of identifying all pairs of densities corresponding to positive random variables that can be called size-bias convolution pairs in the sense that their convolution is just a size-biased version of one of the densities in the pair. If we turn to consider non-negative integer valued random variables, as we shall in Section 4, not unexpected parallel results involving geometric variables can be formulated. Analogous Poisson characterizations can also be identified. In fact, in Sub-section 4.3, very general characterization results will be proved for any distribution with support equal to the non-negative integers. Similar

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results are obtained for bounded non-negative integer valued random variables. It is tempting to propose that parallel general results will be available for general absolutely continuous positive random variables. This ambitious conjecture remains open, except for a few exponential cases.

2. Exponential characterizations

Suppose that X_1 and X_2 are i.i.d exponential random variables and that we define $S_2 = X_1 + X_2$. A comprehensive survey of distributional properties of exponential variables may be found in the volume dedicated to the exponential distribution that includes [1]. The density function of S_2 is that of a gamma distributed random variable and thus is a size biased version of the density of the X_i's. That this is a characteristic property of the exponential distribution is readily confirmed as follows.

THEOREM 1. *Suppose that X_1 and X_2 are i.i.d positive absolutely continuous random variables and that $S_2 = X_1 + X_2$. Suppose that, for some positive c we have*

$$f_{S_2}(x) = cxf_{X_1}(x), \quad x > 0. \tag{2.1}$$

It follows that X_1 has an exponential distribution with mean $1/c$.

PROOF. Denote the Laplace transform of a positive random random variable X by $L_X(s) = E(e^{-sX}), \quad s > 0$. Chapter 13 of [3] will provide adequate discussion of Laplace transforms for our current purposes. Since $L_{S_2}(s) = [L_{X_1}(s)]^2$, we can conclude from (2.1) that

$$[L_{X_1}(s)]^2 = \int_0^\infty e^{-sx} cxf_{X_1}(x)dx = -c(d/ds)L_{X_1}(s).$$

However, this is a simple "variables-separable" differentiable equation with general solution of the form $L_{X_1}(s) = (k + s/c)^{-1}$. Since $L_{X_1}(0) = 1$ it follows that $k = 1$, and that X_1 has an exponential density with $\lambda = c$.

Several closely related characterizations can be formulated. A sample of five such possibilities follows. Proofs will be supplied for three of them, while the other two at present lack proofs and are labeled as (plausible) conjectures.

THEOREM 2. *Let $\{X_i\}_{i=1}^\infty$ be i.i.d. positive absolutely continuous random variables and for each n define $S_n = \sum_{i=1}^n X_i$ If, for some $c > 0$ and some positive integer k, we have*

$$f_{S_{k+1}}(x) = cxf_{S_k}(x) \tag{2.2}$$

then X_1 has an exponential distribution with mean $1/ck$.

PROOF. Using Laplace transforms we may rewrite (2.2) in the form

$$L_{X_1}^{k+1}(s) = -c(d/ds)L_{X_1}^k(s) = -ckL_{X_1}^{k-1}(s)(d/ds)L_{X_1}(s)$$

Dividing both sides by $L_{X_1}^{k-1}(s)$ yields an equation identical to that encountered in the proof of Theorem 1 with c replaced by ck. It follows that X_1 has an exponential density with $\lambda = ck$.

In the next items we will use the standard notation for a convolution of two densities, f_1 and f_2, namely $f_1 * f_2$.

THEOREM 3. *If $f * g(x) = cxf(x)$ where g is an exponential density with intensity λ, then f is a gamma density.*

PROOF. Using Laplace transforms we have, by hypothesis,

$$L_f(s)(1+s/\lambda)^{-1} = -c(d/ds)L_f(s).$$

The general solution to this differential equation is $L_f(s) = (1+s/\lambda)^{-\alpha}$ where $\alpha > 0$, indicating that f is a gamma density.

Instead of using x as a weighting or biasing function. we may ask what happens when x is replaced by a power of x. For the case involving x^2, we have the following result.

THEOREM 4. *Suppose that X_1 and X_2 are i.i.d positive absolutely continuous random variables and that $S_2 = X_1 + X_2$. Suppose that, for some positive c we have*

$$f_{S_2}(x) = cx^2 f_{X_1}(x), \quad x > 0. \tag{2.3}$$

Provided that $var(X_1) = (1/2)E^2(X_1)$, it follows that X_1 has a gamma distribution with shape parameter 2.

PROOF. First, note that it is readily verified that if $X_1 \sim \Gamma(2,\beta)$ then (2.3) holds. Suppose now that (2.3) holds. Evidently we must have $c = 1/\mu_2 = 1/E(X_1^2) < \infty$. Rewriting this in terms of $L(s)$, the Laplace transform of X_1, we have

$$L''(s) = \mu_2 L^2(s). \tag{2.4}$$

Multiplying both sides of this equality by $2L'(s)$, we have

$$2L'(s)L''(s) = \frac{2\mu_2}{3} 3L^2(s)L'(s)$$

Integrating over the interval $(0,t)$ and recalling that $L(0) = 1$ and $L'(0) = -\mu = -E(X_1)$, we have

$$[L'(t)]^2 - (-\mu)^2 = \frac{2\mu_2 3^3}{L}(t) - \frac{2\mu_2}{3}.$$

This will simplify when we apply the condition, stated in the hypothesis of the theorem, that $var(X_1) = (1/2)E^2(X_1)$, equivalently that $\mu_2 - (3/2)\mu^2 = 0$. Under this assumption, we have

$$[L'(t)]^2 = \frac{2\mu_2}{3} L^3(t).$$

However, from (2.4) we can write

$$L''(t)L(t) = \mu_2 L^3(t),$$

and consequently

$$[L'(t)]^2 = (2/3)L''(t)L(t).$$

we may rearrange this to obtain

$$\frac{3}{2}\frac{L'(t)}{L(t)} = \frac{-L''(t)}{-L'(t)}.$$

Integrating with respect to t over the interval $(0,s)$ yields

$$(3/2)\log L(s) = \log[-L'(s)] - \log \mu,$$

so that $-\mu = [L(s)^{-3/2}L'(s)$. Integrating with respect to s over the interval $(0,t)$ produces

$$-\mu t = -2\{[L(t)]^{-(3/2)-1} - 1\},$$

so that $L(t) = (1+\mu t/2)^{-2}$ and consequently $X_1 \sim \Gamma(2,\mu/2)$. Since in this expression, μ can take on any positive value, we conclude that, if (2.3) holds and if $var(X_1) = (1/2)E^2(X_1)$, then $X_1 \sim \Gamma(2/\beta)$ for some $\beta > 0$. After viewing this result, it is inevitable that one would consider the following unproved conjecture.

CONJECTURE 1. *If $f * f(x) = cx^\alpha f(x)$ then, subject to regularity conditions involving certain moments of the density, f is a gamma density with shape parameter α.*

Motivated by the fact that for the exponential case, convolution corresponds to size biasing (when $n_1 = 1$ and $n_2 = 2$ below) we have the following quite general conjecture.

CONJECTURE 2. *Let $\{X_i\}_{i=1}^\infty$ be i.i.d. positive absolutely continuous random variables and for each n define $S_n = \sum_{i=1}^n X_i$ Claim: If for a fixed pair $1 \le n_1 < n_2$ we have*

$$f_{S_{n_2}}(x) = cx^{n_2-n_1} f_{S_{n_1}}(x)$$

$\forall x$, and for some $c > 0$ then X_1 has an exponential distribution. Here too, it is likely that it will be necessary to invoke regularity conditions involving certain moments of the density of X_1.

A proof or disproof of this last conjecture has eluded us. However, as we shall see below, better results are available in discrete cases. To introduce the discussion of non-negative integer valued random variables, we will first consider geometric and Poisson examples. But before leaving the absolutely continuous case, we will consider the general problem of identifying all cases, not just exponential and gamma cases, in which convolution is equivalent to size biasing.

3. Size-bias convolution pairs of densities

Throughout this section we will be dealing with density functions corresponding to positive absolutely continuous random variables which are positive throughout the interval $(0, \infty)$. If f is such a density, we will denote its Laplace transform by $L_f(s)$, thus

$$L_f(s) = \int_0^\infty e^{-sx} f(x)dx, \quad s \in (0, \infty).$$

We know that if f is a gamma density with shape parameter α and scale parameter $1/\lambda$ and if g is an exponential density with mean $1/\lambda$,, then the convolution$f * g$ is again a gamma density. In fact we have the following situation:

$$f * g(x) = cxf(x), \quad x > 0. \tag{3.1}$$

for some positive c. In the particular case just mentioned we have $c = [\int_0^\infty xf(x)dx]^{-1}$.

If a pair of densities (f, g) satisfies equation (3.1), we will call it a size-bias-convolution (or sbc) pair. We have seen one example of an sbc pair. The name comes from the fact that when (3.1) holds then the convolution of f and g produces a size biased version of f.

Our goal is to characterize all valid sbc pairs of densities.

3.1. Laplace transforms corresponding to a size-bias-convolution pair of densities

If f and g are legitimate densities satisfying the sbc equation (3.1), then the corresponding Laplace transforms can readily be shown to be related by

$$L_g(s) = -c\left[\frac{d}{ds}\log L_f(s)\right], \tag{3.2}$$

or, equivalently

$$L_f(s) = \exp\left[-\frac{1}{c}\int_0^s L_g(t)dt\right]. \tag{3.3}$$

Note that in (3.2), in order that $L_g(0) = 1$ we must set $c = 1/\mu_f$ where μ_f is the necessarily finite mean of the density f.

Uniqueness considerations: For a given density f it is clear that if there exists a density g with (f, g) being an sbc pair. then g is the unique density with this property. Likewise, for a given density g it is clear that if there exists a density f with (f, g) being an sbc pair. then f is the unique density with this property.

3.2. Identifying sbc pairs

It might be hoped that every density f, with Laplace transform $L_f(s)$ will form part of an sbc pair. We may consider a candidate choice of g to be that density with a Laplace transform given by equation (3.2). This will be a solution provided that the expression on the right side of (3.2) is a valid Laplace transform, i.e,. if it is completely monotone. Alternatively, it might be possible to recognize the right hand side of (3.2) as the Laplace transform of some well-known density. It is not at all obvious that the right hand side of (3.2) will always be completely monotone. We know it is for certain choices for f of the gamma form. But are there other cases ?

It turns out that the key result that allows us to resolve our identification problem is a characterization of infinite divisibility of distributions on $(0,\infty)$ provided by [3]. The result in question is as follows. The Laplace transform $L_f(s)$ corresponds to an infinitely divisible f if and only if the function $-\log(L_f(s))$ has a completely monotone derivative. However this is precisely the condition necessary for $L_g(s)$ defined by (3.2) to be a valid Laplace transform.

Note [2] made use of this characterization to verify the infinite divisibility of generalized inverse Gaussian densities.

We are able then to characterize the set of all valid sbc pairs (f,g) to consist of all pairs in which f is infinitely divisible and a corresponding g has its Laplace transform determined by (3.2).

Example 1. If we choose f to correspond to a gamma density, which is infinitely divisible, then from (3.2) we can identify the choice of g to yield a valid sbc pair will be an exponential density. This observations (and analogous observations involving different sbc pairs) can be rephrased as characterizations of distributions. For example, we might wish to identify all possible densities g such that (f,g) constitutes an sbc pair with f being a gamma density. It follows that g must be an exponential density. This particular characterization appeared in [4], see also [5] and [6].

Example 2. If f is taken to correspond to an inverse Gaussian distribution with parameters μ and λ denoted by $IG(\mu,\lambda)$, which is known to be infinitely divisible, then its Laplace transform is of the form

$$L_f(s)=\exp[(\lambda/\mu)(1-\sqrt{1+2\mu^2\lambda^{-1}s})].$$

Differentiating with respect to s yields

$$L_f^{'}(s)=\exp[(\lambda/\mu)(1-\sqrt{1+2\mu^2\lambda^{-1}s})]\left\{-\mu\left(1+\frac{2\mu^2}{\lambda}s\right)^{-1/2}\right\}.$$

The corresponding density g to form an sbc pair is, from (3.2), one with Laplace transform given by

$$L_g(s)=-c\frac{d}{ds}\log L_f(s)=-c\frac{L_f^{'}(s)}{L_f(s)}=c\mu\left(1+\frac{2\mu^2}{\lambda}s\right)^{-1/2}=\left(1+\frac{2\mu^2}{\lambda}s\right)^{-1/2}.$$

where c has been chosen equal to $1/\mu$ to ensure that $L_g(0)=1$. Thus g is a gamma density with shape parameter $\alpha=1/2$ and scale parameter $(2\mu^2/\lambda)$, i.e. corresponding to a random variable $Y=(\mu^2/\lambda)U$ where U has a chi-squared distribution with one degree of freedom.

Example 3. If f is taken to correspond to a generalized inverse Gaussian distribution with parameters a,b and p denoted by $GIG(a,b,p)$, which is also known to be infinitely divisible, then its Laplace transform is of the form

$$L_f(s)\propto(a+2s)^{-p/2}K_p(\sqrt{b(a+2s)}),$$

where $K_p(u)$ is a modified Bessel function of the second kind. Differentiating with respect to s yields

$$L_f^{'}(s)\propto(-p)(a+2s)^{-p/2-1}K_p(\sqrt{b(a+2s)})+b(a+2s)^{-(p+1)/2}K_p^{'}(\sqrt{b(a+2s)}).$$

The corresponding density g to form an sbc pair is, from (3.2), one with Laplace transform given by

$$L_g(s) = -c\frac{L_f^{`}(s)}{L_f(s)}$$

where c is chosen to ensure that $L_g(0) = 1$. We can then recognize the density g as a linear combination of gamma densities.

4. Analogous discrete characterizations

We now turn to consider a selection of discrete characterizations suggested as natural analogs of the absolutely continuous results in Section 2.

4.1. Geometric characterizations

Parallel to the situation for exponential variables, in the geometric case, convolution essentially corresponds to size biasing. For a sample of size two, we have the following geometric characterization.

THEOREM 5. *Let X_1 and X_2 be i.i.d. non-negative integer valued random variables. Suppose that for each k and some $c > 0$*

$$P(X_1 + X_2 = k) = c(k+1)P(X_1 = k), \tag{4.1}$$

it follows that X_1 has a geometric distribution.

PROOF. Let $P(s)$ be the probability generating function of X_1. Then, from (4.1) we have

$$P^2(s) = csP'(s) + cP(s).$$

Rearranging this becomes:

$$\frac{P'(s)}{P(s)[P(s) - c]} = \frac{1}{cs}$$

i.e., writing dP/ds for $P'(s)$ and P for $P(s)$, as is usual in differential equations,

$$\frac{dP}{P(P-c)} = \frac{ds}{cs}.$$

Using partial fractions applied to $1/P(P-c)$ this is equivalent to

$$\frac{ds}{s} = \frac{dP}{P-c} - \frac{dP}{P}.$$

Integrating we get

$$log(s) = log(P-c) - log(P) + k.$$

Thus

$$s = \tilde{k}\frac{P-c}{P}.$$

From this we have

$$P = \frac{c}{1 - \frac{s}{\tilde{k}}}.$$

However, we know that $P(0) = p_0$ and that $P(1) = 1$, so that finally we get

$$P(s) = \frac{p_0}{1-(1-p_0)s},$$

i.e., X_1 has a *geometric*(p_0) distribution.

A more general result is available. First note that if we have i.i.d. *geometric*(p) random variables, then for any $n \geq 2$,we have

$$P(S_n = k) = p\left(1 + \frac{k}{n-1}\right)P(S_{n-1} = k). \quad k = 0.1, 2, ...$$

where $S_n = \sum_{i=1}^{n} X_i$.

THEOREM 6. *Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. non-negative integer valued random variables. For each n define $S_n = \sum_{i=1}^{n} X_i$. If for a fixed integer $n \geq 2$ and for every k we have*

$$P(S_n = k) = c\left(1 + \frac{k}{n-1}\right)P(S_{n-1} = k). \quad k = 0.1, 2, ... \tag{4.2}$$

for some positive c, then the X_i's have a common geometric distribution.

PROOF. Let $P(s)$ be the probability generating function of X_1, so that the generating function of S_n is $[P(s)]^n$. From (4.2) we then have

$$[P(s)]^n = c[P(s)]^{n-1} + \frac{cs}{n-1}\frac{d}{ds}[P(s)]^{n-1} = c[P(s)]^{n-1} + cs[P(s)]^{n-2}P'(s).$$

Consequently we have

$$[P(s)]^2 = c[P(s)] + csP'(s).$$

But this is exactly the equation solved in the case $n = 2$ and we can conclude that

$$P(s) = \frac{p_0}{1-(1-p_0)s},$$

i.e., X_1 has a *geometric*(p_0) distribution.

In fact, in the geometric case, we are able to prove an even more general result which is parallel to the conjectured exponential characterization described in the previous section.

THEOREM 7. *Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. non-negative integer valued random variables. For each n define $S_n = \sum_{i=1}^{n} X_i$*

If for a fixed pair $1 \leq n_1 < n_2$ we have

$$P(S_{n_2} = k) = c\frac{(n_2+k-1)!}{(n_1+k-1)!}P(S_{n_1} = k) \tag{4.3}$$

$\forall k$, for some $c > 0$ then X_1 has a geometric distribution.

We will defer proving this result until Section 5, where will prove an even more general result as follows.

Consider a sequence of non-negative integer valued random variables $\{X_i\}_{i=1}^{\infty}$ and define

$$A(n,k) = \frac{P(\sum_{i=1}^{n} X_i = k)}{P(\sum_{i=1}^{n-1} X_i = k)}.$$

Claim : For any fixed $n \geq 2$, the sequence $A(n,k)$ determines the common distribution of the X_i's.

We will illustrate a special case of this claim in the following Section where a Poisson sequence is considered.

4.2. Poisson characterizations

As usual, let the X_i's be i.i.d. non-negative integer valued r.v.'s and for each n define $S_n = \sum_{i=1}^n X_i$. It is readily verified that if the X_i's have a common $Poisson(\lambda)$ distribution then for a fixed pair $1 \le n_1 < n_2$ we have

$$P(S_{n_2} = k) = c\left(\frac{n_2}{n_1}\right)^k P(S_{n_1} = k), \tag{4.4}$$

$\forall k$, for $c = e^{-(n_2-n_1)\lambda}$.

This observation leads to the following characterization of the Poisson distribution.

Theorem 8. *Let the X_i's be i.i.d. non-negative integer valued random variables and for each n define $S_n = \sum_{i=1}^n X_i$. If for a fixed pair $1 \le n_1 < n_2$ we have*

$$P(S_{n_2} = k) = c\left(\frac{n_2}{n_1}\right)^k P(S_{n_1} = k) \tag{4.5}$$

$\forall k$, for some $c > 0$, i.e., if (4.4) holds for some $c > 0$, then X_1 has a Poisson distribution.

Proof. It is tempting to try to resolve this issue by using probability generating functions. The generating function of X_1, denoted by $P(s)$ must satisfy

$$P^{n_2}(s) = cP^{n_1}(\frac{n_2}{n_1}s).$$

However, it is not obvious how to solve this equation, even in the case in which $n_1 = 1$ and $n_2 = 2$.

We can make progress by considering equation(4.5) for a series of values of k. We will denote $P(X_1 = i)$ by p_i for $i = 0, 1, 2...$ Next denote the ratio between p_1 and p_0 by λ. The case $k = 2$ of (4.5) simplifies to yield $p_2 = p_1^2/(2p_0) = \lambda^2 p_0/2$. Next if we consider $k = 3$ we obtain an equation for p_3 as a function of p_0, p_1 and p_2 which can be solved to yield $p_3 = \lambda^3 p_0/3!$. We may then conclude that $p_i = \lambda^i p_0/i!$ for every i by using an induction argument whereby we assume that for $j < i$ we have $p_j = \lambda^j p_0/j!$ and, inserting these values in equation (4.5) for $k = i$, we may verify that $p_i = \lambda^i p_0/i!$. The value of p_0 is then determined by the requirement that $\sum_{i=0}^\infty p_i = 1$. Thus we find $p_0 = e^{-\lambda}$ and confirm that X_1 has a $Poisson(\lambda)$ distribution.

4.3. General characterizations of discrete distributions

Conjecture 2 in Section 2 was an instance in which for a sequence of i.i.d. X_i's with sums defined by $S_n = \sum_{i=1}^n X_i$, it was felt to be plausible that the ratio of densities of S_{n_1} and S_{n_2} would determine the density of the X_i's. In the absolutely continuous case, the conjecture remains open. However, progress can be made in the case in which the X_i's are non-negative integer valued random variables.

Suppose that X_i^*'s are i.i.d. random variables with $P(X_i^* = k) = p_k^* > 0, \quad k = 0, 1, 2, ...$ The corresponding sums will be denoted by $S_n^* = \sum_{i=1}^n X_i^*$. For $1 \le n_1 < n_2$ the corresponding ratio of densities of sums will be denoted by

$$A^*(n_1.n_2, k) = \frac{P(\sum_{i=1}^{n_2} X_i^* = k)}{P(\sum_{i=1}^{n_1} X_i^* = k)} \tag{4.6}$$

We claim that if another sequence $\{X_i\}_{i=1}^\infty$ has the same ratio of densities of sums as do the X_i^*'s and if $P(X_1 = 1) = P(X_1^* = 1)$ then $X_1 \stackrel{d}{=} X_1^*$.

THEOREM 9. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d non-negative integer valued random variables with $P(X_1=k)=p_k>0,\ k=0,1,2,...$ and with $p_1=p_1^*$ as defined above. Suppose that for some pair n_1,n_2 with $1\le n_1<n_2$ and every $k=0,1,2,...$ we have*

$$P(S_{n_2}=k)=A^*(n_1,n_2,k)P(S_{n_1}=k). \tag{4.7}$$

It follows that $X_1 \stackrel{d}{=} X_1^*$.

PROOF. Note that (4.7) holds for the S_n^*'s as well as for the S_n's.
Consider the case in which $k=0$, we have

$$p_0^{n_2}=P(\sum_{i=1}^{n_2}X_i=0)=A^*(n_1.n_2,0)P(\sum_{i=1}^{n_1}X_i=0)=A^*(n_1.n_2,0)p_0^{n_1},$$

so that p_0 is determined by $A^*(n_1.n_2,0)$, and indeed $p_0=p_0^*$.
Next consider $k=1$, we have

$$n_2p_1p_0^{n_2-1}=P(\sum_{i=1}^{n_2}X_i=1)=A^*(n_1.n_2,1)P(\sum_{i=1}^{n_1}X_i=1)=A^*(n_1.n_2,1)n_1p_1p_0^{n_1-1}.$$

Note that p_1 cancels and is not determined by this equation. However, by one of our hypotheses, $p_1=p_1^*$. Next consider $k=2$,

$$\begin{aligned}[n_2p_2p_0^{n_2-1}+n_2(n_2-1)p_1^2p_0^{n_2-2}]&=P(\sum_{i=1}^{n_2}X_i=2)=A^*(n_1.n_2,2)P(\sum_{i=1}^{n_1}X_i=2)\\&=A^*(n_1.n_2,2)[n_2p_2p_0^{n_2-1}+n_2(n_2-1)p_1^2p_0^{n_2-2}].\end{aligned}$$

This gives p_2 as a linear function with coefficients that are functions of p_0 and p_1. Thus p_2 is determined by $A^*(n_1,n_2,2)$ and indeed $p_2=p_2^*$.

Now each successive value of k will introduce a new p_k which will be a linear function with coefficients that are known functions of the preceding p_i's. By an inductive argument the full sequence $p_0,p_1,p_2,p_3,....$ is determined by the sequence $A^*(n_1,n_2,k)$. Thus we conclude that $X_1 \stackrel{d}{=} X_1^*$.

COROLLARY 1. *If the random variables, the X_i's have bounded support say 0,1,2,...,M, then for any fixed $1\le n_1<n_2$, the finite sequence $\{A^*(n_1,n_2,k)\}_{k=0}^{M}$ determines the common distribution of the X_i's.*

PROOF. Just the same as in the theorem, except that we only need to consider values of k that are less than or equal to M.

5. Conclusions

Almost inevitably, when characterization results are presented to a statistical audience, the question of possible application of the results is raised. One strong argument for the study of characterizations is that they often enable researchers to realize interesting consequences of distributional assumptions that they routinely make. Characterizations often can be used to apply quick preliminary tests of certain distributional assumptions. In reliability settings, it will be of interest to know whether a size biased version of the lifetime distribution (the lifetime of an item in service) really behaves like a sum of two independent device lifetimes. If it doesn't, then a desirable assumption of exponentially distributed lifetimes must be set aside. If it does, then we can be more comfortable about the common distributional assumption.

References

[1] Balakrishnan, N. and Basu, A.P. (1995). Basic distributional results and properties. in Balakrishnan, N. and Basu, A.P. (Eds.) *The Exponential Distribution: Theory, Methods and Applications.* (7-15). Gordon and Breach, Amsterdam.

[2] Barndorff-Nielsen, O. and Halgreen, C. (1977). Infinite divisibility of the hyperbolic and generalized inverse Gaussian distributions. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 38, 309-311.

[3] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II. Second edition,* Wiley, New York.

[4] Khattree, R. (1989). Characterization of inverse-Gaussian and gamma distributions through their length-biased distributions . *IEEE Transactions on Reliability*, 38, 610-611.

[5] Pakes, A. G. and Khattree, R. (1992). Length-biasing characterization of laws and the moment problem. *Australian Journal of Statistics*, 34, 307-322.

[6] Sen, A. and Khattree, R. (1996). Length biased distribution, equilibrium distribution and characterization of probability laws. *Journal of Applied Statistical Science*, 3, 239-252.