Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (6) (2022), 1535 – 1549
DOI : 10.15672/hujms.1034883

Research Article

Ricci curvature for pointwise semi-slant warped products in non-Sasakian generalized Sasakian space forms and its applications

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Abstract

We find Ricci curvature bounds for pointwise semi-slant warped products submanifolds in non-Sasakian generalized Sasakian space forms in this work, and analyze the equality case of the inequality. The derived inequality is also used to develop a number of applications.

Mathematics Subject Classification (2020). 53C05, 53A40, 53C40

Keywords. warped products, pointwise semi-slant submanifolds, generalized Sasakian space forms, Chen-Ricci inequality, harmonic function, Hessian function, Dirichlet energy

1. Introduction

Alegre et al.[1] proposed the concept of a generalized Sasakian space form as a generalization of Sasakian space form, Kenmotsu space form and cosymplectic space form. They used geometric constructions such as Riemannian submersions, warped products, and D-conformal deformations to produce several non-trivial examples of generalized Sasakian space forms. Many fascinating outcomes have been demonstrated in these ambient areas since then [2–7, 15–18, 20].

On the other hand, since J. F. Nash's famous theory of isometric immersion of a Riemannian manifold into a suitable Euclidean space provides a powerful motivation to view each Riemannian manifold as a submanifold in a Euclidean space, one of the most fundamental problems in submanifold theory is to find simple basic relationships between intrinsic and extrinsic invariants of a Riemannian submanifold. The major extrinsic invariant is the squared mean curvature, whereas the key intrinsic invariants are the Ricci curvature and the scalar curvature.

The theory of product manifolds contains crucial physical and geometrical ramifications, in addition to Hermitian geometry. In physics, Einstein's general relativity spacetime can be thought of as a product of three-dimensional space and one-dimensional time, both of which have their own metrics, and hence its topology is determined by these metrics.

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Received: 09.12.2021; Accepted: 17.05.2022

Kaluza Klein theory, brane theory, and gauge theory all have interesting applications of product manifolds. In 1969, R. L. Bishop et al. [8] introduced a generalized case of Riemannian product manifolds to study manifolds of negative sectional curvature called warped product manifold. They defined warped products as follows:

Let us consider a Riemannian manifolds N_T of dimension d_1 with Riemannian metric g_1, N_θ of dimension d_2 with Riemannian metric g_2 and σ be positive differentiable functions on N_T. Consider the warped product $N_T \times N_\theta$ with its projections $\iota_1 : N_T \times N_\theta \to N_T$ and $\iota_2 : N_T \times N_\theta \to N_\theta$. Then, their warped product manifold $M = N_T \times_\sigma N_\theta$ is the product manifold equipped with the structure

$$g(X,Y) = g_1(\iota_{1*}X, \iota_{1*}Y) + (\sigma \circ \iota_1)^2 g_2(\iota_{2*}X, \iota_{2*}Y),$$

for any vector fields X, Y on M, where $*$ denotes the symbol for tangent maps.

Due to its usefulness many research article has been published in this area [9–12, 14, 19, 21, 22].

The major goal of this paper is to establish a relationship between Ricci curvature and mean curvature vectors of warped product pointwise semi-slant submanifolods of non-Sasakian generalized Sasakian space forms. Further, we derived some applications of the result in physics.

2. Preliminaries

Let $\tilde{M}$ be a $(2p+1)$-dimensional almost contact metric manifold with an almost contact structure (ϕ, ξ, η, g). The $(1,1)$ tensor field ϕ, the structure vector field ξ, the 1-form η, and the Riemannian metric g on $\tilde{M}$ are all known to satisfy the relations

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

The above condition also imply that

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(X) = g(X, \xi),$$
$$g(\phi X, Y) + g(X, \phi Y) = 0,$$

where $X, Y \in T\tilde{M}$. Here, $T\tilde{M}$ denotes the Lie algebra of vector fields on $\tilde{M}$.

Let $(\tilde{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold whose curvature tensor satisfies

$$\begin{aligned} \overline{R}(X,Y)Z \;=\; & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.1}$$

for all vector fields X, Y, Z, where f_1, f_2, f_3 are differentiable functions on $\tilde{M}$, then $\tilde{M}(f_1, f_2, f_3)$ is said to be a generalized Sasakian space form.

Remark 2.1. It's worth noting that the generalized Sasakian space forms encompass the following well-known spaces:

(1) Sasakian space forms and in this case

$$f_1 = \frac{(c+3)}{4}, f_2 = f_3 = \frac{(c-1)}{4}.$$

(2) Kenmotsu space forms and in this case

$$f_1 = \frac{(c-3)}{4}, f_2 = f_3 = \frac{(c+1)}{4}.$$

(3) Cosymplectic space forms and in this case

$$f_1 = f_2 = f_3 = \frac{c}{4}.$$

Let M be an d-dimensional submanifold of a generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ of dimension $2p+1$. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection on M and $\tilde{M}(f_1, f_2, f_3)$ respectively. The Gauss and Weingarten equations are defined as

$$\tilde{\nabla}_X Y = \nabla_X Y + \zeta(X, Y),$$

$$\tilde{\nabla}_X \xi = -A_N X + \nabla^{\perp}_X Y,$$

for vector fields $X, Y \in TM$ and $N \in T^{\perp}M$, where ζ, A_N and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. The equation

$$g(\zeta(X, Y), N) = g(A_N X, Y), \quad X, Y \in TM, \quad N \in T^{\perp}M$$

connects the second fundamental form with the shape operator.
Let R be the curvature tensor of M and let $\tilde{R}$ be the curvature tensor of $\tilde{M}(f_1, f_2, f_3)$, then the Gauss equation is given by

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) + g(\zeta(X, Z), \zeta(Y, W)) \\ &\quad - g(\zeta(X, W), \zeta(Y, Z)) \end{aligned} \tag{2.2}$$

for $X, Y, Z, W \in TM$.

We can write

$$\phi X = PX + FX, \tag{2.3}$$

and

$$\phi N = tN + fN, \tag{2.4}$$

for any $X \in TM$ and $N \in T^{\perp}M$, where $PX(resp.tN)$ is the tangential component and $FX(resp.fN)$ is normal component of $\phi X(resp.\phi N)$. When F is identically zero, a submanifold M is said to be invariant, and when P is identically zero, it is said to be anti-invariant.

Let $\{e_1, \ldots, e_d\}$ and $\{e_{d+1}, \ldots, e_{2p+1}\}$ be the tangent and normal orthonormal frames on M, respectively. Then, the mean curvature vector field is given by

$$\mathcal{H} = \frac{1}{d}\sum_{i=1}^{d} \zeta(e_i, e_i), \quad d^2||\mathcal{H}||^2 = \sum_{i,j}^{d} g(\zeta(e_i, e_i), \zeta(e_j, e_j)). \tag{2.5}$$

Also, for D_{θ_1}- minimality, we have

$$d^2||\mathcal{H}||^2 = \sum_{\gamma=d+1}^{2p+1} (\zeta^r_{d_1+1 d_1+1} + \cdots + \zeta^r_{dd})^2. \tag{2.6}$$

Further, we set

$$\zeta^{\gamma}_{ij} = g(\zeta(e_i, e_j), e_\gamma), \quad \|\zeta\|^2 = \sum_{i,j=1}^{d} g(\zeta(e_i, e_j), \zeta(e_i, e_j)). \tag{2.7}$$

The second fundamental form, ζ, has various geometric features as a result of which we have the following submanifold classes.

Definition 2.2. A submanifold is said to be totally geodesic submanifold if the second fundamental form ζ vanishes identically, that is $\zeta = 0$.

Definition 2.3. A submanifold is said to be minimal submanifold if the mean curvature vector $\mathcal{H}$ vanishes identically, that is $\mathcal{H} = 0$.

Let $K(\pi)$ denotes the sectional curvature of a Riemannian manifold M of the plane section $\pi \subset T_xM$ at a point $x \in M$. If $\{e_1, \ldots, e_d\}$ be the orthonormal basis of T_xM and $\{e_{d+1}, \ldots, e_{2p+1}\}$ be the orthonormal basis of $T_x^\perp M$ at any $x \in M$, then

$$\tau(x) = \sum_{1 \le i < j \le d} K(e_i \wedge e_j), \tag{2.8}$$

where τ is the scalar curvature.

Then, in view of gauss equation, we have

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{\gamma=d+1}^{2p+1} \Big(\zeta_{ii}^r \zeta_{jj}^r - (\zeta_{ij}^r)^2\Big), \tag{2.9}$$

where $K(e_i \wedge e_j)$ and $\tilde{K}(e_i \wedge e_j)$ denotes the sectional curvature of the plane section spanned by e_i and e_j in the submanifold M and the ambient manifold $\tilde{M}$, respectively, at a point x.

Further,

$$2\tau(x) = 2\tilde{\tau}(T_xM) + d^2||\mathcal{H}||^2 - ||\zeta||^2, \tag{2.10}$$

where

$$\tilde{\tau}(T_xM) = \sum_{1 \le i < j \le < d} \tilde{K}(e_i \wedge e_j)$$

is the scalar curvature of the d-plane section T_xM in $\tilde{M}$, this is achieved by adding across the orthonormal frame of M's tangent space in the last equation.

Moreover, a k-Ricci curvature $Ric\Pi_k$ of a k-plane section $\Pi_k(2 \le k \le d)$ at e_a is defined by

$$Ric\Pi_k = \sum_{i \neq a} K_{ai}, \tag{2.11}$$

for a fixed integer $a \in \{1, \ldots, k\}$, where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j and e_a is a unit vector field from the orthonormal basis $\{e_1, \ldots, e_k\}$ of the k-plane section Π_k.

Definition 2.4. A submanifold M of an almost contact manifold $\tilde{M}$ is said to be a pointwise slant submanifold if for any $x \in M$ and a nonzero vector $X \in M_x$, the angle $\theta = \theta(X)$ between ϕX and M_x is constant, where $M_x := \{X \in T_xM | g(X, \xi(x)) = 0\}$.

Definition 2.5. A submanifold M of be an almost contact metric manifold $\tilde{M}$ is said to be a pointwise semi-slant submanifold, if there exist two orthogonal distributions D_1 and D_2 such that

(i) $TM = D_1 \oplus D_2 \oplus \xi$.

(ii) D_1 is invariant.

(iii) D_2 is a pointwise slant with a slant function θ.

Finally, we conclude the section with the following relation by B. Y. Chen [**?**]. According to him, we have

$$\sum_{1 \le i \le d_1} \sum_{d_1+1 \le j \le d} K(e_i \wedge e_j) = d_2 \frac{\Delta\sigma}{\sigma} = d_2\Big(\Delta(ln\sigma) - ||\nabla\sigma||^2\Big), \tag{2.12}$$

where Δ is the Laplacian operator.

3. Ricci curvature on warped products $N_T \times_\sigma N_\theta$

The proof of the major finding is the focus of this section.

Theorem 3.1. *Let $M = N_T \times_\sigma N_\theta \to \tilde{M}(f_1, f_2, f_3)$ be an isometric immersion of an d-dimensional pointwise semi-slant warped products submanifold M in non-Sasakian generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$. Then, the following inequalities exist for each unit vector $e_a \in T_xM$ orthogonal to ξ:*

(1) For each unit vector $e_a \in T_xM$ orthogonal to ξ, we have
(i) If e_a is tangent to N_T, then

$$\frac{1}{4}d^2||\mathcal{H}||^2 \geq Ric(e_a) + d_2\frac{\Delta\sigma}{\sigma} - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2 + f_3(d_2 + 1). \tag{3.1}$$

(ii) If e_a is tangent to N_θ, then

$$\frac{1}{4}d^2||\mathcal{H}||^2 \geq Ric(e_a) + d_2\frac{\Delta\sigma}{\sigma} - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2 + 1). \tag{3.2}$$

where d_1 and d_2 are dimensions of N_T and N_θ, respectively.

(2) If $\vec{\mathcal{H}}(x) = 0$, then there is a unit tangent vector $e_\circ$ at each point x in M that meets the equality condition in (1) then M is mixed totally geodesic and $e_\circ$ is in the relative null space $\mathcal{N}_x$ at x and conversely.

(3) For the equality cases, we have
(a) the equality case of (3.1) holds identically for all unit tangent vectors to N_T at each $x \in M$ then M is mixed totally geodesic and D-totally geodesic pointwise semi-slant warped product submanifold in $\tilde{M}(f_1, f_2, f_3)$ and conversely,
(b) the equality case of (3.2) holds identically for all unit tangent vectors to N_θ at each $x \in M$ then M is mixed totally geodesic and either D_θ-totally geodesic pointwise semi-slant warped product or M is a D_θ-totally umbilical in $\tilde{M}(f_1, f_2, f_3)$ with $dimN_\theta = 2$ and conversely,
(c) the equality case of (1) holds identically for all unit tangent vectors to M at each $x \in M$ then M is mixed totally geodesic submanifold, or M is a mixed totally geodesic, totally umbilical and D-totally geodesic submanifolds with $dimN_\theta = 2$ and conversely.

Proof. From (2.1) and (2.10), we derive

$$d^2||\mathcal{H}||^2 = 2\tau + ||\zeta||^2 - \Big[f_1(d(d-1)) + 3f_2((d_1 - 1) + d_2cos^2\theta) - 2f_3(d-1)\Big]. \tag{3.3}$$

If we use a unit vector field $e_a \in \{e_1, \dots, e_d\}$ for a fixed index $a \in \{1, \dots, d\}$, then (3.3) implies

$$\begin{aligned}
d^2||\mathcal{H}||^2 &= 2\tau + \sum_{\gamma=d+1}^{2p+1} \Big[(\zeta_{aa}^r)^2 + (\zeta_{11}^r + \cdots + \zeta_{dd}^r - \zeta_{aa}^\gamma)^2 + 2 \sum_{1\le i<j\le d} (\zeta_{ij}^\gamma)^2 \Big] \\
&- 2 \sum_{\gamma=d+1}^{2p+1} \sum_{1\le i<j\le d (i,j\neq a)} \zeta_{ii}^\gamma \zeta_{jj}^\gamma \\
&- \Big[f_1\big(d(d-1)\big) + 3f_2\big((d_1-1) + d_2 cos^2\theta\big) - 2f_3(d-1) \Big] \\
&= 2\tau + \frac{1}{2} \sum_{\gamma=d+1}^{2p+1} \Big[(\zeta_{11}^\gamma + \cdots + \zeta_{dd}^\gamma)^2 + (\zeta_{aa}^\gamma + (-\zeta_{11}^\gamma - \cdots - \zeta_{dd}^r)^2 + (\zeta_{aa}^\gamma)^2 \Big] \\
&+ 2 \sum_{\gamma=d+1}^{2p+1} \sum_{1\le i<j\le d} (\zeta_{ij}^\gamma)^2 - 2 \sum_{\gamma=d+1}^{2p+1} \sum_{1\le i<j\le d (i,j\neq a)} \zeta_{ii}^\gamma \zeta_{jj}^\gamma \\
&- \Big[f_1\big(d(d-1)\big) + 3f_2\big((d_1-1) + d_2 cos^2\theta\big) - 2f_3(d-1) \Big]. \qquad (3.4)
\end{aligned}$$

From here we got the two cases:

Case 1: If e_a is tangent to N_{θ_1}, then we require to fix a unit vector field from $\{e_1, \dots, e_{d_1}\}$ to be e_a, and consider $e_a = e_1$, hence from (2.9) and (2.11), we deduce that

$$\begin{aligned}
d^2||\mathcal{H}||^2 &\ge Ric(e_a) + \frac{1}{2} \sum_{\gamma=d+1}^{2p+1} (\zeta_{d_1+1 d_1+1}^r + \cdots + \zeta_{dd}^r)^2 + d_2 \frac{\Delta\sigma}{\sigma} \\
&+ \frac{1}{2} \sum_{\gamma=d+1}^{2p+1} \Big(2\zeta_{11}^\gamma - (\zeta_{d_1+1 d_1+1}^\gamma + \cdots + \zeta_{dd}^\gamma) \Big)^2 \\
&+ \sum_{\gamma=d+1}^{2p+1} \sum_{1\le\alpha<\beta\le d_1} (\zeta_{\alpha\alpha}^\gamma \zeta_{\beta\beta}^\gamma - (\zeta_{\alpha\beta}^\gamma)^2) + \sum_{\gamma=d+1}^{2p+1} \sum_{d_1+1\le s<t\le d} (\zeta_{ss}^\gamma \zeta_{tt}^\gamma - (\zeta_{st}^\gamma)^2) \\
&+ \sum_{\gamma=d+1}^{2p+1} \sum_{1\le i<j\le d_1} (\zeta_{ij}^\gamma)^2 - \sum_{\gamma=d+1}^{2p+1} \sum_{2\le i<j\le d} \zeta_{ii}^\gamma \zeta_{jj}^\gamma \\
&- \Big[f_1\big(d(d-1)\big) + 3f_2\big((d_1-1) + d_2 cos^2\theta\big) - 2f_3(d-1) \Big] \\
&+ \Big[\frac{1}{2} f_1\big((d-1)(d-2)\big) + \frac{3}{2} f_2\big((d_1-2) + d_2 cos^2\theta\big) - f_3(d-2) \Big] \\
&+ \Big[\frac{1}{2} f_1\big(d_1(d_1-1)\big) + \frac{3}{2} f_2(d_1-1) - f_3(d_1-1) \Big] \\
&+ \Big[\frac{1}{2} f_1\big(d_2(d_2-1)\big) + \frac{3}{2} f_2 d_2 cos^2\theta \Big]. \qquad (3.5)
\end{aligned}$$

A straight forward computations, equation (3.5) yields

$$\begin{aligned} d^2||\mathcal{H}||^2 &\geq Ric(e_a) + \frac{1}{2}d^2||\mathcal{H}||^2 + d_2\frac{\Delta\sigma}{\sigma} \\ &+ \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}\Big(2\zeta_{11}^{\gamma} - (\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma})\Big)^2 + \sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}(\zeta_{ij}^{\gamma})^2 \\ &+ \sum_{\gamma=d+1}^{2p+1}\sum_{\beta=2}^{d_1}\zeta_{11}^{\gamma}\zeta_{\beta\beta}^{\gamma} - \sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}\zeta_{ii}^{\gamma}\zeta_{jj}^{\gamma} \\ &- f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2 + f_3(d_2+1)\Big]. \end{aligned} \tag{3.6}$$

Alternatively, it can be effortlessly seen that

$$\sum_{\gamma=d+1}^{2p+1}\sum_{\beta=2}^{d_1}\zeta_{11}^{\gamma}\zeta_{\beta\beta}^{\gamma} = -\sum_{\gamma=d+1}^{2p+1}(\zeta_{11}^{\gamma})^2 \tag{3.7}$$

and

$$\sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}\zeta_{ii}^{\gamma}\zeta_{jj}^{\gamma} = \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta_{11}^{\gamma}\zeta_{jj}^{\gamma}. \tag{3.8}$$

Using (3.7) and (3.8) in (3.6), we find

$$\begin{aligned} d^2||\mathcal{H}||^2 &\geq Ric(e_a) + \frac{1}{2}d^2||\mathcal{H}||^2 + d_2\frac{\Delta\sigma}{\sigma} \\ &+ \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}\Big(2\zeta_{11}^{\gamma} - (\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma})\Big)^2 \\ &+ \sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}(\zeta_{ij}^{\gamma})^2 - \sum_{\gamma=d+1}^{2p+1}(\zeta_{11}^{\gamma})^2 + \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta_{11}^{\gamma}\zeta_{jj}^{\gamma} \\ &- f_1(n + d_1d_2 - 1) - \frac{3}{2}f_2 + f_3(d_2+1)\Big]. \end{aligned} \tag{3.9}$$

Simplifying the fifth term in the right hand side of (3.9) and using (2.6), we have

$$\begin{aligned} &\frac{1}{2}\sum_{\gamma=d+1}^{2p+1}\Big(2\zeta_{11}^{\gamma} - (\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma})\Big)^2 \\ &= 2\sum_{\gamma=d+1}^{2p+1}(\zeta_{11}^{\gamma})^2 + \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}(\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma})^2 \\ &- 2\sum_{\gamma=d+1}^{2p+1}\Big(\zeta_{11}^{\gamma}(\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma}) \\ &= 2\sum_{\gamma=d+1}^{2p+1}(\zeta_{11}^{\gamma})^2 + \frac{1}{2}d^2||\mathcal{H}||^2 - 2\sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta_{11}^{\gamma}\zeta_{jj}^{\gamma}). \end{aligned} \tag{3.10}$$

We derive from (3.9) and (3.10) that

$$\begin{aligned}\frac{1}{2}d^2||\mathcal{H}||^2 \geq Ric(e_a) &+ \sum_{\gamma=d+1}^{2p+1}(\zeta_{11}^{\gamma})^2 + d_2\frac{\Delta\sigma}{\sigma}\\ &+ \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}(\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma})^2\\ &+ \sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}(\zeta_{ij}^{\gamma})^2 - \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta_{11}^{\gamma}\zeta_{jj}^{\gamma}\\ &- f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2 + f_3(d_2 + 1).\end{aligned} \tag{3.11}$$

On simplification of (3.11), one can get

$$\begin{aligned}\frac{1}{4}d^2||\mathcal{H}||^2 &\geq Ric(e_a) + \sum_{\gamma=d+1}^{2p+1}(\zeta_{11}^{\gamma} - \frac{1}{2}(\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma}))^2\\ &+ d_2\frac{\Delta\sigma}{\sigma} - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta_1 + f_3(d_2 + 1)\\ &= Ric(e_a) + d_2\frac{\Delta\sigma}{\sigma}\\ &- f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2 + f_3(d_2 + 1),\end{aligned} \tag{3.12}$$

which proves the required inequality (3.1).

Case 2: If e_a is tangent to N_{θ_2}, then we need to fix a unit vector field from $\{e_{d_1+1}, \ldots, e_{2q} = e_d\}$, we fix e_a as unit vector field say $e_a = e_d$. Then from (3.4), we get

$$\begin{aligned}d^2||\mathcal{H}||^2 \geq Ric(e_a) &+ \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}(\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma})^2 + d_2\frac{\Delta\sigma}{\sigma}\\ &+ \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}\Big((\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma}) - 2\zeta_{dd}^{\gamma}\Big)^2\\ &+ \sum_{\gamma=d+1}^{2p+1}\sum_{1\leq\alpha<\beta\leq d_1}(\zeta_{\alpha\alpha}^{\gamma}\zeta_{\beta\beta}^{\gamma} - (\zeta_{\alpha\beta}^{\gamma})^2) + \sum_{\gamma=d+1}^{2p+1}\sum_{1\leq i<j\leq d}(\zeta_{ij}^{\gamma})^2\\ &+ \sum_{\gamma=d+1}^{2p+1}\sum_{d_1+1\leq s<t\leq d}(\zeta_{ss}^{\gamma}\zeta_{tt}^{\gamma} - (\zeta_{st}^{\gamma})^2) - \sum_{\gamma=d+1}^{2p+1}\sum_{1\leq i<j\leq d-1}\zeta_{ii}^{\gamma}\zeta_{jj}^{\gamma}\\ &- \Big[f_1\big(d(d-1)\big) + 3f_2\big((d_1-1) + d_2cos^2\theta\big) - 2f_3(d-1)\Big]\\ &+ \Big[\frac{1}{2}f_1\big((d-1)(d-2)\big) + \frac{3}{2}f_2\big((d_1-1) + (d_2-1)cos^2\theta\big) - f_3(d-2)\Big]\\ &+ \Big[\frac{1}{2}f_1\big(d_1(d_1-1)\big) + \frac{3}{2}f_2(d_1-1) - f_3(d_1-1)\Big]\\ &+ \Big[\frac{1}{2}f_1\big(d_2(d_2-1)\big) + \frac{3}{2}f_2d_2cos^2\theta\Big].\end{aligned} \tag{3.13}$$

Analogous to Case 1, we obtain

$$\begin{aligned}
d^2||\mathcal{H}||^2 \geq\ & Ric(e_a) + \frac{1}{2}d^2||\mathcal{H}||^2 + d_2\frac{\Delta\sigma}{\sigma} \\
&+ \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}\Big((\zeta^{\gamma}_{d_1+1d_1+1} + \cdots + \zeta^{\gamma}_{dd}) - 2\zeta^{\gamma}_{dd}\Big)^2 \\
&+ \sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}(\zeta^{\gamma}_{ij})^2 + \sum_{\gamma=d+1}^{2p+1}\sum_{t=d_1+1}^{d-1}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{tt} \\
&- \sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d-1}\zeta^{\gamma}_{ii}\zeta^{\gamma}_{jj} - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2+1).
\end{aligned} \tag{3.14}$$

Also, it is easy to see that

$$\sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d-1}\zeta^{\gamma}_{ii}\zeta^{\gamma}_{jj} = 0. \tag{3.15}$$

From equations (3.14) and (3.15), we get

$$\begin{aligned}
d^2||\mathcal{H}||^2 \geq\ & Ric(e_a) + \frac{1}{2}d^2||\mathcal{H}||^2 + d_2\frac{\Delta\sigma}{\sigma} \\
&+ \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}\Big((\zeta^{\gamma}_{d_1+1d_1+1} + \cdots + \zeta^{\gamma}_{dd}) - 2\zeta^{\gamma}_{dd}\Big)^2 \\
&+ \sum_{\gamma=d+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}(\zeta^{\gamma}_{ij})^2 + \sum_{\gamma=d+1}^{2p+1}\sum_{t=d_1+1}^{d-1}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{tt} \\
&- f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2+1).
\end{aligned} \tag{3.16}$$

Now consider

$$\begin{aligned}
&\frac{1}{2}\sum_{\gamma=d+1}^{2p+1}\Big((\zeta^{\gamma}_{d_1+1d_1+1} + \cdots + \zeta^{\gamma}_{dd}) - 2\zeta^{\gamma}_{dd}\Big)^2 + \sum_{\gamma=d+1}^{2p+1}\sum_{t=d_1+1}^{d-1}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{tt} \\
&\quad= \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}(\zeta^{\gamma}_{d_1+1d_1+1} + \cdots + \zeta^{\gamma}_{dd})^2 + 2\sum_{\gamma=d+1}^{2p+1}(\zeta^{\gamma}_{dd})^2 - \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{jj} \\
&\quad- \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d-1}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{jj} + \sum_{\gamma=d+1}^{2p+1}\sum_{t=d_1+1}^{d-1}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{tt} \\
&\quad= \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}(\zeta^{\gamma}_{d_1+1d_1+1} + \cdots + \zeta^{\gamma}_{dd})^2 + 2\sum_{\gamma=d+1}^{2p+1}(\zeta^{\gamma}_{dd})^2 \\
&\quad- \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{jj} - \sum_{\gamma=d+1}^{2p+1}(\zeta^{\gamma}_{dd})^2 \\
&\quad= \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}(\zeta^{\gamma}_{d_1+1d_1+1} + \cdots + \zeta^{\gamma}_{dd})^2 + \sum_{\gamma=d+1}^{2p+1}(\zeta^{\gamma}_{dd})^2 - \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta^{\gamma}_{dd}\zeta^{\gamma}_{jj}.
\end{aligned} \tag{3.17}$$

Further, with the help of (3.16) and (3.17), we conclude

$$\begin{aligned}\frac{1}{2}d^2||\mathcal{H}||^2 \geq\ & Ric(e_a) + \sum_{\gamma=d+1}^{2p+1}(\zeta_{dd}^{\gamma})^2 - \sum_{\gamma=d+1}^{2p+1}\sum_{j=d_1+1}^{d}\zeta_{dd}^{\gamma}\zeta_{jj}^{\gamma} \\ & + \frac{1}{2}\sum_{\gamma=d+1}^{2p+1}(\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma})^2 + d_2\frac{\Delta\sigma}{\sigma} \\ & + \sum_{\gamma=d+1}^{2(p-l)}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}(\zeta_{ij}^{\gamma})^2 + \sum_{\gamma=2(p-l)+1}^{2p+1}\sum_{i=1}^{d_1}\sum_{j=d_1+1}^{d}(\zeta_{ij}^{\gamma})^2 \\ & - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2 + 1). \end{aligned} \tag{3.18}$$

Applying the same approach as in Case 1's proof, equation (3.18) leads to

$$\begin{aligned}\frac{1}{4}d^2||\mathcal{H}||^2 \geq\ & Ric(e_a) + \sum_{\gamma=d+1}^{2p+1}(\zeta_{dd}^{\gamma} - \frac{1}{2}(\zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma}))^2 \\ & + d_2\frac{\Delta\sigma}{\sigma} - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2 + 1) \\ & = Ric(e_a) + d_2\frac{\Delta\sigma}{\sigma} \\ & - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2 + 1), \end{aligned} \tag{3.19}$$

which is the required inequality (3.2).

Now, we will verify the equality case of the inequalities. To begin, note that the relative null space, $\mathcal{N}_x$, of the submanifold M^d in the complex space form $\tilde{M}^m$ at a point $x \in M^d$ was defined in [13] as:

$$\mathcal{N}_x = \{X \in T_xM : \zeta(X, Y) = 0 \quad \forall \quad Y \in T_xM\}. \tag{3.20}$$

For $\circ \in \{1, \ldots, d\}$, a unit vector $e_\circ$ to M^d ar x satisfies the equality sign of (3.1) identically then the following three conditions hold

$$\begin{cases} \sum_{a=1}^{d_1}\sum_{A=d_1+1}^{d}(\zeta_{aA}^{\gamma})^2 = 0, \\ \sum_{\substack{j=1 \\ j\neq\circ}}^{d}(\zeta_{\circ j}^{\gamma})^2 = 0, \\ 2\zeta_{\circ\circ}^{\gamma} = \zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma}, \quad \gamma \in \{d+1, \ldots, 2p+1\} \end{cases} \tag{3.21}$$

and conversely. The first requirement in (3.21) leads to mixed totally geodesy, however the last two conditions, as well as the pointwise semi-slant warped product submanifolds, lead to the conclusion that $e_\circ$ is in the relative null space $\mathcal{N}_x$. This proves assertion since the converse is trivial (2).

For all unit tangent vectors to N_{θ_1} at x for a pointwise semi-slant warped product submanifold the equality sign of (3.1) holds then

$$\begin{cases} \sum_{a=1}^{d_1}\sum_{A=d_1+1}^{d}(\zeta_{aA}^{\gamma})^2 = 0, \\ \sum_{j=1}^{d}\sum_{\substack{j=1 \\ (j\neq a)}}^{d}(\zeta_{aj}^{\gamma})^2 = 0, \\ 2\zeta_{aa}^{\gamma} = \zeta_{d_1+1d_1+1}^{\gamma} + \cdots + \zeta_{dd}^{\gamma}, \quad a \in \{1, \ldots, d_1\} \end{cases} \tag{3.22}$$

and conversely.

The final requirement of the above condition means that

$$\zeta^{\gamma}_{aa} = 0, \quad \forall a \in \{1, \ldots, d_1\} \tag{3.23}$$

since M^d is a warped product bi-slant submanifold.

Moreover, it is easy to verify that M^d is D-totally geodesic pointwise semi-slant warped product submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$ using the second condition in (3.22), and (3.23), while the mixed totally geodesy derives from the first condition in (3.22), proving (a) in assertion (3).

The equality sign of (3.2) holds identically for all unit tangent vectors to N_θ at x for a pointwise semi-slant warped product submanifold then the following conditions are met

$$\begin{cases} \sum_{a=1}^{d_1} \sum_{A=d_1+1}^{d} (\zeta^{\gamma}_{aA})^2 = 0, \\ \sum_{j=1}^{d} \sum_{\substack{A=d_1+1 \\ (j \neq A)}}^{d} (\zeta^{\gamma}_{Aj})^2 = 0, \\ 2\zeta^{\gamma}_{AA} = \zeta^{\gamma}_{d_1+1 d_1+1} + \cdots + \zeta^{\gamma}_{dd}, \quad A \in \{d_1+1, \ldots, d\} \end{cases} \tag{3.24}$$

and conversely.

M^d is a mixed totally geodesic submanifold of $M^{2p+1}(f_1, f_2, f_3)$, according to the first condition in the preceding relation.

The third condition of the aforementioned relations offers two options:

$$\zeta^{\gamma}_{AA} = 0, \tag{3.25}$$

or, $dimN_\theta = 2$.

If (3.25) is true, M^d is a D_θ-totally geodesic warped product submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$, based on the second condition in (3.24). This is the first situation in part (b) of the theorem's statement (3).

In the other case, consider that M^d in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$ is not D_θ-totally geodesic warped product submanifold and $dimN_\theta = 2$.. As a result, we can conclude from the second condition of (3.24) that M^d is a D_θ-totally umbilical warped product submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$, it is the second scenario in this part. As a result, portion (b) of (3) is fully demonstrated.

To demonstrate (c), we first combine (3.22) and (3.23). As a result, we can make use of sections (a) and (b) of (3). Assume that $dimN_T \neq 2$ in the first instance of this section.

Since (a) of statement (3) implies that M^d is D-totally geodesic and (b) of statement (3) implies that M^d is D_θ-totally geodesic submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$. As a result, M^d is a totally geodesic submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$.

In the other case, let the first situation is not true. As a consequence, parts (a) and (b) immediately show that M^d is mixed totally geodesic and D-totally geodesic submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$ with $dimN_\theta = 2$.

To demonstrate that M^d is a totally umbilical submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$, it is sufficient to know that M^d is D_θ-totally umbilical warped product submanifold in $\tilde{M}^{2p+1}(f_1, f_2, f_3)$ from (b) and D-totally geodesic from (a), which leads to the claim of part (c). As a result, the theorem has been fully shown. □

4. Some applications of the result

In this section we discuss various applications of the main results.

4.1. Results on warped products pointwise semi-slant submanifolds related with compact N_T

From theory of integration we recall that if M is an orientable compact invariant submanifold, then for the volume element $d\mathcal{V}$ of M

$$\int_M \Delta\sigma d\mathcal{V} = 0, \tag{4.1}$$

Using this fact we arrive to the following result.

Theorem 4.1. *Let $M = N_T \times_\sigma N_\theta \to \tilde{M}(f_1, f_2, f_3)$ be an isometric immersion of an d-dimensional pointwise semi-slant warped products submanifold M in non-Sasakian generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ with compact N_T and $q \in N_\theta$. Then, the following inequalities exist for each unit vector $e_a \in T_xM$ orthogonal to ξ:*

(1) If e_a is tangent to N_T, then

$$\int_{N_T\times\{q\}} \Big(\frac{1}{4}d^2|\mathcal{H}||^2 - Ric(e_a)\Big)d\mathcal{V} \geq \Big[f_3(d_2+1) - f_1(d+d_1d_2-1) - \frac{3}{2}f_2\Big]vol(N_T). \tag{4.2}$$

(2) If e_a is tangent to N_θ, then

$$\int_{N_T\times\{q\}} \Big(\frac{1}{4}d^2|\mathcal{H}||^2 - Ric(e_a)\Big)d\mathcal{V} \geq \Big[f_3(d_2+1) - f_1(d+d_1d_2-1) - \frac{3}{2}f_2cos^2\theta\Big]vol(N_T). \tag{4.3}$$

where $vol(N_T)$ is the volume N_T.

Proof. For compact N_T, from (3.1), we have

$$\begin{aligned}\int_{N_T\times\{q\}} \frac{1}{4}d^2|\mathcal{H}||^2 d\mathcal{V} \geq &\int_{N_T\times\{q\}} Ric(e_a)\Big)d\mathcal{V} \\ &+ \int_{N_T\times\{q\}} d_2\frac{\Delta\sigma}{\sigma} \\ &+ \Big[f_3(d_2+1) - f_1(d+d_1d_2-1) - \frac{3}{2}f_2\Big]vol(N_T),\end{aligned} \tag{4.4}$$

for each $q \in N_\theta$.

Using Hopf's lemma and (2.12), we obtain

$$\begin{aligned}\int_{N_T\times\{q\}} \frac{1}{4}d^2|\mathcal{H}||^2 d\mathcal{V} \geq &\int_{N_T\times\{q\}} Ric(e_a)\Big)d\mathcal{V} - d_2\int_{N_T\times\{q\}} ||\Delta(ln\sigma)||^2 d\mathcal{V} \\ &+ \Big[f_3(d_2+1) - f_1(n+d_1d_2-1) - \frac{3}{2}f_2\Big]vol(N_T),\end{aligned} \tag{4.5}$$

which implies the required inequality (4.2). Similarly we find the inequality (4.3). □

4.2. Results on warped product poimtwise semi-slant submanifolds with harmonic function

Theorem 4.2. *Let $M = N_T \times_\sigma N_\theta \to \tilde{M}(f_1, f_2, f_3)$ be an isometric immersion of an d-dimensional pointwise semi-slant warped products submanifold M in non-Sasakian generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$. Then, if the warping functions σ is harmonic function, the following inequalities exist for each unit vector $e_a \in T_xM$ orthogonal to ξ:*

(1) If e_a is tangent to N_T, then

$$\frac{1}{4}d^2|\mathcal{H}|||^2 \geq Ric(e_a) - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2 + f_3(d_2 + 1). \tag{4.6}$$

(2) If e_a is tangent to N_θ, then

$$\frac{1}{4}d^2|\mathcal{H}|||^2 \geq Ric(e_a) - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2 + 1). \tag{4.7}$$

Proof. If σ_1 and σ_2 are harmonic functions, then $\Delta\sigma = 0$. Using this fact with (3.1) and (3.2) yields the results. □

4.3. Results on doubly warped product poimtwise bi-slant submanifolds related to Hessian functions

Let ϕ be a positive differentiable C^∞-differentiable function. Then the Hessian tensor of function ϕ is a symmetric 2-covariant tensor field on M^d defined by

$$\mathbb{H}^\phi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M) \tag{4.8}$$

such that

$$\mathbb{H}^\phi(X, Y) = \mathbb{H}^\phi_{ij}X^iY^j, \tag{4.9}$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathbb{H}^\phi_{ij}$ can be expressed as

$$\mathbb{H}^\phi_{ij} = \frac{\partial^2\phi}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial\phi}{\partial x_k}. \tag{4.10}$$

Let us assume that $\phi = ln\sigma$. Then as a consequence of the Theorem 3.1 and the above relation, we conclude the following result.

Theorem 4.3. *Let $M = N_T \times_\sigma N_\theta \to \tilde{M}(f_1, f_2, f_3)$ be an isometric immersion of an d-dimensional pointwise semi-slant warped products submanifold M in non-Sasakian generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$. Then, the following inequalities exist for each unit vector $e_a \in T_xM$ orthogonal to ξ:*

(1) If e_a is tangent to N_T, then

$$\frac{1}{4}d^2|\mathcal{H}|||^2 \geq Ric(e_a) + d_2\frac{trace\mathbb{H}^\phi}{\sigma} - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2 + f_3(d_2 + 1). \tag{4.11}$$

(2) If e_a is tangent to N_θ, then

$$\frac{1}{4}d^2|\mathcal{H}|||^2 \geq Ric(e_a) + d_2\frac{trace\mathbb{H}^\phi}{\sigma} - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta + f_3(d_2 + 1). \tag{4.12}$$

4.4. Results on warped product poimtwise semi-slant submanifolds related to Dirichlet energy functions

A great motivation of bound of Ricci curvature is to express the Dirichlet energy of the warping functions σ, which is a helpful instrument in physics. On a compact manifold M, the Dirichlet energy of any function ς is defined as:

$$E(\varsigma) = \frac{1}{2}\int_M ||\nabla\varsigma||^2 d\mathcal{V}, \tag{4.13}$$

where $d\mathcal{V}$ denotes the volume element and $\nabla\varsigma$ the gradient of ς.

Theorem 4.4. *Let $M = N_T \times_\sigma N_\theta \to \tilde{M}(f_1, f_2, f_3)$ be an isometric immersion of an d-dimensional pointwise semi-slant warped products submanifold M in non-Sasakian generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ with compact N_T and $q \in N_\theta$. Then, the following inequalities exist for each unit vector $e_a \in T_xM$ orthogonal to ξ:*

(1) If e_a is tangent to N_T, then

$$\begin{aligned} d_2E(ln\sigma) \geq& \frac{1}{2}\int_{N_T\times\{q\}} \Big(Ric(e_a) - \frac{1}{4}d^2|\mathcal{H}||^2\Big)d\mathcal{V} \\ &+ \frac{1}{2}\Big[f_3(d_2+1) - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2\Big]vol(N_T). \end{aligned} \tag{4.14}$$

(2) If e_a is tangent to N_θ, then

$$\begin{aligned} d_2E(ln\sigma) \geq& \frac{1}{2}\int_{N_T\times\{q\}} \Big(Ric(e_a) - \frac{1}{4}d^2|\mathcal{H}||^2\Big)d\mathcal{V} \\ &+ \frac{1}{2}\Big[f_3(d_2+1) - f_1(d + d_1d_2 - 1) - \frac{3}{2}f_2cos^2\theta\Big]vol(N_T). \end{aligned} \tag{4.15}$$

where $vol(N_T)$ is the volume N_T.

Proof. Making use of (4.13) into (4.5) we obtain the desired inequality (4.14). To obtain the inequality (4.15), first we integrate (3.2) over $N_T \times \{q\}$. Then making use of Hopf's lemma and (4.13) we get the required inequality (4.15). □

References

[1] P. Alegre, D.E. Blair and A. Carriazo, *Generalized Sasakian space forms*, Israel J. Math. **141**, 157–183, 2004.

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian-space-forms*, Differ. Geom. Appl. **26** (6), 656–666, 2008.

[3] P. Alegre and A. Carriazo, *Submanifolds of generalized Sasakian space forms*, Taiwan. J. Math. **13**, 923–941, 2009.

[4] P. Alegre and A. Carriazo, *Generalized Sasakian space forms and conformal changes of the metric*, Results Math. **59**, 485–493, 2011.

[5] R. Al-Ghefari, F.R. Al-Solamy and M.H. Shahid, *CR-submanifolds of generalized Sasakian space-forms*, JP J. Geom. Topol. **6**, 151–166, 2006.

[6] M. Aquib, M.N. Boyom, M.H. Shahid and G. Vîlcu, *The first fundamental equation and generalized wintgen-type inequalities for submanifolds in generalized space forms*, Mathematics, **7**, (2019) 1151. doi:10.3390/math7121151.

[7] M. Atçeken , C*ontact CR-warped product submanifolds in Sasakian space forms*, Hacet. J. Math. Stat. **44** (1), 23–32, 2015.

[8] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145**, 1–49, 1967.

[9] B.Y. Chen, *On isometric minimal immersions from warped products into real space forms*, Proc. Edinb. Math. Soc. **45**, 579–587, 2002.

[10] B.Y. Chen, *Geometry of warped products as Riemannian submanifolds and related problem*, Soochow J. Math. **28**, 125–157, 2002.

[11] B.Y. Chen, *CR-warped products in complex projective spaces with compact holomorphic factor*, Monatsh. Math. **141** (3), 177–186, 2004.

[12] B.Y. Chen, *Geometry of warped product submanifolds: a survey*, J. Adv. Math. Stud. **6** (2), 1–43, 2013.

[13] B.Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Scientific, 2011.

[14] A. Mustafa, A. De and S. Uddin, *Characterization of warped product submanifolds in Kenmotsu manifolds*, Balkan. J. Geom. Appl. **20**, 86–97, 2015.

[15] M. Falcitelli, *Locally conformal C_6-manifolds and generalized Sasakian-space-forms*, Mediterr. J. Math. **7**, 19–36, 2010.

[16] S.K. Hui, S. Uddin, A. Alkhaldi and P. Mandal, *Invariant submanifolds of generalized Sasakian-space-forms*, Int. J. Geom. Methods Mod. Phys. **15** (9), 2018.

[17] U.K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note Mat. **26**, 55–67, 2006.

[18] F. Malek and V. Nejadakbary, *A lower bound for the Ricci curvature of submanifolds in generalized Sasakian space forms*, Adv. Geom. **13**, 695–711, 2013.

[19] M.I. Munteanu, *Warped product contact CR-submanifolds of Sasakian space forms*, Publ. Math. Debrecen, **66**, 75–120, 2005.

[20] C. Ozgur and S. Guvenc, *On some classes of biharmonic Legendre curves in generalized Sasakian space forms*, Collect. Math. **65**, 203–218, 2014.

[21] S. Uddin, B.Y. Chen and F.R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math. **14** (2), (2017) 11.

[22] S. Uddin and F.R. Al-Solamy, *Warped product pseudo-slant immersions in Sasakian manifolds*, Publ.Math. Debrecen **91** ((3-4)), 331–348 2017.