Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (6) (2022), 1674 – 1679
DOI : 10.15672/hujms.1035373

Research Article

Uniformly continuous cosine families properties around weak demicompactness concept

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Abstract

In this paper, we use the concept of weak demicompactness in order to give some properties for the uniformly continuous cosine families. Our theoretical results will be illustrated by investigating the spectral inclusion for a uniformly continuous cosine family for an upper semi-Fredholm spectrum.

Mathematics Subject Classification (2020). 47D09, 47A53, 47A10

Keywords. cosine family, weakly demicompact operator, upper semi-Fredholm spectrum, spectral inclusion

1. Introduction

Throughout this work we will denote by X a Banach space. The set of all closed densely defined (resp. bounded) linear operators on X is denoted by $\mathcal{C}(X)$ (resp. $\mathcal{L}(X)$). Let $A \in \mathcal{C}(X)$ with domain $\mathcal{D}(A)$, kernel $N(A)$ and range $R(A)$ in X. If λ belongs to the resolvent set of A, denoted by $\rho(A)$, then $R(\lambda, A)$ denotes the resolvent operator $(\lambda I - A)^{-1}$. We say that A is upper semi-Fredholm if $R(A)$ is closed and $\dim N(A) < \infty$. We denote by $\Phi_+(X)$ the set of upper semi-Fredholm operators. The upper semi-Fredholm spectrum of A is defined by

$$\sigma_{uf}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_+(X)\}, \text{ (see [10]).}$$

Recall from [1] that a strongly continuous cosine family is a family$(C(t))_{t\in\mathbb{R}}$ of $\mathcal{L}(X)$ satisfying the following conditions:

(1) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$;
(2) $C(0) = I$;
(3) $C(t)x$ is continuous in t from $\mathbb{R}$ to X for each fixed $x \in X$.

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Received: 11.12.2021; Accepted: 16.06.2022

It then follows that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq Me^{\omega|t|}$ for all $t \in \mathbb{R}$. The generator $(A, \mathcal{D}(A))$ of $(C(t))_{t\in\mathbb{R}}$ is the linear operator in X defined by $Ax = \lim_{t\to 0} \frac{2}{t^2}(C(t)x - x)$, $x \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \Big\{x \in X \text{ such that } \lim_{t\to 0} \frac{2}{t^2}(C(t)x - x) \text{ exists}\Big\}.$$

The resolvent $R(\lambda^2, A)$ exists for $\lambda > \omega$. Recall that $(C(t))_{t\in\mathbb{R}}$ is a uniformly continuous cosine family if it is a strongly continuous cosine family such that $\lim_{t\to 0} \|C(t) - I\| = 0$ holds. Further, with the strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$, we associate the sine family $(S(t))_{t\in\mathbb{R}}$ which is the family of $\mathcal{L}(X)$ defined by $S(t)x = \int_0^t C(s)x\, ds,\ t \in \mathbb{R},\ x \in X$.

Many equations of mathematical physics can be cast in the abstract form:

$$u''(t) = Au(t), \quad u(0) = u_0, \quad u'(0) = u_1, \quad t \in \mathbb{R}, \quad (CP2)$$

on a Banach space X. Here A is a given linear operator with domain $\mathcal{D}(A)$. The problem $(CP2)$ is well-posed if and only if A generates a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ (see [1]). In order to understand the behavior of the solutions in terms of the data concerning A, one seeks information about the spectrum of $C(t)$ in terms of the spectrum of A. The problem of determining the spectrum, given by the following spectral inclusion

$$\cosh\left(t\sqrt{\sigma(A)}\right) \subseteq \sigma(C(t)), \quad t \in \mathbb{R},$$

has been widely studied for various reduced spectra, see [6, 7].

Let us recall from [8] that an operator $A \in \mathcal{C}(X)$ is said to be demicompact if, for every bounded sequence $\{x_n\}$ in $\mathcal{D}(A)$ such that $\{x_n - Ax_n\}$ converges to $x \in X$, there is a convergent subsequence of $\{x_n\}$. In [3], Jeribi used this class to obtain some results on Fredholm and spectral theories.

Recall from [5] that an operator $A \in \mathcal{C}(X)$ is said to be weakly demicompact if, for every bounded sequence $\{x_n\}$ in $\mathcal{D}(A)$ such that $\{x_n - Ax_n\}$ weakly converges in X, has a weakly convergent subsequence. We denote by $\mathcal{WDC}(X)$ the class of weakly demicompact operators. This class includes both the weakly compact and demicompact operators. Note that the sum of weakly demicompact operators is not necessarily weakly demicompact.

Remark 1.1. Let X be a Banach space and $A \in \mathcal{C}(X)$.
(i) If $A \in \mathcal{WDC}(X)$ and $B : X \longrightarrow X$ is weakly compact, then $A + B \in \mathcal{WDC}(X)$.
(ii) $I - A \in \mathcal{WDC}(X)$ if and only if $I + A \in \mathcal{WDC}(X)$.

In the present work, it is our intention to use the framework of the theory of cosine families to provide a systematic approach to the weak demicompactness criteria. In this paper, we introduce the notion of weakly demicompact of a cosine family operators (see Definition 2.1). After that, we discuss the relationship between the weak demicompactness of $I - A$, $C(t)$, $\lambda^2 R(\lambda^2, A)$ and $I - (tI - S(t))$ (see Theorem 2.3 and Corollary 2.6). Furthermore, we apply the obtained results to study the spectral inclusion for a uniformly continuous cosine family for an upper semi-Fredholm spectrum (see Corollary 2.10).

2. Main results

We start this section by the following definition.

Definition 2.1. A uniformly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ is said to be weakly demicompact if the operator $C(t)$ is weakly demicompact for any $t \in \mathbb{R}^*$.

Remark 2.2. It is worth noting that $C(t) \in \mathcal{WDC}(X)$ for every $t \in \mathbb{R}^*$ if and only if $C(t) \in \mathcal{WDC}(X)$ for every $t > 0$.

Let A be the generator of a uniformly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on a Banach space X and $\|C(t)\| \leq Me^{w|t|}$ for some constants $M \geq 1$, $w \in \mathbb{R}$. Let $(S(t))_{t\in\mathbb{R}}$ be the associated sine family of $(C(t))_{t\in\mathbb{R}}$.

Theorem 2.3. *The following statements are equivalent:*

(1) $I - A \in \mathcal{WDC}(X)$.
(2) *There exists* $\alpha > 0$ *such that* $C(t) \in \mathcal{WDC}(X)$ *for every* $0 < t < \alpha$.
(3) $\lambda^2 R(\lambda^2, A) \in \mathcal{WDC}(X)$ *for some (and then, for all)* $\lambda > \omega$.
(4) *There exists* $\beta > 0$ *such that* $I - (tI - S(t)) \in \mathcal{WDC}(X)$ *for every* $0 < t < \beta$.

Proof. (1) $\Longrightarrow$ (2) Let $\{x_n\}$ be a bounded sequence of X such that $x_n - C(t)x_n$ converges weakly to y. We have to show that $\{x_n\}$ contains a weakly convergent subsequence. Thanks to [9, Proposition 2.2], we have

$$(C(t) - I)x_n = A\int_0^t S(s)\ ds\ x_n.$$

Then $A\int_0^t S(s)\ ds\ x_n$ converges weakly to $-y$. Thus, the weak demicompactness of $I - A$ asserts that $\int_0^t S(s)\ ds\ x_n$ contains a weakly convergent subsequence $\int_0^t S(s)\ ds\ x_{n_k}$ which converges weakly to z. On the other hand, since $C(t)$ is uniformly continuous for $t \geq 0$. Hence, for every $\varepsilon > 0$ there is $\alpha > 0$ such that for all $0 < t < \alpha$ we have $\|C(t) - I\| < \varepsilon$. Then,

$$\left\|\frac{1}{t}S(t) - I\right\| = \left\|\frac{1}{t}\int_0^t (C(s) - I)\ \ ds\right\| \leq \frac{1}{t}\int_0^t \left\|C(s) - I\right\|\ ds < \varepsilon.$$

Accordingly, for all $0 < t < \alpha$, we have

$$\left\|\frac{2}{t^2}\int_0^t S(s)\ ds - I\right\| = \left\|\frac{2}{t^2}\int_0^t s\left(\frac{1}{s}S(s) - I\right)\ ds\right\| \leq \frac{2}{t^2}\int_0^t s\left\|\frac{1}{s}S(s) - I\right\|\ ds < \varepsilon.$$

Thus, for $\varepsilon = 1$, we obtain for every $0 < t < \alpha$, $\int_0^t S(s)\ ds$ is invertible. Hence x_{n_k} converges weakly to $\left(\int_0^t S(s)ds\right)^{-1} z$ and so $C(t) \in \mathcal{WDC}(X)$ for all $0 < t < \alpha$.

(2) $\Longrightarrow$ (3) Suppose that there exists $\alpha > 0$ such that $C(t)$ is weakly demicompact for every $0 < t < \alpha$. Consider $t_0 \in]0, \alpha[$ such that $\int_0^{t_0} S(\tau)\ d\tau$ is invertible. Let $\{x_n\}$ be a bounded sequence of X such that $x_n - \lambda^2 R(\lambda^2, A)x_n$ converges weakly to y for some $\lambda > \omega$. Since we can write

$$\begin{aligned} x_n - \lambda^2 R(\lambda^2, A)x_n &= -R(\lambda^2, A)Ax_n \\ &= -R(\lambda^2, A)(C(t_0) - I)\Big(\int_0^{t_0} S(\tau)\ d\tau\Big)^{-1} x_n. \end{aligned}$$

This implies that $(I - C(t_0))\Big(\int_0^{t_0} S(\tau)\ d\tau\Big)^{-1} x_n$ converges weakly to $(\lambda^2 - A)y$. From the weak demicompactness of $C(t_0)$, it follows that $\{\Big(\int_0^{t_0} S(\tau)\ d\tau\Big)^{-1} x_n\}$ contains a weakly convergent subsequence $\{\Big(\int_0^{t_0} S(\tau)\ d\tau\Big)^{-1} x_{n_k}\}$ which converges weakly to z. Therefore, x_{n_k} converges weakly to $\Big(\int_0^{t_0} S(\tau)\ d\tau\Big)z$.

(3) $\Longrightarrow$ (4) Let $\{x_n\}$ be a bounded sequence of X such that $tx_n - S(t)x_n$ converges weakly to y. Since we can write

$$(S(t) - tI)x_n = \int_0^t (C(s) - I)\ ds\ x_n = A \int_0^t \int_0^s S(\tau)\ d\tau\ ds\ x_n,$$

then for some $\lambda > \omega$, we have

$$\begin{aligned} R(\lambda^2, A)(S(t) - tI)x_n &= R(\lambda^2, A)A(\int_0^t \int_0^s S(\tau)\ d\tau\ ds\ )x_n \\ &= (\lambda^2 R(\lambda^2, A) - I) \int_0^t \int_0^s S(\tau)\ d\tau\ ds\ x_n. \end{aligned}$$

This implies that $(\lambda^2 R(\lambda^2, A) - I) \int_0^t \int_0^s S(\tau)\ d\tau\ ds\ x_n$ converges weakly to $-R(\lambda^2, A)y$. Setting $y_n = \int_0^t \int_0^s S(\tau)\ d\tau\ ds\ x_n$. The fact that $\lambda^2 R(\lambda^2, A)$ is weakly demicompact then $\{y_n\}$ has a weakly convergent subsequence $\{y_{n_k}\}$ which converges weakly to z. On the other hand, there exists a constant $\alpha > 0$ such that for every $0 < t < \alpha$ we have $\left\| \frac{2}{t^2} \int_0^t S(\tau)\ d\tau - I \right\| < \varepsilon$. Accordingly, for all $0 < t < \alpha$, we have

$$\begin{aligned} \left\| \frac{6}{t^3} \int_0^t \int_0^s S(\tau)\ d\tau\ ds - I \right\| &= \left\| \frac{6}{t^3} \left[\int_0^t \int_0^s S(\tau)\ d\tau\ ds - \frac{t^3}{6} \right] \right\| \\ &= \left\| \frac{6}{t^3} \left[\int_0^t \left(\int_0^s S(\tau)\ d\tau - \frac{s^2}{2} \right)\ ds \right] \right\| \\ &\leq \frac{6}{t^3} \left[\int_0^t \frac{s^2}{2} \left\| \frac{2}{s^2} \int_0^s S(\tau)\ d\tau - I \right\|\ ds \right] \\ &< \varepsilon. \end{aligned}$$

Hence, for ε small enough, $\int_0^t \int_0^s S(\tau)\ d\tau\ ds$ is invertible for all $0 < t < \alpha$. Taking $\beta = \alpha$. Thus, we have x_{n_k} converges weakly to $\left(\int_0^t \int_0^s S(\tau)\ d\tau\ ds \right)^{-1} z$ and so $I - (tI - S(t))$ is weakly demicompact for all $0 < t < \beta$.

(4) $\Longrightarrow$ (1) Let $\{x_n\}$ be a bounded sequence of X such that Ax_n converges weakly to y. In view of [1, Theorem 3.1], we get

$$(S(t) - tI)x_n = \int_0^t (C(s) - I)\ ds\ x_n = \int_0^t \int_0^s (s - \tau)C(\tau)\ d\tau\ ds\ Ax_n,$$

then $tx_n - S(t)x_n$ converges weakly to $-\int_0^t \int_0^s (s - \tau)C(\tau)\ d\tau\ ds\ y$. Thus, the weak demicompactness of $I - (tI - S(t))$ implies that $\{x_n\}$ has a weakly convergent subsequence. Hence, $I - A$ is weakly demicompact. □

The following result is devoted to investigate the weak demicompactness properties of a uniformly continuous cosine family on intervals.

Proposition 2.4. *Let $\alpha' > 0$. Suppose that for every $t \geq \alpha'$ there exists $t_0 \in]0, \alpha'[$ such that $I - C(t_0)C(t)$ is weakly compact. If $C(t) \in \mathcal{WDC}(X)$ for every $0 < t < \alpha'$, then $C(t) \in \mathcal{WDC}(X)$ for every $t > 0$.*

Proof. Let $t \geq \alpha'$ and $\{x_n\}$ be a bounded sequence in X such that $x_n - C(t)x_n$ converges weakly to y. As $t \geq \alpha'$ then, by hypothesis, there exists $t_0 \in]0, \alpha'[$ such that $I - C(t_0)C(t)$ is weakly compact. Hence, $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\{(I-C(t_0)C(t))x_{n_k}\}$ converges weakly to z. Hence, from the relation

$$\begin{array}{rcl} x_{n_k} - C(t_0)x_{n_k} & = & x_{n_k} - C(t_0)C(t)x_{n_k} + C(t_0)C(t)x_{n_k} - C(t_0)x_{n_k} \\ & = & (I - C(t_0)C(t))x_{n_k} - C(t_0)(I - C(t))x_{n_k}, \end{array}$$

it follows that $x_{n_k} - C(t_0)x_{n_k}$ converges weakly to $z - C(t_0)y$. The fact that $C(t_0)$ is weakly demicompact, we obtain that $\{x_{n_k}\}$ contains a weakly convergent subsequence. Hence, $C(t)$ is weakly demicompact for every $t \geq \alpha'$. □

Remark 2.5. Note that in the assumption of Proposition 2.4, t and t_0 are separated. However, if t tends to t_0 then $I - C^2(t_0)$ is weakly compact. Hence from the relation $I - C(2t_0) = 2(I - C^2(t_0))$ and in view of Remark 1.1, It follows that $I = I - C(2t_0) + C(2t_0)$ is weakly demicompact which is impossible, except if the dimension of X is finite.

As a consequence of Theorem 2.3 and Proposition 2.4, we obtain:

Corollary 2.6. *Assume that we have the same hypotheses as Proposition 2.4. Then, the following statements are equivalent:*

(1) $I - A \in \mathcal{WDC}(X)$.
(2) *There exists* $\alpha > 0$ *such that* $C(t) \in \mathcal{WDC}(X)$ *for every* $0 < t < \alpha$.
(3) $C(t) \in \mathcal{WDC}(X)$ *for every* $t > 0$.
(4) $\lambda^2 R(\lambda^2, A) \in \mathcal{WDC}(X)$ *for some (and then, for all)* $\lambda > \omega$.
(5) *There exists* $\beta > 0$ *such that* $I - (tI - S(t)) \in \mathcal{WDC}(X)$ *for every* $0 < t < \beta$.

To prove our next theorem, we need to recall the following lemma:

Lemma 2.7. [7, Lemma 4] *Let* A *be the generator of the uniformly continuous cosine family* $(C(t))_{t\in\mathbb{R}}$. *For* $t \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$, *let*

$$S_\lambda(t)x = \int_0^t \sinh(\lambda(t-s))C(s)x\, ds, \; x \in X.$$

Then $S_\lambda(t) \in \mathcal{L}(X)$ *is an operator that commutes with* A, *and*

$$(A - \lambda^2)S_\lambda(t)x = \lambda(C(t) - \cosh(\lambda t))x \text{ for all } x \in X.$$

Theorem 2.8. *Let* X *be a Banach space and* $(C(t))_{t\in\mathbb{R}}$ *be a uniformly continuous cosine family on* X *with infinitesimal generator* A. *If* $I - (\cosh(\lambda t) - C(t)) \in \mathcal{WDC}(X)$ *for all* $t \neq 0$, *then* $I - (\lambda^2 - A) \in \mathcal{WDC}(X)$.

Proof. Let $t \neq 0$ and suppose that $I - (\cosh(\lambda t) - C(t))$ is weakly demicompact.
Case 1: If $\lambda = 0$, then $I - A$ is weakly demicompact (see Theorem 2.3). Thus, following Remark 1.1, we obtain that $I + A$ is weakly demicompact.
Case 2: Let $\lambda \neq 0$ and $\{x_n\}$ be a bounded sequence in X such that $(\lambda^2 - A)x_n$ converges weakly to y. So, by using Lemma 2.7, we infer that $(\cosh(\lambda t) - C(t))x_n$ converges to $\frac{1}{\lambda}S_\lambda(t)y$. Since $I - (\cosh(\lambda t) - C(t)) \in \mathcal{WDC}(X)$, we obtain that $\{x_n\}$ contains a weakly convergent subsequence. Hence, $I - (\lambda^2 - A) \in \mathcal{WDC}(X)$. □

As a consequence, we can apply the result obtained in Theorem 2.8 to investigate the spectral inclusion for a uniformly continuous cosine family for an upper semi-Fredholm spectrum. First, we need to present the following theorem:

Theorem 2.9. [5, Corollary 2.3.1] *Let* X *be a Banach space and* $A \in \mathcal{L}(X)$. *Assume for some* $k \in \{1, 2, ...\}$ *that* $A^k \in \mathcal{WDC}(X)$ *and* A^k *is a Dunford-Pettis operator. Then,* $I - A$ *is an upper semi-Fredholm operator.*

Corollary 2.10. *Let X be a Banach space and $(C(t))_{t\in\mathbb{R}}$ be a uniformly continuous cosine family on X with infinitesimal generator A. Suppose that $I-(\lambda^2-A)$ is a Dunford-Pettis operator for every λ. Then, for all $t\neq 0$,*

$$\cosh\left(t\sqrt{\sigma_{uf}(A)}\right)\subseteq \sigma_{uf}(C(t)).$$

Proof. Let $t\neq 0$ and assume that $\cosh(\lambda t)-C(t)\in\Phi_+(X)$. By [4, Theorem 2.1], we obtain that $I-(\cosh(\lambda t)-C(t))$ is demicompact. From [2, Lemma 3.1], it is clear that $I-(\cosh(\lambda t)-C(t))\in\mathcal{WDC}(X)$. Thus, following Theorem 2.8, we infer that $I-(\lambda^2-A)\in\mathcal{WDC}(X)$. Since $I-(\lambda^2-A)$ is a Dunford-Pettis operator and in view of Theorem 2.9, we obtain $\lambda^2-A\in\Phi_+(X)$. ☐

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