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Analysis of a System of Nonlinear Hadamard Type Fractional Boundary Value Problems

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Abstract

The aim of this work is to analyze the existence of positive solutions for a coupled system of Hadamard type fractional boundary value problems. By using the five functional fixed point theorem, the conditions for the existence of positive solutions are derived. Finally, to show the applicability of the main result, an illustrative example is also involved.

1. Introduction

Fractional calculus and fractional differential equations have recently gained significance due to the expansion of the application fields against real world problems in the areas of applied mathematics, engineering, physics, system control, etc. One reason for such interest is that fractional differential equations can explain more precise results with respect to integer order models, see [1]-[5]. Moreover, a lot of scientists have been studying on the existence results of positive solutions for fractional boundary value problems and the systems of fractional differential equations by means of methods of nonlinear analysis. The importance of the area of coupled systems of fractional order differential equations comes from that they can be observed in a large number of problems of applied nature. For details and examples on the topic, see [6]-[15] and the references therein.

Other than the commonly mentioned Riemann-Liouville and Caputo fractional differential equations, there is a gap in investigation of Hadamard fractional differential equations and coupled systems under different boundary conditions on an bounded/unbounded domain. One of the main speciality of Hadamard fractional derivative is that the definition contains logarithmic function of arbitrary exponent. For some recent results on boundary value problems of Hadamard fractional differential equations and coupled systems, we refer to [16]-[33].

In [23], Zhai and Wang investigated the following coupled Hadamard type fractional boundary value problems:

$$\begin{cases} {}^{H}D^{p}u(t)+f(t,v(t))=a, \quad 1<p\le 2, \quad t\in(1,e), \\ {}^{H}D^{q}v(t)+g(t,u(t))=b, \quad 1<q\le 2, \quad t\in(1,e), \\ u(1)=0, \quad {}^{H}D^{p-1}u(e)=\displaystyle\sum_{i=1}^{m}\mu_i {}^{H}I^{\alpha_i}v(\eta), \\ v(1)=0, \quad {}^{H}D^{q-1}v(e)=\displaystyle\sum_{j=1}^{n}\sigma_j {}^{H}I^{\beta_j}u(\xi), \end{cases}$$

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where ${}^{H}D$ denotes the Hadamard-type fractional derivative; ${}^{H}I$ is the Hadamard-type fractional integral. By the use of increasing $\varphi-(h,r)$ concave operators, the authors obtained the existence and uniqueness of solutions for Hadamard fractional differential systems.

Motivated particularly by the above mentioned papers, we are interested in investigating a coupled system of Hadamard fractional differential equations, which include both integral boundary conditions and m-point fractional integral boundary conditions:

$$\begin{cases} {}^{H}D_{1^+}^{\vartheta_1}u(t)+f_1(t,u(t),v(t))=0, \quad n-1<\vartheta_1\le n, \quad t\in(1,e),\\ {}^{H}D_{1^+}^{\vartheta_2}v(t)+f_2(t,u(t),v(t))=0, \quad m-1<\vartheta_2\le m, \quad t\in(1,e),\\ u(1)=u'(1)=...=u^{(n-2)}(1)=0, \qquad {}^{H}D_{1^+}^{\vartheta_1-1}u(e)=\int_1^e g_1(t)u(t)\frac{dt}{t}+\sum_{i=1}^{p}\lambda_i^{H}I_{1^+}^{\beta_i}u(\sigma_1^*),\\ v(1)=v'(1)=...=v^{(m-2)}(1)=0, \qquad {}^{H}D_{1^+}^{\vartheta_2-1}v(e)=\int_1^e g_2(t)v(t)\frac{dt}{t}+\sum_{j=1}^{q}\sigma_j^{H}I_{1^+}^{\alpha_j}v(\sigma_2^*), \end{cases} \tag{1.1}$$

where $n,m\in\mathbb{N}$, $n,m\ge 3$, ${}^{H}D_{1^+}^{\vartheta_1}$ and ${}^{H}D_{1^+}^{\vartheta_2}$ are the Hadamard-type fractional derivatives of order ϑ_1, ϑ_2, respectively. ${}^{H}I_{1^+}^{\beta_i}$ and ${}^{H}I_{1^+}^{\alpha_j}$ are the Hadamard-type fractional integrals of order $\beta_i>0$ $(i=1,2,...,p)$, $\alpha_j>0$ $(j=1,2,...,q)$, $f_1,f_2\in C([1,e]\times[0,\infty)\times[0,\infty),[0,\infty))$, $g_1,g_2\in C([1,e],(0,\infty))$ and $\lambda_i\ge 0$ $(i=1,2,...,p)$, $\sigma_j\ge 0$ $(j=1,2,...,q)$, $\sigma_1^*,\sigma_2^*\in(1,e)$ are given constants.

We deal with the analysis of existence result of positive solutions for Hadamard differential systems. We accentuate that there are a lot of studies on Riemann-Liouville or Caputo type fractional differential systems. To the best authors' knowledge, there are a little number of papers which are studied on the systems of nonlinear Hadamard fractional differential equations. Here, unlike other papers, we attempt to study new Hadamard differential systems which consist of both integral boundary conditions and m-point fractional integral boundary conditions on an bounded domain.

We prepare the following sections of this paper as follows: Section 2 includes some preliminaries. We also summarize all properties of the corresponding Green's function. We indicate the existence of positive solutions of the problem and an example illustrating our result is ensured in Section 3.

2. Preliminaries

In this section, basic concepts, notations and some lemmas about the Hadamard-type fractional calculus are demonstrated for the convenience of the readers.

Definition 2.1 ([4]). The Hadamard fractional derivative of fractional order $\nu>0$ for a function $k:[1,\infty)\to\mathbb{R}$ is defined as

$${}^{H}D_{1^+}^{\nu}k(t)=\frac{1}{\Gamma(n-\nu)}\Big(t\frac{d}{dt}\Big)^n\int_1^t\Big(\log\frac{t}{s}\Big)^{n-\nu-1}k(s)\frac{ds}{s}, \quad n-1<\nu<n,\ n=[\nu]+1,$$

where $[\nu]$ denotes the integer part of the real number ν and $\log(\cdot)=\log_e(\cdot)$.

Definition 2.2 ([4]). The Hadamard fractional integral of order $\nu>0$ for a function $k:[1,\infty)\to\mathbb{R}$ is defined as

$${}^{H}I_{1^+}^{\nu}k(t)=\frac{1}{\Gamma(\nu)}\int_1^t(\log\frac{t}{s})^{\nu-1}k(s)\frac{ds}{s}, \quad \nu>0,$$

provided the integral exists.

Lemma 2.3 ([4]). If $a,\nu,\mu>0$, then

$$({}^{H}I_a^{\nu}(\log\frac{t}{a})^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}(\log\frac{x}{a})^{\mu+\nu-1}, \qquad ({}^{H}D_a^{\nu}(\log\frac{t}{a})^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}(\log\frac{x}{a})^{\mu-\nu-1}.$$

in particular, $({}^{H}D_a^{\nu}(\log\frac{t}{a})^{\nu-j})(x)=0, j=1,2,...,[\nu]+1.$

Lemma 2.4 ([4]). Let $\nu>0$. Assume that $c\in C[1,\infty)\cap L^1[1,\infty)$, then the solution of Hadamard-type fractional differential equation ${}^{H}D_{1^+}^{\nu}c(t)=0$ can be denoted as

$$c(t)=\sum_{i=1}^{m}c_i(\log t)^{\nu-i},$$

and the following formula holds:

$$ {}^{H}I^{\nu}_{1^+}\,{}^{H}D^{\nu}_{1^+}c(t) = c(t) + \sum_{i=1}^{m} c_i(\log t)^{\nu-i}, $$

where $c_i \in \mathbb{R}, i = 1,2,...,n, n-1 < \nu < n, n = [\nu]+1$.

Lemma 2.5. *If $x, y \in C[1,e]$, then, for the functions $u, v \in C[1,e]$, the following system*

$$ \begin{cases} {}^{H}D^{\vartheta_1}_{1^+}u(t) + x(t) = 0, \quad n-1 < \vartheta_1 \le n, \quad t \in (1,e), \\ {}^{H}D^{\vartheta_2}_{1^+}v(t) + y(t) = 0, \quad m-1 < \vartheta_2 \le m, \quad t \in (1,e), \\ u(1) = u'(1) = ... = u^{(n-2)}(1) = 0, \qquad {}^{H}D^{\vartheta_1-1}_{1^+}u(e) = \int_1^e g_1(t)u(t)\frac{dt}{t} + \sum_{i=1}^{p} \lambda_i^{H} I^{\beta_i}_{1^+}u(\sigma_1^*), \\ v(1) = v'(1) = ... = v^{(m-2)}(1) = 0, \qquad {}^{H}D^{\vartheta_2-1}_{1^+}v(e) = \int_1^e g_2(t)v(t)\frac{dt}{t} + \sum_{j=1}^{q} \sigma_j^{H} I^{\alpha_j}_{1^+}v(\sigma_2^*), \end{cases} \tag{2.1} $$

can be given in the integral representations of the form

$$ u(t) = \int_1^e H_1(t,s)x(s)\frac{ds}{s}, $$

$$ v(t) = \int_1^e H_2(t,s)y(s)\frac{ds}{s}, $$

where

$$ H_1(t,s) = G_1(t,s) + G_2(t,s), \tag{2.2} $$

$$ H_2(t,s) = G_3(t,s) + G_4(t,s), \tag{2.3} $$

and

$$ G_1(t,s) = g_1(t,s) + \sum_{i=1}^{p} \frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)} g_1^{\beta_i}(\sigma_1^*, s), $$

$$ G_2(t,s) = \frac{(\log t)^{\vartheta_1-1}}{\Upsilon_1}\int_1^e G_1(t,s)g_1(t)\frac{dt}{t}, $$

$$ G_3(t,s) = g_2(t,s) + \sum_{j=1}^{q} \frac{\sigma_j(\log t)^{\vartheta_2-1}}{\Upsilon^*\Gamma(\vartheta_2+\alpha_j)} g_2^{\alpha_j}(\sigma_2^*, s), $$

$$ G_4(t,s) = \frac{(\log t)^{\vartheta_2-1}}{\Upsilon_1^*}\int_1^e G_3(t,s)g_2(t)\frac{dt}{t}, $$

with

$$ g_1(t,s) = \frac{1}{\Gamma(\vartheta_1)} \begin{cases} (\log t)^{\vartheta_1-1} - (\log \frac{t}{s})^{\vartheta_1-1}, & 1 \le s \le t \le e, \\ (\log t)^{\vartheta_1-1}, & 1 \le t \le s \le e, \end{cases} \tag{2.4} $$

$$ g_2(t,s) = \frac{1}{\Gamma(\vartheta_2)} \begin{cases} (\log t)^{\vartheta_2-1} - (\log \frac{t}{s})^{\vartheta_2-1}, & 1 \le s \le t \le e, \\ (\log t)^{\vartheta_2-1}, & 1 \le t \le s \le e, \end{cases} \tag{2.5} $$

$$ g_1^{\beta_i}(\sigma_1^*, s) = \begin{cases} (\log \sigma_1^*)^{\vartheta_1+\beta_i-1} - (\log \frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}, & 1 \le s \le \sigma_1^* \le e, \\ (\log \sigma_1^*)^{\vartheta_1+\beta_i-1}, & 1 \le \sigma_1^* \le s \le e, \end{cases} $$

$$ g_2^{\alpha_j}(\sigma_2^*, s) = \begin{cases} (\log \sigma_2^*)^{\vartheta_2+\alpha_j-1} - (\log \frac{\sigma_2^*}{s})^{\vartheta_2+\alpha_j-1}, & 1 \le s \le \sigma_2^* \le e, \\ (\log \sigma_2^*)^{\vartheta_2+\alpha_j-1}, & 1 \le \sigma_2^* \le s \le e, \end{cases} $$

where $\Upsilon = \Gamma(\vartheta_1) - \sum_{i=1}^{p} \frac{\lambda_i \Gamma(\vartheta_1)}{\Gamma(\vartheta_1+\beta_i)} (\log \sigma_1^*)^{\vartheta_1+\beta_i-1}$ *and* $\Upsilon^* = \Gamma(\vartheta_2) - \sum_{j=1}^{q} \frac{\sigma_j \Gamma(\vartheta_2)}{\Gamma(\vartheta_2+\alpha_j)} (\log \sigma_2^*)^{\vartheta_2+\alpha_j-1}$

$$\Upsilon_1 = \Upsilon - \int_1^e g_1(t)(\log t)^{\vartheta_1-1} \frac{dt}{t} > 0,$$

and

$$\Upsilon_1^* = \Upsilon^* - \int_1^e g_2(t)(\log t)^{\vartheta_2-1} \frac{dt}{t} > 0.$$

Proof. Using Lemma (2.4), the above system (2.1) can be given by

$$u(t) = -\frac{1}{\Gamma(\vartheta_1)} \int_1^t (\log \frac{t}{s})^{\vartheta_1-1} x(s) \frac{ds}{s} + c_1(\log t)^{\vartheta_1-1} + c_2(\log t)^{\vartheta_1-2} + ... + c_n(\log t)^{\vartheta_1-n},$$

$$v(t) = -\frac{1}{\Gamma(\vartheta_2)} \int_1^t (\log \frac{t}{s})^{\vartheta_2-1} y(s) \frac{ds}{s} + d_1(\log t)^{\vartheta_2-1} + d_2(\log t)^{\vartheta_2-2} + ... + d_m(\log t)^{\vartheta_2-m},$$

where $c_i, d_j \in \mathbb{R}$, $i = 1,..,n$ and $j = 1,..,m$. Using the boundary conditions, we derive $c_2 = c_3 = ... = c_n = 0$ and $d_2 = d_3 = ... = d_m = 0$. Then,

$$u(t) = -\frac{1}{\Gamma(\vartheta_1)} \int_1^t (\log \frac{t}{s})^{\vartheta_1-1} x(s) \frac{ds}{s} + c_1(\log t)^{\vartheta_1-1}. \tag{2.6}$$

$$v(t) = -\frac{1}{\Gamma(\vartheta_2)} \int_1^t (\log \frac{t}{s})^{\vartheta_2-1} y(s) \frac{ds}{s} + d_1(\log t)^{\vartheta_2-1}. \tag{2.7}$$

By using Lemma (2.3)

$$^H D_{1^+}^{\vartheta_1-1} u(t) = c_1 \Gamma(\vartheta_1) - \int_1^t x(s) \frac{ds}{s},$$

$$^H D_{1^+}^{\vartheta_2-1} v(t) = d_1 \Gamma(\vartheta_2) - \int_1^t y(s) \frac{ds}{s}.$$

Using $^H D_{1^+}^{\vartheta_1-1} u(e) = \int_1^e g_1(t) u(t) \frac{dt}{t} + \sum_{i=1}^{p} \lambda_i^H I_{1^+}^{\beta_i} u(\sigma_1^*)$, and $^H D_{1^+}^{\vartheta_2-1} v(e) = \int_1^e g_2(t) v(t) \frac{dt}{t} + \sum_{j=1}^{q} \sigma_j^H I_{1^+}^{\alpha_j} v(\sigma_2^*)$, we have

$$c_1 = \frac{1}{\Upsilon} (\int_1^e g_1(t) u(t) \frac{dt}{t} + \int_1^e x(s) \frac{ds}{s} - \sum_{i=1}^{p} \frac{\lambda_i}{\Gamma(\vartheta_1 + \beta_i)} \int_1^{\sigma_1^*} (\log \frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1} x(s) \frac{ds}{s}). \tag{2.8}$$

$$d_1 = \frac{1}{\Upsilon^*} (\int_1^e g_2(t) v(t) \frac{dt}{t} + \int_1^e y(s) \frac{ds}{s} - \sum_{j=1}^{q} \frac{\sigma_j}{\Gamma(\vartheta_2 + \alpha_j)} \int_1^{\sigma_2^*} (\log \frac{\sigma_2^*}{s})^{\vartheta_2+\alpha_j-1} y(s) \frac{ds}{s}). \tag{2.9}$$

Substituting (2.8) into (2.6), we get

$$\begin{aligned}
u(t) =& \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t} - \frac{1}{\Gamma(\vartheta_1)}\int_1^t (\log\frac{t}{s})^{\vartheta_1-1}x(s)\frac{ds}{s}\\
&-\sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
=& \frac{(\log t)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \frac{(\Gamma(\vartheta_1)-\Upsilon)(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t}\\
&- \frac{1}{\Gamma(\vartheta_1)}\int_1^t (\log\frac{t}{s})^{\vartheta_1-1}x(s)\frac{ds}{s} - \sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
=& \frac{(\log t)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^e (\log\sigma_1^*)^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
&+ \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t} - \frac{1}{\Gamma(\vartheta_1)}\int_1^t (\log\frac{t}{s})^{\vartheta_1-1}x(s)\frac{ds}{s}\\
&- \sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
=& \int_1^e g_1(t,s)x(s)\frac{ds}{s} + \sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^e g_1^{\beta_i}(\sigma_1^*,s)x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t}\\
=& \int_1^e G_1(t,s)x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t}.
\end{aligned}$$

Similarly, substituting (2.9) into (2.7), we get

$$v(t) = \int_1^e G_2(t,s)y(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_2-1}}{\Upsilon^*}\int_1^e g_2(t)v(t)\frac{dt}{t}.$$

Furthermore,

$$\begin{aligned}
\int_1^e g_1(t)u(t)\frac{dt}{t} =& \int_1^e g_1(t)\int_1^e G_1(t,s)x(s)\frac{ds}{s}\frac{dt}{t}\\
&+\frac{1}{\Upsilon}\int_1^e g_1(t)(\log t)^{\vartheta_1-1}\frac{dt}{t}\int_1^e g_1(t)u(t)\frac{dt}{t},
\end{aligned}$$

and

$$\begin{aligned}
\int_1^e g_2(t)v(t)\frac{dt}{t} =& \int_1^e g_2(t)\int_1^e G_3(t,s)y(s)\frac{ds}{s}\frac{dt}{t}\\
&+\frac{1}{\Upsilon^*}\int_1^e g_2(t)(\log t)^{\vartheta_2-1}\frac{dt}{t}\int_1^e g_2(t)v(t)\frac{dt}{t},
\end{aligned}$$

which provide

$$\int_1^e g_1(t)u(t)\frac{dt}{t} = \frac{\Upsilon}{\Upsilon_1}\int_1^e g_1(t)\int_1^e G_1(t,s)x(s)\frac{ds}{s}\frac{dt}{t},$$

$$\int_1^e g_2(t)v(t)\frac{dt}{t} = \frac{\Upsilon^*}{\Upsilon_1^*}\int_1^e g_2(t)\int_1^e G_3(t,s)y(s)\frac{ds}{s}\frac{dt}{t}.$$

Then,

$$\begin{aligned}
u(t) =& \int_1^e G_1(t,s)x(s)\frac{ds}{s} + \int_1^e G_2(t,s)x(s)\frac{ds}{s}\\
=& \int_1^e H_1(t,s)x(s)\frac{ds}{s},
\end{aligned}$$

$$\begin{aligned}
v(t) =& \int_1^e G_3(t,s)y(s)\frac{ds}{s} + \int_1^e G_4(t,s)y(s)\frac{ds}{s}\\
=& \int_1^e H_2(t,s)y(s)\frac{ds}{s}.
\end{aligned}$$

The proof is completed. □

Lemma 2.6. *The functions $g_i(t,s)$, (i=1,2) given by (2.4) and (2.5) satisfy*

(i) $g_i(t,s)$ are continuous functions and $g_i(t,s) \geq 0$ for any $t,s \in [1,e]$, $i = 1,2$.

(ii)$g_i(t,s) \leq g_i(e,s)$ for any $t,s \in [1,e]$, $i = 1,2$.

(iii)$g_1(t,s) \geq (\frac{1}{4})^{\vartheta_1-1} g_1(e,s)$ and $g_2(t,s) \geq (\frac{1}{4})^{\vartheta_2-1} g_2(e,s)$ for any $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$ and $s \in [1,e]$.

Proof. To show (*i*), it is easy to check that the functions $g_i(t,s)$, (i=1,2) are continuous functions. Next, for $1 \leq s \leq t \leq e$, we have

$$\begin{aligned} g_1(t,s) &= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1-1} - (\log\frac{t}{s})^{\vartheta_1-1}) \\ &= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1-1} - (\log t)^{\vartheta_1-1}(1-\frac{\log s}{\log t})^{\vartheta_1-1}) \\ &\geq \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1-1}(1-(1-\log s)^{\vartheta_1-1})) \\ &\geq 0. \end{aligned}$$

For $1 \leq t \leq s \leq e$, $g_1(t,s) = \dfrac{1}{\Gamma(\vartheta_1)}(\log t)^{\vartheta_1-1} \geq 0$. Using a similar proof, we obtain $g_2(t,s) \geq 0$ for any $t,s \in [1,e]$. To prove (*ii*), for $1 \leq s \leq t \leq e$, we get

$$\begin{aligned} g_{1t}(t,s) &= \frac{(\vartheta_1-1)(\log t)^{\vartheta_1-2}\frac{1}{t} - (\vartheta_1-1)(\log\frac{t}{s})^{\vartheta_1-2}\frac{1}{t}}{\Gamma(\vartheta_1)} \\ &\geq \frac{(\vartheta_1-1)(\log t)^{\vartheta_1-2}\left[1-(1-\log s)^{\vartheta_1-2})\right]}{\Gamma(\vartheta_1)t} \geq 0. \end{aligned}$$

Then, $g_{1t}(t,s)$ is increasing on $[s,e]$ according to t. That is, $g_1(t,s) \leq g_1(e,s)$ is obtained. It is easy to see that $g_1(t,s) \leq g_1(e,s)$ when $1 \leq t \leq s \leq e$. Thus, $g_1(t,s) \leq g_1(e,s)$ for any $t,s \in [1,e]$. Similarly, we have $g_2(t,s) \leq g_2(e,s)$ for any $t,s \in [1,e]$. To demonstrate (*iii*), for $1 \leq s \leq t \leq e$ and $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$,

$$\begin{aligned} g_1(t,s) &= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1-1} - (\log\frac{t}{s})^{\vartheta_1-1}) \\ &= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1-1} - (\log t)^{\vartheta_1-1}(1-\frac{\log s}{\log t})^{\vartheta_1-1}) \\ &\geq \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1-1}(1-(1-\log s)^{\vartheta_1-1})) \\ &\geq (\frac{1}{4})^{\vartheta_1-1} g_1(e,s). \end{aligned}$$

It is clear that for $1 \leq t \leq s \leq e$ and $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$, $g_1(t,s) \geq (\frac{1}{4})^{\vartheta_1-1} g_1(e,s)$. In a similar manner, we get $g_2(t,s) \geq (\frac{1}{4})^{\vartheta_2-1} g_2(e,s)$ for any $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$ and $s \in [1,e]$. The proof is completed. □

Lemma 2.7. *Let $K_1(s) = g_1(e,s) + \sum_{i=1}^{p} \frac{\lambda_i}{\Upsilon\Gamma(\vartheta_1+\beta_i)} g_1^{\beta_i}(\sigma_1^*, s)$, $K_2(s) = g_2(e,s) + \sum_{j=1}^{q} \frac{\sigma_j}{\Upsilon^*\Gamma(\vartheta_2+\alpha_j)} g_2^{\alpha_j}(\sigma_2^*, s)$, for $s \in [1,e]$ and $\varpi_1 = 1 + \frac{1}{\Upsilon_1}\int_1^e g_1(t)\frac{dt}{t}$, $\varpi_2 = 1 + \frac{1}{\Upsilon_1^*}\int_1^e g_2(t)\frac{dt}{t}$. Then, the functions $H_i(t,s)$, $i = 1,2$ defined by (2.2) and (2.3) ensure the following properties:*

(i) $H_i(t,s)$ are continuous and $H_i(t,s) \geq 0$, for $(t,s) \in [1,e] \times [1,e]$, $i = 1,2$;

(ii)$H_1(t,s) \leq K_1(s)\varpi_1$, for $(t,s) \in [1,e] \times [1,e]$;

(iii)$\min_{t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]} H_1(t,s) \geq (\frac{1}{4})^{2\vartheta_1-2} K_1(s)\varpi_1$, for $s \in [1,e]$;

(iv) $H_2(t,s) \leq K_2(s)\varpi_2$, for $(t,s) \in [1,e] \times [1,e]$;

(v) $\min_{t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]} H_2(t,s) \geq (\frac{1}{4})^{2\vartheta_2-2} K_2(s)\varpi_2$, for $s \in [1,e]$.

Proof. We can evidently see that (i) holds. To show (ii), for $(t,s) \in [1,e]\times[1,e]$, we have,

$$\begin{aligned}
H_1(t,s) &= G_1(t,s) + G_2(t,s) \\
&= g_1(t,s) + \sum_{i=1}^{p} \frac{\lambda_i (\log t)^{\vartheta_1 - 1}}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s) \\
&\quad + \frac{(\log t)^{\vartheta_1 - 1}}{\Upsilon_1} \int_1^e G_1(t,s) g_1(t) \frac{dt}{t} \\
&\le g_1(e,s) + \sum_{i=1}^{p} \frac{\lambda_i}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s) \\
&\quad + \frac{1}{\Upsilon_1} \int_1^e (g_1(e,s) + \sum_{i=1}^{p} \frac{\lambda_i}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s)) g_1(t) \frac{dt}{t} \\
&= K_1(s)\varpi_1.
\end{aligned}$$

To prove (iii), for $(t,s) \in [1,e]\times[1,e]$, we get,

$$\begin{aligned}
\min_{t\in[e^{\frac14},e^{\frac34}]} H_1(t,s) &= \min_{t\in[e^{\frac14},e^{\frac34}]} \Big[g_1(t,s) + \sum_{i=1}^{p} \frac{\lambda_i (\log t)^{\vartheta_1 - 1}}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s) \\
&\quad + \frac{(\log t)^{\vartheta_1 - 1}}{\Upsilon_1} \int_1^e G_1(t,s) g_1(t) \frac{dt}{t} \Big] \\
&\ge (\frac{1}{4})^{\vartheta_1 - 1} g_1(e,s) + \sum_{i=1}^{p} \frac{\lambda_i (\frac14)^{\vartheta_1 - 1}}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s) \\
&\quad + \frac{(\frac14)^{\vartheta_1 - 1}}{\Upsilon_1} \int_1^e \Big(g_1(t,s) + \sum_{i=1}^{p} \frac{\lambda_i (\log t)^{\vartheta_1 - 1}}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s) \Big) g_1(t) \frac{dt}{t} \\
&\ge (\frac{1}{4})^{\vartheta_1 - 1} K_1(s) + \frac{(\frac14)^{\vartheta_1 - 1}}{\Upsilon_1} \int_1^e \Big((\frac{1}{4})^{\vartheta_1 - 1} g_1(e,s) + \sum_{i=1}^{p} \frac{\lambda_i (\frac14)^{\vartheta_1 - 1}}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s) \Big) g_1(t) \frac{dt}{t} \\
&\ge (\frac{1}{4})^{\vartheta_1 - 1} K_1(s) + \frac{(\frac14)^{2\vartheta_1 - 2}}{\Upsilon_1} K_1(s) \int_1^e g_1(t) \frac{dt}{t} \\
&\ge (\frac{1}{4})^{2\vartheta_1 - 2} K_1(s)\varpi_1.
\end{aligned}$$

The proofs of the parts (iv) and (v) can be shown similar to the proofs above (ii) and (iii).
The proof is completed. □

We deal with the Banach space $E = C[1,e]\times C[1,e]$ with the norm $\|(u,v)\|_E = \|u\| + \|v\|$ for $(u,v)\in E$ and $\|u\| = \max_{t\in[1,e]} |u(t)|$. We introduce the cone $P \subset E$,

$$P = \left\{ (u,v) \in E : u(t) \ge 0, v(t) \ge 0, \forall t \in [1,e], \min_{t\in[e^{\frac14},e^{\frac34}]} (u(t) + v(t)) \ge \Psi \|(u,v)\| \right\}, \tag{2.10}$$

where $\Psi = min\left\{(\frac14)^{2\vartheta_1 - 2}, (\frac14)^{2\vartheta_2 - 2}\right\}$. Define the operator $F : P \to E$ by

$$F(u,v)(t) = (F_1(u,v)(t), F_2(u,v)(t)), \text{for all } t \in [1,e], \tag{2.11}$$

with $F_1, F_2 : P \to C[1,e]$ are given by

$$F_1(u,v)(t) = \int_1^e H_1(t,s) f_1(s,u(s),v(s)) \frac{ds}{s}, \tag{2.12}$$

$$F_2(u,v)(t) = \int_1^e H_2(t,s) f_2(s,u(s),v(s)) \frac{ds}{s}.$$

Lemma 2.8. *Consider that (u,v) is a positive solution of the system (1.1) if and only if (u,v) is a fixed point of the operator F.*

Proof. It is obvious that a positive solution of the system (1.1) is a fixed point of the operator F.
In fact, if $u(t) = F_1(u,v)(t)$, by applying the operator ${}^HD^{\vartheta_1}_{1^+}$ on both sides of (2.12), after some arrangement, for $x(s) = f_1(s,u(s),v(s),\ s\in[1,e]$ in Lemma (2.5), we get

$$\begin{aligned}{}^HD^{\vartheta_1}_{1^+}F_1(u,v)(t) =& \frac{{}^HD^{\vartheta_1}_{1^+}(\log t)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \sum_{i=1}^{p}\frac{\lambda_i{}^HD^{\vartheta_1}_{1^+}(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^e(\log\sigma_1^*)^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\ &-({}^HD^{\vartheta_1}_{1^+}{}^HI^{\vartheta_1}_{1^+}x)(t) - \sum_{i=1}^{p}\frac{\lambda_i{}^HD^{\vartheta_1}_{1^+}(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}k(s)\frac{ds}{s}\\ &+\frac{{}^HD^{\vartheta_1}_{1^+}(\log t)^{\vartheta_1-1}}{\Upsilon_1}\Big(\int_1^e g_1(t)\int_1^e G_1(t,s)x(s)\frac{ds}{s}\frac{dt}{t}\Big).\end{aligned}$$

Applying Lemma (2.3), we have

$$ {}^HD^{\vartheta_1}_{1^+}F_1(u,v)(t) = -x(t),$$

which implies that the system (1.1) is satisfied. Then by a direct computation, it follows that u satisfies the boundary conditions of (1.1). Similarly, we obtain that $v(t) = F_2(u,v)(t)$ is a solution of the system (1.1). The proof is completed. □

Lemma 2.9. *$F : \mathrm{P}\to\mathrm{P}$ is a completely continuous operator.*

Proof. Let us indicate that $F(P)\subset P$. The continuity of H_1,H_2, f_1, f_2, it follows that F is continuous. Lemma (2.7) and the nonnegativity of f_1 and f_1 ensure that $F_1(u,v)(t)\geq 0$, $F_2(u,v)(t)\geq 0$ for $t\in[1,e]$. Also, for $(u,v)\in P$

$$\begin{aligned}\|F_1(u,v)\| &\leq \varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s},\\ \|F_2(u,v)\| &\leq \varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s},\end{aligned}$$

and

$$\begin{aligned}\min_{t\in[e^{\frac14},e^{\frac34}]} F_1(u,v)(t) &\geq (\frac14)^{2\vartheta_1-2}\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}\\ &\geq(\frac14)^{2\vartheta_1-2}\|F_1(u,v)\|.\end{aligned}$$

Similary, we get $\min_{t\in[e^{\frac14},e^{\frac34}]} F_2(u,v)(t)\geq(\frac14)^{2\vartheta_2-2}\|F_2(u,v)\|$. Hence,

$$\begin{aligned}\min_{t\in[e^{\frac14},e^{\frac34}]}\{F_1(u,v)(t)+F_2(u,v)(t)\} &\geq (\frac14)^{2\vartheta_1-2}\|F_1(u,v)\| + (\frac14)^{2\vartheta_2-2}\|F_2(u,v)\|\\ &\geq \Psi[\|F_1(u,v)\|+\|F_2(u,v)\|]\\ &= \Psi\|F(u,v)\|,\end{aligned}$$

so $F : P\to P$. Moreover, we can use the Arzela–Ascoli theorem, we obtain that F is a completely continuous operator. The proof is completed. □

Let Φ, Λ, θ be nonnegative continuous convex functionals on P and κ, ψ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers k,s,d,l and h, we define the following convex sets:

$$\begin{aligned}&P(\Phi,h) = \{\vartheta\in P : \Phi(\vartheta) < h\},\\ &P(\Phi,\kappa,s,h) = \{\vartheta\in P : s\leq\kappa(\vartheta),\Phi(\vartheta)\leq h\},\\ &Q(\Phi,\Lambda,l,h) = \{\vartheta\in P : \Lambda(\vartheta)\leq l,\Phi(\vartheta)\leq h\},\\ &P(\Phi,\theta,\kappa,s,d,h) = \{\vartheta\in P : s\leq\kappa(\vartheta),\theta(\vartheta)\leq d,\Phi(\vartheta)\leq h\},\\ &Q(\Phi,\Lambda,\psi,k,l,h) = \{\vartheta\in P : k\leq\psi(\vartheta),\Lambda(\vartheta)\leq l,\Phi(\vartheta)\leq h\}.\end{aligned}$$

In ensuring positive solutions of (1.1), the following theorem will be essential.

Lemma 2.10. *[see [34]] Let P be a cone in a real Banach space E. Assume there exist $h > 0$ and $\mathrm{M} > 0$, nonnegative, continuous, concave functionals κ and ψ on P, and nonnegative, continuous, convex functionals Φ, Λ, and θ on P, satisfying*

$$\kappa(\vartheta) \leq \Lambda(\vartheta) \text{ and } \|\vartheta\| \leq \mathrm{M}\Phi(\vartheta)$$

for all $\vartheta \in \overline{P(\Phi,h)}$. If

$$S : \overline{P(\Phi,h)} \to \overline{P(\Phi,h)}$$

is completely continuous and there exist nonnegative numbers k,l,d,s with $0 < l < s$ such that:

(i) $\{\vartheta \in P(\Phi,\theta,\kappa,s,d,h) : \kappa(\vartheta) > s\} \neq \emptyset$ and $\kappa(S\vartheta) > s$ for $\vartheta \in P(\Phi,\theta,\kappa,s,d,h)$,

(ii) $\{\vartheta \in \mathrm{Q}(\Phi,\Lambda,\psi,k,l,h) : \Lambda(\vartheta) < l\} \neq \emptyset$ and $\Lambda(S\vartheta) < l$ for $\vartheta \in \mathrm{Q}(\Phi,\Lambda,\psi,k,l,h)$,

(iii) $\kappa(S\vartheta) > s$ for $\vartheta \in P(\Phi,\kappa,s,h)$ with $\theta(S\vartheta) > d$,

(vi) $\Lambda(S\vartheta) < l$ for $\vartheta \in \mathrm{Q}(\Phi,\Lambda,l,h)$ with $\psi(S\vartheta) < k$.

Then, S has at least three fixed points $\vartheta_1,\vartheta_2,\vartheta_3 \in \overline{P(\Phi,h)}$ satisfying;

$$\Lambda(\vartheta_1) < l, s < \kappa(\vartheta_2),$$

and

$$l < \Lambda(\vartheta_3) \text{ with } \kappa(\vartheta_3) < s.$$

For the readers convenience, let us denote

$$W = \min\Big\{\Big[\varpi_1 \int_1^e K_1(s)\frac{ds}{s}\Big]^{-1}, \Big[\varpi_2 \int_1^e K_2(s)\frac{ds}{s}\Big]^{-1}\Big\},$$

$$V = \max\Big\{\Big[(\frac{1}{4})^{2\vartheta_1-2}\varpi_1 \int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_1(s)\frac{ds}{s}\Big]^{-1}, \Big[(\frac{1}{4})^{2\vartheta_2-2}\varpi_2 \int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_2(s)\frac{ds}{s}\Big]^{-1}\Big\}.$$

Now, we introduce the nonnegative continuous concave functionals ξ, ψ and the nonnegative continuous convex functionals β, θ, σ on P by

$$\xi(u,v) = \psi(u,v) = \min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]} (u(t)+v(t)), \quad \theta(u,v) = \max_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]} (u(t)+v(t)),$$

$$\beta(u,v) = \sigma(u,v) = \max_{t\in[1,e]} (u(t)+v(t)).$$

3. Main result

Theorem 3.1. *Assume that there exist constants $0 < \ell < \kappa < \frac{\kappa}{\Psi} < h$ such that $\kappa V < hW$. If f_i, $i = 1,2$ satisfy the following conditions:*

(M_1) $f_i(t,u,v) < \dfrac{\ell W}{2}$ for $t \in [1,e]$, $(u+v) \in [0,\ell]$,

(M_2) $f_i(t,u,v) > \dfrac{\kappa V}{2}$ for $t \in [e^{\frac{1}{4}},e^{\frac{3}{4}}]$, $(u+v) \in [\kappa, \frac{\kappa}{\Psi}]$,

(M_3) $f_i(t,u,v) \leq \dfrac{hW}{2}$ for $t \in [1,e]$, $(u+v) \in [0,h]$.

Then the problem (1.1) has at least three positive solutions (u_i,v_i) $(i = 1,2,3)$ such that $\beta(u_1,v_1) < \ell$, $\kappa < \xi(u_2,v_2)$, $\ell < \beta(u_3,v_3)$ with $\xi(u_1,v_1) < \kappa$.

Proof. We introduce P and F as above equations (2.10) and (2.11). For any $(u,v) \in P$,

$$\xi(u,v) \leq \beta(u,v),$$

$$\|(u,v)\| \leq \frac{1}{\Psi} \min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]} (u(t)+v(t)) \leq \frac{1}{\Psi} \max_{t\in[1,e]} (u(t)+v(t)) = \frac{1}{\Psi}\sigma(u,v).$$

Next, we denote that the operator F ensures all conditions in Lemma (2.10). According to Lemma (2.9), F is completely continuous. As a beginning, we prove that $F : \overline{P(\sigma,h)} \to \overline{P(\sigma,h)}$. If $(u,v) \in \overline{P(\sigma,h)}$, then $\sigma(u,v) \leq h$, $0 \leq \|u\| + \|v\| \leq h$. With respect to (M_3), we obtain that,

$$
\begin{aligned}
\sigma(F(u,v)) &= \max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\le \varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}+\varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&\le \frac{hW}{2}\varpi_1\int_1^e K_1(s)\frac{ds}{s}+\frac{hW}{2}\varpi_2\int_1^e K_2(s)\frac{ds}{s}\\
&\le \frac{h}{2}+\frac{h}{2}=h.
\end{aligned}
$$

Hence, we ensure $F:\overline{P(\sigma,h)}\to\overline{P(\sigma,h)}$.
To verify condition (i) of Lemma (2.10), by choosing, $(\frac{\kappa\Psi+\kappa}{4\Psi},\frac{\kappa\Psi+\kappa}{4\Psi})$, we get that $(\frac{\kappa\Psi+\kappa}{4\Psi},\frac{\kappa\Psi+\kappa}{4\Psi})\in P(\sigma,\theta,\xi,\kappa,\frac{\kappa}{\Psi},h)$ and $\xi(u,v)>\kappa$. Thus, $\Big\{(u,v)\in P(\sigma,\theta,\xi,\kappa,\frac{\kappa}{\Psi},h):\xi(u,v)>\kappa\Big\}\neq\emptyset$. Let $(u,v)\in P(\sigma,\theta,\xi,\kappa,\frac{\kappa}{\Psi},h)$, then $(u(t)+v(t))\in[\kappa,\frac{\kappa}{\Psi}]$ for any $t\in[e^{\frac14},e^{\frac34}]$. By ($M2$), we obtain

$$
\begin{aligned}
\xi(F(u,v)) &= \min_{t\in[e^{\frac14},e^{\frac34}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\ge (\frac14)^{2\vartheta_1-2}\varpi_1\int_{e^{\frac14}}^{e^{\frac34}} K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}+(\frac14)^{2\vartheta_2-2}\varpi_2\int_{e^{\frac14}}^{e^{\frac34}} K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&> \frac{\kappa V}{2}(\frac14)^{2\vartheta_1-2}\varpi_1\int_{e^{\frac14}}^{e^{\frac34}} K_1(s)\frac{ds}{s}+\frac{\kappa V}{2}(\frac14)^{2\vartheta_2-2}\varpi_2\int_{e^{\frac14}}^{e^{\frac34}} K_2(s)\frac{ds}{s}\\
&> \frac{\kappa}{2}+\frac{\kappa}{2}=\kappa.
\end{aligned}
$$

Then, the condition (i) of Lemma (2.10) is satisfied. Now, we demonstrate that the condition (ii) of Lemma (2.10) is fulfilled. Let $(\frac{\Psi\ell+\ell}{4},\frac{\Psi\ell+\ell}{4})$, then $(\frac{\Psi\ell+\ell}{4},\frac{\Psi\ell+\ell}{4})\in Q(\sigma,\beta,\psi,\Psi\ell,\ell,h)$ and $\beta(u,v)<\ell$. Hence, $\{(u,v)\in Q(\sigma,\beta,\psi,\Psi\ell,\ell,h):\beta(u,v)<\ell\}\neq\emptyset$. Let $(u,v)\in Q(\sigma,\beta,\psi,\Psi\ell,\ell,h)$, then $(u(t)+v(t))\in[0,\ell]$ for any $t\in[1,e]$. By ($M1$), we obtain

$$
\begin{aligned}
\beta(F(u,v)) &= \max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\le \varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}+\varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&< \frac{\ell W}{2}\varpi_1\int_1^e K_1(s)\frac{ds}{s}+\frac{\ell W}{2}\varpi_2\int_1^e K_2(s)\frac{ds}{s}\\
&< \frac{\ell}{2}+\frac{\ell}{2}=\ell.
\end{aligned}
$$

Now, we can show that the condition (iii) of Lemma (2.10) is satisfied. Let $(u,v)\in P(\sigma,\xi,\kappa,h)$ with $\theta(F(u,v))>\frac{\kappa}{\Psi}$. Then, we have,

$$
\begin{aligned}
\xi(F(u,v)) &= \min_{t\in[e^{\frac14},e^{\frac34}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\ge (\frac14)^{2\vartheta_1-2}\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}+(\frac14)^{2\vartheta_2-2}\varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&\ge \Psi\Big[\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}+\varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\ge \Psi\max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\ge \Psi\max_{t\in[e^{\frac14},e^{\frac34}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&= \Psi\theta(F(u,v))=\kappa.
\end{aligned}
$$

Finally, we can verify that the condition (iv) of Lemma (2.10) ensures. Let $(u,v)\in Q(\sigma,\beta,\ell,h)$ with $\psi(F(u,v))<\Psi\ell$,

$$\begin{aligned}\beta(F(u,v)) &= \max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\ &\le \frac{1}{\Psi}\Big[\Psi\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}+\Psi\varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\ &\le \frac{1}{\Psi}\min_{t\in[e^{\frac14},e^{\frac34}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}+\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\ &= \frac{1}{\Psi}\psi(F(u,v))<\ell.\end{aligned}$$

Because the conditions of Lemma (2.10) are satisfied, the system (1.1) has at least three positive solutions (u_i,v_i) $(i=1,2,3)$ such that $\beta(u_1,v_1)<\ell$, $\kappa<\xi(u_2,v_2)$, $\ell<\beta(u_3,v_3)$ with $\xi(u_1,v_1)<\kappa$. The proof is completed. □

Example 3.2. *Consider the system of Hadamard fractional differential equations*

$$\begin{cases} {}^H D_{1^+}^{\frac52}u(t)+f_1(t,u(t),v(t))=0, & t\in(1,e),\\ {}^H D_{1^+}^{\frac52}v(t)+f_2(t,u(t),v(t))=0, & t\in(1,e),\\ u(1)=u'(1)=0, & {}^H D_{1^+}^{\frac32}u(e)=\int_1^e u(t)\frac{dt}{t}+\frac12\,{}^H I_{1^+}^{\frac12}u(e^{\frac12})+{}^H I_{1^+}^{\frac32}u(e^{\frac12}),\\ v(1)=v'(1)=0, & {}^H D_{1^+}^{\frac32}v(e)=\int_1^e v(t)\frac{dt}{t}+\frac32\,{}^H I_{1^+}^{\frac32}u(e^{\frac13})+2\,{}^H I_{1^+}^{\frac72}u(e^{\frac13}),\end{cases}\tag{3.1}$$

in which $\vartheta_1=\vartheta_2=\frac52$, $n=m=3$, $p=q=2$, $\lambda_1=\frac12,\lambda_2=1$, $\sigma_1=\frac32,\sigma_2=2$, $\sigma_1^*=e^{\frac12},\sigma_2^*=e^{\frac13}$, $\beta_1=\frac12,\beta_2=\frac32$, $\alpha_1=\frac32,\alpha_2=\frac72$, $g_1(t)=g_2(t)=1$ *for* $t\in[1,e]$,

$$f_1(t,u,v)=\begin{cases}\frac{t}{10}+\frac{(u+v)}{4}, & t\in[1,e], \quad (u+v)\in[0,4],\\ \frac{t}{10}+170(u+v)-679, & t\in[1,e], \quad (u+v)\in[4,6],\\ \frac{t}{10}+\frac{10(u+v)+270694}{794}, & t\in[1,e], \quad (u+v)\in[6,\infty),\end{cases}$$

$$f_2(t,u,v)=\begin{cases}\frac{t}{20}+\frac{(u+v)}{4}, & t\in[1,e], \quad (u+v)\in[0,4],\\ \frac{t}{20}+170(u+v)-679, & t\in[1,e], \quad (u+v)\in[4,6],\\ \frac{t}{20}+\frac{(u+v)+135371}{397}, & t\in[1,e], \quad (u+v)\in[6,\infty).\end{cases}$$

By direct calculation, we get $\Psi=0,015625$,

$$\begin{aligned} W&=\min\Big\{\Big[\varpi_1\int_1^e K_1(s)\frac{ds}{s}\Big]^{-1},\Big[\varpi_2\int_1^e K_2(s)\frac{ds}{s}\Big]^{-1}\Big\}\approx\min\{0.8842,1.0435\}=0.8842,\\ V&=\max\Big\{\Big[(\frac14)^{2\vartheta_1-2}\varpi_1\int_{e^{\frac14}}^{e^{\frac34}}K_1(s)\frac{ds}{s}\Big]^{-1},\Big[(\frac14)^{2\vartheta_2-2}\varpi_2\int_{e^{\frac14}}^{e^{\frac34}}K_2(s)\frac{ds}{s}\Big]^{-1}\Big\}\\ &\approx\max\{106.383,111.1111\}=111.1111.\end{aligned}$$

Choosing the constants as $\ell=4,\kappa=6,h=800$, *then* $0<\ell<\kappa<\frac{\kappa}{\Psi}<h$ *such that* $\kappa V<hW$. *Then,* f_i, $i=1,2$ *satisfy the following conditions:*

(M_1) $f_i(t,u,v)<\frac{\ell W}{2}\approx1.7684$ *for* $t\in[1,e]$, $(u+v)\in[0,4]$,

(M_2) $f_i(t,u,v)>\frac{\kappa V}{2}\approx333.3333$ *for* $t\in[e^{\frac14},e^{\frac34}]$, $(u+v)\in[6,384]$,

(M_3) $f_i(t,u,v)\le\frac{hW}{2}\approx353.63$ *for* $t\in[1,e]$, $(u+v)\in[0,800]$.

Then, all the hypotheses of Theorem (3.1) *are satisfied. Thus, the system of fractional differential equations* (3.1) *has at least three positive solutions.*

4. Conclusion

In our main result, it is obtained positive solutions for Hadamard differential systems. By using the five functionals fixed point theorem, the conditions for the existence of positive solutions are derived. There are a little number of papers which are studied on the systems of nonlinear Hadamard fractional differential equations. Here, unlike other papers, we attempt to study new Hadamard differential systems which consist of both integral boundary conditions and m-point fractional integral boundary conditions on an bounded domain.

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