International Journal of Informatics and Applied Mathematics
e-ISSN:2667-6990 Vol. 5, No. 1, 52-61

Free Surface Flow Between Two Semi-Infinite Inclined Plate Without Gravity Effect

Abdelkader Laiadi

Mohamed Khider University, Biskra 7000, Algeria
abdelkader.laiadi@univ-biskra.dz

Abstract. In this paper, we study the problem of two-dimensional free surface flows past an inclined plate of an infinite length. The Riabouchinsky model is considered. The fluid is assumed to be inviscid and incompressible and the flow to be steady and irrotational. For this study, the effects of gravity and surface tension are negligible. This problem is solved by using two methods. The first method allow us to compute analytical solution by employing Schwarz-Christoffel mapping. The other is numerically using the series truncation method. The obtained results showed a good agreement between the numerical solution and the exact solution.

Keywords: Free Surface Flow · Zero Gravity · Potential Flow · Schwarz-Christoffel Transformation · Series Truncation.

1 Introduction

In this paper, we consider the flow of an incompressible inviscid fluid past a semi-infinite inclined plate. The two-dimensional flow is assumed to be irrotational and steady. The Riabouchinsky model is adopted. This model were studied by Riabouchinsky [11], Daboussy [5] and my work [9] and for a detailed description of this problem, one can refer to the books Birkhoff and Zarantello [2], Gurevich [7], Milne-Thomson [10] and others. In the absence of gravity and surface tension, the problem of Riabouchinsky's flow past a plate of a finite length has an exact solution calculated by authors such as [2], [7], [10] and [11]. In this work, we study the problem of flow past a plate of an infinite length. The flow configuration is shown in Figure 1. In the case of zero gravity, we compute analytical solution by using the classical theory, depend on the conformal mapping such as Schwarz-Christoffel transformation and hodograph variable. We also solve this problem numerically via a series truncation method. This method has been used extensively by many researchers, Vanden-Broeck [12,13], Daboussy [5,6], Alex [1], Charles [3], Dias [4]. The problem is formulated in Sect.2. The solution presented is computed using the conformal mapping described in Sect. 3. However, we also compute the numerical solution using the series truncation method from Sect. 4. The agreement between the analytical solution and numerical scheme provides a convincing check on the methods used.

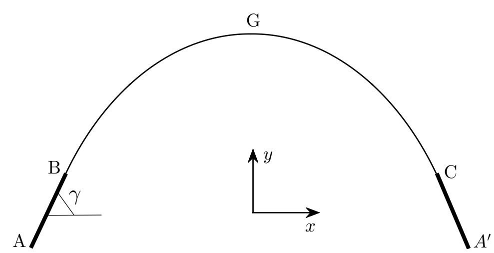

Fig. 1. The physical $z-$plane $z = x + iy$.

2 Problem formulation

We consider the steady two-dimensional flow past an infinite inclined plate. The flow configuration shown in Figure 1 is considered. The flow domain is bounded above by the walls AB and CA' and the free surface BC. The walls are inclined by the angle γ with the horizontal where $0 < \gamma \leq \frac{\pi}{2}$. It is assumed that the flow is symmetric about the $y-$axis. We suppose that the flow is irrotational and the fluid is incompressible and inviscid. Far downstream and far upstream the velocity is infinite if $\gamma > 0$. Let us introduce Cartesian coordinates with the $x-$axis along the bottom and the $y-$axis directed vertically upwards. Let's introduce the velocity potential ϕ and the stream function ψ by defining the complex potential function $f = \phi + i\psi$. Without loss of generality, let $\phi = 0$ at the point G on the line of symmetry and let $\psi = 0$ along the free streamline BC. By conformal transformation, the flow domain in $z-$plane can be mapped onto the upper half $f-$plane. The flow in the $f-$plane is shown in Figure 2. We define dimensionless variables by taking $\frac{\phi_C}{V_G}$ as unit length and V_G as unit velocity, where ϕ_C is the value of the potential at the point C and V_G the velocity at the point G.

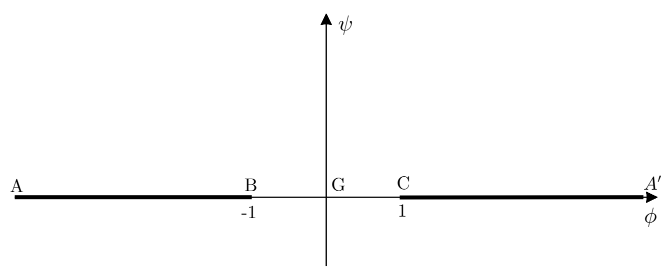

Fig. 2. The complex potential $f-$plane $f = \phi + i\psi$.

Next we define the complex velocity

$$\xi = \frac{df}{dz} = u - iv \qquad (1)$$

where u and v denote the horizontal and vertical components of the velocity. The function ξ is an analytic function of the complex potential f inside the flow domain. When the surface tension and the gravity are negligible, the mathematical problem is to determine the function ϕ who verifies the following conditions

$$\Delta\phi = 0 \qquad (2)$$

in the fluid domain.

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = Cte \qquad (3)$$

The Bernoulli's equation on the free surface BC.

$$\frac{\partial\phi}{\partial y} = tan\gamma\frac{\partial\phi}{\partial x} \qquad (4)$$

The kinematic condition along the walls.

3 Analytical solution

We will present an exact solution for the flow configuration shown in Figure 1. The problem is formulated in previous section. First, we introduce the hodograph variable

$$\Omega = log\frac{dz}{df} = log\frac{1}{q} + i\theta \tag{5}$$

Where q and θ are the modulus and argument of the velocity respectively. In the solution of problems involving polygonal boundaries the required mappings can always be exhibited in closed form, except perhaps for the determination of certain parameters. The principal tool is the Schwarz-Christoffel formula which, through explicit mappings on a half plane. The Schwarz-Christoffel transformation in its usual form asserts that a simple plane polygon P in the $Z-$ plane, with vertices B_k having interior angles $\alpha_k\pi, k = 1, ..., n$ is mapped conformally from a half-plane by the formula

$$z(t) = A\int\prod_{k=1}^{n}(t-b_k)^{\alpha_k - 1}dt + B, Imt > 0 \tag{6}$$

where A and B are real constants, and the b_k are points on the real $t-$axis whose images are the respective vertices B_k. In this formula, three of the constants b_k can be chosen arbitrarily, and if one is placed at infinity the corresponding factor does not appear in the integrand. The shape of free surface is determined by quadrature

$$z(f) = \int e^{\Omega}df \tag{7}$$

By using (5), the flow field in the $f-$plane (see Figure 2) is mapped to the semi infinite band in the $\Omega-$plane (see Figure 3). We obtain from the Schwarz-Christoffel transformation (6)

$$\frac{d\Omega}{df} = \frac{M}{\sqrt{f^2-1}} \tag{8}$$

Where M is constant. After integration, we obtain

$$\Omega = \frac{2\gamma}{\pi}arccoshf - i\gamma \tag{9}$$

By using (7) and (9), we obtain

$$z(f) = e^{-i\gamma}\int(f + i\sqrt{1-f^2})^{\frac{2\gamma}{\pi}}df \tag{10}$$

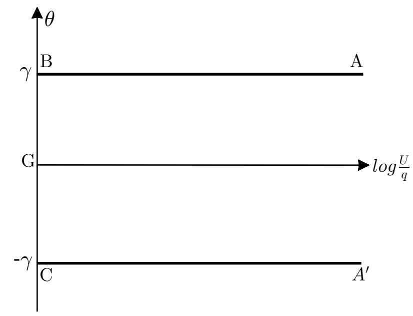

Fig. 3. The $\Omega-$plane

On the free surface BC $(-1 \leq f \leq 1)$, introducing the change of variables $f = \phi = \cos\theta$, after integration (10), the parametric equations of the form of free surface BC become

$$\begin{cases} x\,(\theta) = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) - \frac{\pi}{4} \\ y\,(\theta) = -\frac{1}{4}\cos(2\theta) + \frac{1}{4} \end{cases} \tag{11}$$

for $\gamma = \frac{\pi}{2}$ and

$$\begin{cases} x\,(\theta) = -\frac{1}{2}\left(\frac{1}{1+2\pi\gamma}\cos((1+2\pi\gamma)\theta - \gamma) + \frac{1}{1-2\pi\gamma}\cos((1-2\pi\gamma)\theta + \gamma)\right) \\ y\,(\theta) = \frac{1}{2}\left(\frac{1}{1+2\pi\gamma}\sin((1+2\pi\gamma)\theta - \gamma) - \frac{1}{1-2\pi\gamma}\sin((1-2\pi\gamma)\theta + \gamma)\right) \end{cases} \tag{12}$$

for each value γ where $0 < \gamma < \frac{\pi}{2}$ and $0 < \theta < \pi$.

4 Numerical solution

We consider steady two-dimensional flow past an infinite inclined plate (see Figure 1). In this section, we will present a numerical scheme based on series truncation to compute fully non-linear solutions with both surface tension and gravity are negligible. Following Birkhoff and Zarantello [2] and Daboussy [5] we define a new variable by the relation

$$f(t) = \frac{1+t^2}{2t} \tag{13}$$

This transformation maps the flow domain in $f-$plane (see Figure 2) into the upper half of the unit disc in the complex $t-$plane so that the free surface on the circumference (see Figure 4). The image of the solid boundaries is the real diameter. The images of the five points A, B, C, G, A' labelled in Figures 3

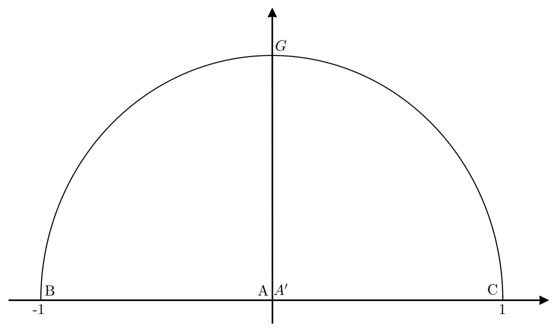

Fig. 4. The complex $t-$plane.

and 4 are $t = 0, t = -1, t = 1, t = i, t = 0$. The points of the free surface in the $t-$plane are given by the relation:

$$t = e^{i\sigma}, 0 < \sigma < \pi \tag{14}$$

We introduce the function $\tau - i\theta$ as

$$\xi = \frac{df}{dz} = e^{\tau - i\theta} \tag{15}$$

In these new variables, the Bernoulli equation in condition (3) becomes

$$e^{\tau} = 1 \tag{16}$$

The behaviour of the complex velocity at infinity is $\xi \sim z^p$ where $2\alpha = \pi \frac{p}{p+1}$. It follows that

$$\xi(t) \sim t^{\frac{-2\gamma}{\pi}} \text{ as } t \longrightarrow 0 \tag{17}$$

Next the complex velocity ξ is expanded as

$$\xi(t) = e^{i\gamma} t^{\frac{-2\gamma}{\pi}} \exp\left(\sum_{k=0}^{k=\infty} a_k t^{2k}\right) \tag{18}$$

Where the coefficients a_k are real. The expansion takes advantage of the symmetry of the problem and of the singularity of the velocity at infinity, and satisfies the kinematic condition (4) along the walls. At the point G $(t = i)$, the velocity approaches unity $(V_G \longrightarrow 1)$. Where V_G is the velocity at the point G.

The kinematic condition (4) are satisfied by requiring the coefficients a_k to be real. It can be checked that (18) satisfies (17). Therefore we expect the series in (18) to converge for $|t| \leq 1$. The coefficients a_k must be determined to satisfy the boundary condition (16) on the free surface BC. According to (14) and (18), we have:

$$\begin{cases} \theta(\sigma) = \gamma - \frac{2\gamma}{\pi}\sigma + \sum_{k=1}^{k=\infty} a_k \sin(2(k-1)\sigma) \\ \tau(\sigma) = \sum_{k=1}^{k=\infty} a_k \cos(2(k-1)\sigma) \end{cases} \tag{19}$$

The problem is solved numerically by truncating the infinite series in (18) after N terms. We find the N coefficients a_k by collocation. We introduce the N mesh points on the free surface $(0 < \sigma < \pi)$

$$\sigma_I = \frac{\pi}{N}\left(I - \frac{1}{2}\right), I = 1, ..., N \tag{20}$$

We satisfy the equation (16) at the mesh points (20). This yields N equations of the N unknowns $(a_k, k = 1, ..., N)$. For given values of γ, this system of N non-linear equations with N unknowns is solved by Newton's method.

The profile of the free surface is obtained by integrating numerically the relation

$$\begin{cases} \frac{\partial x}{\partial \sigma} = \sin\sigma e^{-\tau(\sigma)} \cos(\theta(\sigma)) \\ \frac{\partial y}{\partial \sigma} = \sin\sigma e^{-\tau(\sigma)} \sin(\theta(\sigma)) \end{cases} \tag{21}$$

Most of the calculations were performed with $N = 50$.

The numerical scheme based on series truncation was used to compute solutions for different values of the angle γ and $\alpha \longrightarrow \infty$ where $0 < \gamma \leq \frac{\pi}{2}$. For each value of γ, there is a unique solution. We found that the coefficients a_k are vanishes. For example, $a_1 = 3.77 \times 10^{-15}, a_{10} = 1.49 \times 10^{-17}, a_{30} = 5.09 \times 10^{-18}$ and $a_{50} = -6.66 \times 10^{-18}$ for $\gamma = \frac{\pi}{4}$. In precedent section, we computed the solution of this problem analytically for each value of γ where $0 < \gamma \leq \frac{\pi}{2}$. In the case $\gamma = \frac{\pi}{2}$, the comparison between the exact solution in equation (11) and numerical solution is shown in Figure 5. Figures 6-9 show the free surface profiles for some values of the angle γ where $\gamma = \frac{4\pi}{9}, \gamma = \frac{\pi}{4}, \gamma = \frac{\pi}{6}$ and $\gamma = \frac{\pi}{30}$ by using the two precedent methods. These figures provide the good agreement between the exact free-streamline solution and numerical free surface profile.

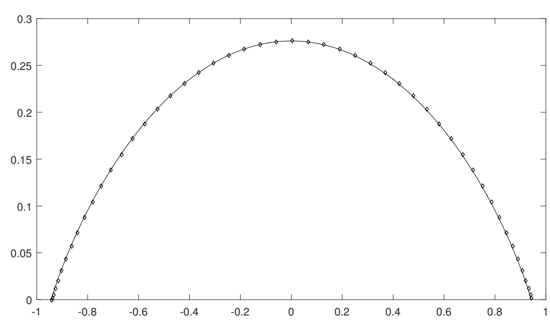

Fig. 5. Comparison of the numerical free surface profile ($\diamond$) with the analytical solution(—) for $\alpha = \infty, \gamma = \frac{\pi}{2}$.

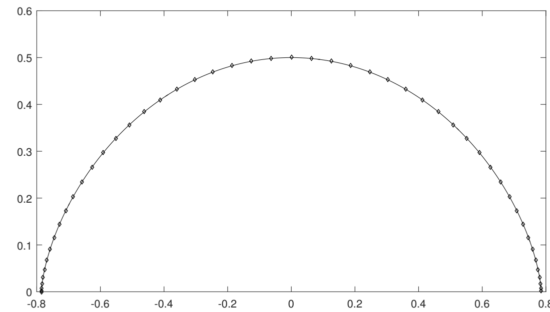

Fig. 6. Comparison of the numerical free surface profile ($\diamond$) with the analytical solution(—) for $\alpha = \infty, \gamma = \frac{4\pi}{9}$.

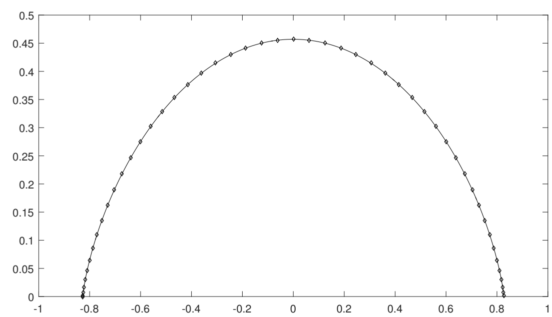

Fig. 7. Comparison of the numerical free surface profile ($\diamond$) with the analytical solution(—) for $\alpha = \infty, \gamma = \frac{\pi}{4}$.

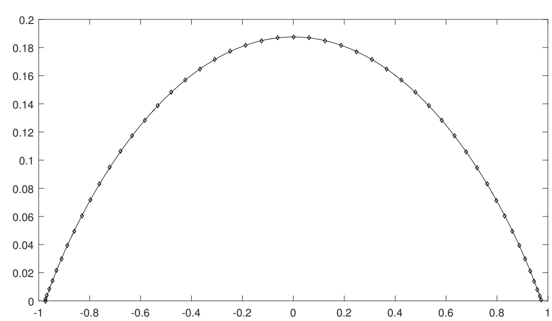

Fig. 8. Comparison of the numerical free surface profile ($\diamond$) with the analytical solution(—) for $\alpha = \infty, \gamma = \frac{\pi}{6}$.

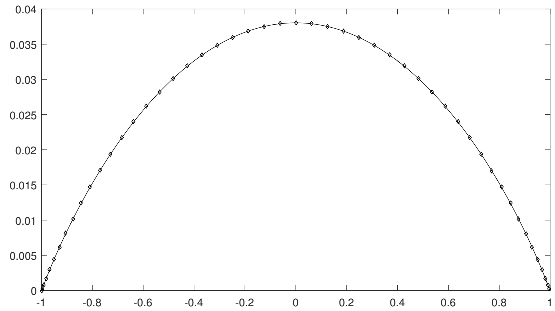

Fig. 9. Comparison of the numerical free surface profile (◇) with the analytical solution(—) for $\alpha = \infty, \gamma = \frac{\pi}{30}$.

5 Conclusion

In this work, we have considered both an analytical method and a numerical method for determining the free-surface flow past an infinite inclined plate. The flow configuration is shown in the Figure 1. We have computed solutions whose gravity and surface tension are negligible. We calculated the analytical solution of this problem using the Schwarz-Christoffel mapping and the numerical solution using the series truncation method. Figures 5-9 show that the comparison of the numerical solution with the analytical solution of which it provides good agreement between them.

6 Acknowledgments

The author would like to express their thanks to the editor and the referees for their helpful comments and suggestions.

References

1. Alex Doak, Vanden-Broeck, J.M.: Nonlinear two-dimensional free surface solutions of flow exiting a pipe and impacting a wedge. J .Eng. Math **126**(8), 1-19 (2021)
2. Birkhoff, G., Zarantonello, E.H.: Jets wakes and cavities. New-York, Academic Press INC (1957)
3. Charles W. Lenau, Robert L. Street: A non-linear theory for symmetric, supercavitating flow in a gravity field. J. Fluid Mech **21**, 257-280 (1965)
4. Dias, F., Vanden Broeck, J.M.: Open channel flows with submerged obstructions. J. Fluid Mech **206**, 155-170 (1989)
5. Daboussy, D., Dias, F., Vanden Broeck, J.M.: Gravity flows with a free surface of finite extent. Eur. J. Mech.B/Fluids **17**(1), 19-31 (1998)
6. Daboussy, D., Dias, F., Vanden Broeck, J.M.: On explicit solutions of the free-surface Euler equations in the presence of gravity. Phys. Fluids **9**(10), 2828-2834 (1997)

7. Gurevich, M.I.: Theory of jets in ideal fluids. Academic Press, New York and London (1965)
8. Lee, J., Vanden Broeck, J.M.: Two-dimensional jets falling from funnels and nozzles. Phys. Fluids A. **5**, 2454-2460 (1993)
9. Laiadi, A., Merzougui, A.: Numerical solution of a cavity problem under surface tension effect. Math. Meth. Appl. Sci **44**(10), 1-9 (2020)
10. Milne-Thomson, L.M.: Theoretical hydrodynamics. Macmillan and Co. Ltd (1962)
11. Riabouchinsky, D.: On steady fluid motions with free surfaces. Proc. London Math. Sot. II **19**, 206-215 (1921)
12. Vanden-Broeck, J.M., Dias, F.: Free-surface flows with two stagnation points. J. Fluid Mech **324**, 393-406 (1996)
13. Vanden-Broeck, J.M.: Gravity-capillary free surface flows. Cambridge University Press (2010)