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Research Article

Bihom-pre-Lie superalgebras and related structures

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Abstract

Throughout this paper, we will study Rota-Baxter operators and super $\mathcal{O}$-operator of BiHom-associative superalgebras, BiHom-Lie superalgebras, BiHom-pre-Lie superalgebras and BiHom-L-dendriform superalgebras. Then we give some properties of BiHom-pre-Lie superalgebras constructed from BiHom-associative superalgebras, BiHom-Lie superalgebras and BiHom-L-dendriform superalgebras.

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Introduction

Pre-Lie algebras are a class of non-associative algebras coming from the study of convex homogeneous cones [48], affine manifolds and affine structures on Lie groups [39], and the aforementioned cohomologies of associative algebras [34]. They also appeared in many fields in mathematics and mathematical physics, such as complex and symplectic structures on Lie groups and Lie algebras [7, 25, 28, 29, 41], phases spaces of Lie algebras [10, 40], integrable systems [19], classical and quantum Yang-Baxter equations [30], combinatorics [31], Poisson brackets and infinite dimensional Lie algebras, vertex algebras, quantum field theory [26], and operads [22]. See [20] for a survey. Recently, pre-Lie superalgebras, the $\mathbb{Z}_2$-graded version of pre-Lie algebras, also appeared in many others fields; see for example [22, 34, 47]. To our knowledge, they were first introduced by Gerstenhaber in 1963 to study the cohomology structure of associative algebras [34]. They are a class of natural algebraical structures appearing in many fields in mathematics and mathematical physics, especially in super-symplectic geometry, vertex superalgebras and graded classical Yang-Baxter equation [11, 12]. Recently, the classifications of complex pre-Lie superalgebras in dimensions two and three were given by R. Zhang and C.M. Bai [16].

The theory of Hom-algebraic structures appeared in the physics literature around 1990 to describe q-deformation of algebras of vector fields, especially Witt and Virasoro algebras (see [5, 24, 27, 42]).

On the other hand,the class of BiHom-algebras was introduced from a categorical approach in [36] as an extension of the class of Hom-algebras. Recall that a BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two

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homomorphisms α and β. More applications of BiHom-Lie algebras, BiHom-algebras, BiHom-Lie superalgebras, BiHom-Lie admissible superalgebras and 3-BiHom-Lie superalgebras can be found in ([18, 23, 46]).

The notion of Rota-Baxter operators is very useful in several constructions. The study of Rota-Baxter algebra appeared for the first time in the work of the mathematician G. Baxter [9] in 1960 and were then intensively studied by F. V. Atkinson [8], J. B. Miller [44], G.-C. Rota [45], P. Cartier [21] and more recently they reappeared in [31, 37, 38].

The notion of dendriform algebras was introduced by Loday ([43]) in 1995 with motivation from algebraic K-theory and has been studied quite extensively with connections to several areas in mathematics and physics, including operads, homology, Hopf algebras, Lie and Leibniz algebras, combinatorics, arithmetic and quantum field theory and so on (see [33] and the references therein). The relationship between dendriform algebras, Rota-Baxter algebras and pre-Lie algebras was given by M. Aguiar and K. Ebrahimi-Fard [2, 31, 32]. C. Bai, L. Liu, L. Guo and X. Ni, generalized the concept of Rota-Baxter operator and introduced a new class of algebras, namely, L-dendriform algebras, in [13–15]. Moreover, there is the following relationship among Lie superalgebras, associative superalgebras, pre-Lie superalgebras and dendriform superalgebras in the sense of commutative diagram of categories:

$$\begin{array}{ccc} \text{Lie superalgebra} & \longleftarrow & \text{pre-Lie superalgebra} \\ \uparrow & & \uparrow \\ \text{associative superalgebra} & \longleftarrow & \text{dendriform superalgebra} \end{array} \tag{0.1}$$

Later quite a few more similar algebra structures have been introduced, such as quadri-algebras of Aguiar and Loday [4]. In order to extend the commutative diagram (0.1) at the level of associative superalgebras (the bottom level of the commutative diagram (0.1)) to the more Loday superalgebras, it is natural to find the corresponding algebraic structures at the level of Lie superalgebras which extends the top level of commutative diagram (0.1). We will show that the L-dendriform superalgebras are chosen in a certain sense such that the following diagram including the diagram (0.1) as a sub-diagram is commutative:

$$\begin{array}{ccccc} \text{Lie superalgebra} & \longleftarrow & \text{pre-Lie superalgebra} & \longleftarrow & L\text{-dendriform superalgebra} \\ \uparrow & & \uparrow & & \uparrow \\ \text{associative superalgebra} & \longleftarrow & \text{dendriform superalgebra} & \longleftarrow & \text{quadri-superalgebra} \end{array} \tag{0.2}$$

Recently, the notion of Rota-Baxter operator on a bimodules was introduced by M. Aguiar [3]. The construction of associative, Lie, pre-Lie and L-dendriform superalgebras are extended to the corresponding categories of bimodules.

The main purpose of this paper is to study, through Rota-Baxter operators and $\mathcal{O}$-operators, the relationship between BiHom-associative superalgebras, BiHom-Lie superalgebras, BiHom-pre-Lie superalgebras and BiHom-L-dendriform superalgebras.

This paper is organized as follows. In Section 1, we recall some definitions of BiHom-associative superalgebras, BiHom-Lie superalgebras and BiHom-pre-Lie superalgebras and we introduce the notion of $\mathcal{O}$-operator of these BiHom superalgebras that generalizes the notion of Rota-Baxter operators. We show that every Rota-Baxter BiHom associative superalgebra of weight $\lambda = -1$ gives rise to a Rota-Baxter BiHom-Lie superalgebra. Moreover, an $\mathcal{O}$-operator on a BiHom-Lie superalgebra (of weight zero) gives rise to a BiHom-pre-Lie superalgebra. In Section 2, we introduce the notion of BiHom-L-dendriform superalgebra and then study some fundamental properties of BiHom-L-dendriform superalgebras in terms of $\mathcal{O}$-operator of BiHom-pre-Lie superalgebras. Their relationship with BiHom-associative superalgebras are also described.

Throughout this paper, all superalgebras are finite-dimensional and are over a field $\mathbb{K}$ of characteristic zero. Let $(\mathcal{A}, \circ, \alpha, \beta)$ be a BiHom superalgebra, then $L_\circ$ and $R_\circ$ denote the even left and right multiplication operators $L_\circ, R_\circ : \mathcal{A} \to End(\mathcal{A})$ defined as $L_\circ(x)(y) = (-1)^{|x||y|}R_\circ(y)(x) = x \circ y$ for all homogeneous element x, y in $\mathcal{A}$. In particular, when $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$ is a BiHom-Lie superalgebra, we let $ad(x)$ denote the adjoint operator, that is, $ad(x)(y) = [x, y]$ for all homogeneous element x, y in $\mathcal{A}$.

1. Rota-Baxter BiHom-associative superalgebras, BiHom-pre-Lie superalgebras and BiHom-Lie superalgebras

Let $(\mathcal{A}, \circ)$ be an algebra over a field $\mathbb{K}$. It is said to be a superalgebra if the underlying vector space of $\mathcal{A}$ is $\mathbb{Z}_2$-graded, that is, $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, and $\mathcal{A}_i \circ \mathcal{A}_j \subset \mathcal{A}_{i+j}$, for $i, j \in \mathbb{Z}_2$. An element of $\mathcal{A}_0$ is said to be even and an element of $\mathcal{A}_1$ is said to be odd. The elements of $\mathcal{A}_j$, $j \in \mathbb{Z}_2$, are said to be homogenous and of parity j. The parity of a homogeneous element x is denoted by $|x|$ and we refer to the set of homogeneous elements of $\mathcal{A}$ by $\mathcal{H}(\mathcal{A})$.

We extend to graded case the concepts of $\mathcal{A}$-bimodule $\mathbb{K}$-algebra and $\mathcal{O}$-operator introduced in [13].

Definition 1.1. An associative superalgebra is a pair $(\mathcal{A}, \mu)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$ and an even linear map $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, (i.e : $\mu(\mathcal{A}_i, \mathcal{A}_j) \subseteq \mathcal{A}_{i+j}, \forall i, j \in \mathbb{Z}_2$) satisfying for any $x, y, z \in \mathcal{A}$

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z).$$

Definition 1.2.

(1) A BiHom-associative superalgebra is a tuple $(\mathcal{A}, \mu, \alpha, \beta)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$, an even linear map $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and two even linear maps $\alpha, \beta : \mathcal{A} \longrightarrow \mathcal{A}$, satisfying the following conditions for all $x, y, z \in \mathcal{H}(\mathcal{A})$.

$$\begin{aligned} \alpha \circ \beta &= \beta \circ \alpha, \\ \alpha(\mu(x, y)) &= \mu(\alpha(x), \alpha(y)) \ \ and \ \ \beta(\mu(x, y)) = \mu(\beta(x), \beta(y)) \ \ (multiplicativity), \\ \mu(\alpha(x), \mu(y, z)) &= \mu(\mu(x, y), \beta(z)) \ \ (BiHom-associativity). \end{aligned}$$

If α and β are invertible then $(\mathcal{A}, \mu, \alpha, \beta)$ is called a regular BiHom-associative superalgebra.

A morphism $f : (\mathcal{A}, \mu_{\mathcal{A}}, \alpha_{\mathcal{A}}, \beta_{\mathcal{A}}) \to (\mathcal{B}, \mu_{\mathcal{B}}, \alpha_{\mathcal{B}}, \beta_{\mathcal{B}})$ of BiHom-associative superalgebras is an even linear map $f : \mathcal{A} \to \mathcal{B}$ such that $\alpha_{\mathcal{B}} \circ f = f \circ \alpha_{\mathcal{A}}$, $\beta_{\mathcal{B}} \circ f = f \circ \beta_{\mathcal{A}}$ and $f \circ \mu_{\mathcal{A}} = \mu_{\mathcal{B}} \circ (f \otimes f)$.

(2) Let $(\mathcal{A}, \mu, \alpha, \beta)$ be a BiHom-associative superalgebra and V be a $\mathbb{Z}_2$-graded vector space. Let $l,\ r\ :\ \mathcal{A} \to End(V)$ be two even linear maps and $\alpha_V,\ \beta_V\ :\ V \to V$ be two even linear maps. The tuple $(V, l, r, \alpha_V, \beta_V)$ is called an $\mathcal{A}$-bimodule if for all $x,\ y \in \mathcal{H}(\mathcal{A})$ and $v \in \mathcal{H}(V)$

$$\begin{aligned} l(\alpha(x))l(y) &= l(\mu(x, y))\beta_V \\ l(\alpha(x))r(y) &= r(\beta(y))l(x) \\ r(\mu(x, y))\alpha_V &= r(\beta(y))r(x) \\ l(\alpha(x))\alpha_V &= \alpha_V(l(x)) \\ r(\alpha(x))\alpha_V &= \alpha_V r(x) \\ l(\beta(x))\beta_V &= \beta_V l(x) \\ r(\beta(x))\beta_V &= \beta_V r(x) \end{aligned}$$

Moreover, if we give a product μ_V on V then, the tuple $(V, \mu_V, l, r, \alpha_V, \beta_V)$ is said to be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra if $(V, l, r, \alpha_V, \beta_V)$ is an $\mathcal{A}$-bimodule compatible with μ_V, that is, for all $x, y \in \mathcal{H}(\mathcal{A})$ and $u, v \in \mathcal{H}(V)$,

$$\begin{aligned} l(\alpha(x))\mu_V(u,v) &= \mu_V(l(x)(u), \beta_V(v)), \\ \mu_V(\alpha_V(u), r(x)(v)) &= r(\beta(x))(\mu_V(u,v)), \\ \mu_V(\alpha_V(u), l(x)(v)) &= \mu_V(r(x)(u), \beta_V(v)). \end{aligned}$$

(3) An even linear map $T : V \longrightarrow \mathcal{A}$ is called an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated to the bimodule $\mathbb{K}$-superalgebra $(V, \mu_V, l, r, \alpha_V, \beta_V)$ if it satisfies

$$\alpha \circ T = T \circ \alpha_V \ \ and \ \ \beta \circ T = T \circ \beta_V,$$

$$\mu(T(u), T(v)) = T\Big(l(T(u))v + (-1)^{|u||v|} r(T(v))u + \lambda \mu_V(u,v)\Big), \quad \forall \ \ u, v \in \mathcal{H}(V). \tag{1.1}$$

In particular, an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated to the bimodule $\mathbb{K}$-superalgebra $(\mathcal{A}, \mu_A, L_\mu, R_\mu, \alpha, \beta)$ is called a Rota-Baxter operator of weight λ on $\mathcal{A}$, that is, R satisfies for all $x, y \in \mathcal{H}(\mathcal{A})$ the following identity:

$$\mu(R(x), R(y)) = R\Big(\mu(R(x), y) + \mu(x, R(y)) + \lambda \mu(x,y)\Big) \tag{1.2}$$

We denote by a tuple $(\mathcal{A}, \mu, R, \alpha, \beta)$ the Rota-Baxter BiHom-associative superalgebra.

Definition 1.3.

(1) A BiHom-Lie superalgebra is a tuple $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$, an even linear map $[\ ,\ ] : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, $([\mathcal{A}_i, \mathcal{A}_j] \subseteq \mathcal{A}_{i+j}, \ \forall \ i, j \in \mathbb{Z}_2)$ and two even morphisms $\alpha, \beta : \mathcal{A} \to \mathcal{A}$ satisfying for all $x, y, z \in \mathcal{H}(\mathcal{A})$,

$$\alpha \circ \beta = \beta \circ \alpha, \tag{1.3}$$

$$[\beta(x), \alpha(y)] = -(-1)^{|x||y|}[\beta(y), \alpha(x)], \quad \text{(BiHom super-skew-symmetry)} \tag{1.4}$$

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}[\beta^2(x), [\beta(y), \alpha(z)]] = 0, \quad \text{(BiHom super-Jacobi identity)}. \tag{1.5}$$

(2) A representation of a BiHom-Lie superalgebra $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$ on a vector superspace V with respect to commuting even linear maps $\alpha_V, \beta_V : V \to V$ is an even linear map $\rho : \mathcal{A} \longrightarrow End(V)$, such that for all $x, y \in \mathcal{H}(\mathcal{A})$, the following equalities are satisfied

$$\rho(\alpha(x)) \circ \alpha_V = \alpha_V \circ \rho(x), \tag{1.6}$$

$$\rho(\beta(x)) \circ \beta_V = \beta_V \circ \rho(x), \tag{1.7}$$

$$\rho([\beta(x), y]) \circ \beta_V = \rho(\alpha\beta(x)) \circ \rho(y) - (-1)^{|x||y|}\rho(\beta(y)) \circ \rho(\alpha(x)). \tag{1.8}$$

The pair $(V, \rho, \alpha_V, \beta_V)$ is said to be an $\mathcal{A}$-module or a representation of $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$. The tuple $(V, [\ ,\ ]_V, \rho, \alpha_V, \beta_V)$, where $[\ ,\ ]_V$ is a BiHom super skew-symmetric bracket, is said to be an $\mathcal{A}$-module $\mathbb{K}$-superalgebra if, for $x \in \mathcal{H}(\mathcal{A})$ and $v, w \in \mathcal{H}(V)$

$$\rho(\beta^2(x))[\beta_V(v), \alpha_V(w)]_V = [\rho(\beta(x))(\alpha_V(v)), \beta_V^2(w)]_V + (-1)^{|x||v|}[\beta_V^2(v), \rho(\beta(x))(\alpha_V(w))]_V.$$

(3) Let $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$ be a regular BiHom-Lie superalgebra (i.e.α, β are invertible) and $(V, \rho, \alpha_V, \beta_V)$ be a representation where α_V, β_V are invertible. An even linear map $T : V \to A$ is called an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated with an $\mathcal{A}$-module $\mathbb{K}$-superalgebra $(V, [\ ,\ ]_V, \rho, \alpha, \beta)$, if it satisfies for all $u, v \in \mathcal{H}(V)$

$$\alpha \circ T = T \circ \alpha_V \ \ and \ \ \beta \circ T = T \circ \beta_V, \tag{1.9}$$

$$[T(u), T(v)] = T\Big(\rho(T(u))v - (-1)^{|u||v|}\rho(T(\alpha_V^{-1}\beta_V(v)))\alpha_V\beta_V^{-1}(u) + \lambda[u,v]_V\Big). \tag{1.10}$$

In particular, an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated to the bimodule $(\mathcal{A}, L_\circ, R_\circ, \alpha, \beta)$ is called a Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$, that is, R satisfies for all x, y, z in $\mathcal{H}(\mathcal{A})$

$$[R(x), R(y)] = R\Big([R(x), y] + [x, R(y)] + \lambda[x,y]\Big). \tag{1.11}$$

The tuple $(\mathcal{A}, [\ ,\ ], R, \alpha, \beta)$ refers to a Rota-Baxter BiHom-Lie superalgebra .

Definition 1.4. Let $(\mathcal{A}, [\ \ ,\ \ ], R, \alpha, \beta)$ and $(\mathcal{A}', [\ \ ,\ \ ]', R', \alpha', \beta')$ be two Rota-Baxter BiHom-Lie superalgebras. An even homomorphism $f : (\mathcal{A}, [\ ,\ ], R, \alpha, \beta) \longrightarrow (\mathcal{A}', [\ ,\ ]', R', \alpha', \beta')$ is said to be a morphism of two Rota-Baxter BiHom-Lie superalgebras if, for all $x, y \in \mathcal{H}(\mathcal{A})$,

$$\begin{aligned}
&f \circ \alpha = \alpha' \circ f \quad \textit{and} \quad f \circ \beta = \beta' \circ f,\\
&f \circ R = R' \circ f,\\
&f([x, y]) = [f(x), f(y)]'.
\end{aligned}$$

Proposition 1.1. Let $(\mathcal{A}, \mu, R, \alpha, \beta)$ be a Rota-Baxter regular BiHom-associative superalgebra of weight $\lambda \in \mathbb{K}$. Then the tuple $(\mathcal{A}, [\ \ ,\ \ ], R, \alpha, \beta)$, where $[x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(\alpha^{-1}\beta(y), \alpha\beta^{-1}(x))$, is a Rota-Baxter BiHom-Lie superalgebra of weight $\lambda \in \mathbb{K}$.

We introduce the notion of $\mathcal{O}$-operators of BiHom-pre-Lie superalgebras and study some properties over BiHom-Lie superalgebras and BiHom-pre-Lie superalgebras.

Definition 1.5 ([1])**.** Let $\mathcal{A}$ be a $\mathbb{Z}_2$ graded vector space and $\circ\ :\ \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ be an even binary operation. The pair $(\mathcal{A}, \circ)$ is called a pre-Lie superalgebra if, for x, y, z in $\mathcal{H}(\mathcal{A})$, the associator

$$as(x, y, z) = (x \circ y) \circ z - x \circ (y \circ z)$$

is super-symmetric in x and y, that is, $as(x, y, z) = (-1)^{|x||y|}as(y, x, z)$, or equivalently

$$(x \circ y) \circ z - x \circ (y \circ z) = (-1)^{|x||y|}\Big((y \circ x) \circ z - y \circ (x \circ z)\Big) \tag{1.12}$$

The identity (1.12) is called pre-Lie super-identity.

Definition 1.6. A BiHom-pre-Lie superalgebra is a tuple $(\mathcal{A}, \circ, \alpha, \beta)$ consisting of a $\mathbb{Z}_2$ graded vector space $\mathcal{A}$, an even binary operation $\circ : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and two even linear maps $\alpha, \beta : A \to A$ such that for any $x, y, z \in \mathcal{H}(\mathcal{A})$

$$\alpha\beta = \beta\alpha, \tag{1.13}$$

$$\alpha(x \circ y) = \alpha(x) \circ \alpha(y) \ \ \textit{and} \ \ \beta(x \circ y) = \beta(x) \circ \beta(y), \tag{1.14}$$

$$\begin{aligned}
&(\beta(x) \circ \alpha(y)) \circ \beta(z) - \alpha\beta(x) \circ (\alpha(y) \circ z)\\
&= (-1)^{|x||y|}\Big((\beta(y) \circ \alpha(x)) \circ \beta(z) - \alpha\beta(y) \circ \big(\alpha(x) \circ z\big)\Big).
\end{aligned} \tag{1.15}$$

The identity (1.15) is called BiHom-pre-Lie super-identity.

Proposition 1.2. Let $(\mathcal{A}, \circ)$ be a pre-Lie superalgebra and $\alpha,\ \beta : \mathcal{A} \longrightarrow \mathcal{A}$ two commuting even morphisms. Then $(\mathcal{A}, \circ_{\alpha,\beta}, \alpha, \beta)$ be a BiHom-pre-Lie superalgebra where

$$x \circ_{\alpha,\beta} y = \alpha(x) \circ \beta(y).$$

Proof. Let $x, y, z \in \mathcal{H}(A)$, we have

$$\begin{aligned}
as_{\alpha,\beta}(x, y, z) &= \big(\beta(x) \circ_{\alpha,\beta} \alpha(y)\big) \circ_{\alpha,\beta} \beta(z) - \alpha\beta(x) \circ_{\alpha,\beta} \big(\alpha(y) \circ_{\alpha,\beta} z\big)\\
&= \alpha\big(\beta(x) \circ_{\alpha,\beta} \alpha(y)\big) \circ \beta^2(z) - \alpha^2\beta(x) \circ \beta\big(\alpha(y) \circ_{\alpha,\beta} z\big)\\
&= \big(\alpha^2\beta(x) \circ \alpha^2\beta(y)\big) \circ \beta^2(z) - \alpha^2\beta(x) \circ \big(\alpha^2\beta(y) \circ \beta^2(z)\big).
\end{aligned}$$

suppose that $X = \alpha^2\beta(x)$, $Y = \alpha^2\beta(y)$ and $Z = \beta^2(z)$, then we get

$$\begin{aligned}
as_{\alpha,\beta}(x, y, z) &= (X \circ Y) \circ Z - X \circ (Y \circ Z)\\
&= (-1)^{|X||Y|}\Big((Y \circ X) \circ Z - Y \circ (X \circ Z)\Big)\\
&= (-1)^{|x||y|}\Big(\big(\alpha^2\beta(y) \circ \alpha^2\beta(x)\big) \circ \beta^2(z) - \alpha^2\beta(y) \circ \big(\alpha^2\beta(x) \circ \beta^2(z)\big)\Big)
\end{aligned}$$

$$= \quad (-1)^{|x||y|}\Big(\big(\beta(y)\circ_{\alpha,\beta}\alpha(x)\big)\circ_{\alpha,\beta}\beta(z)-\alpha\beta(y)\circ_{\alpha,\beta}\big(\alpha(x)\circ_{\alpha,\beta}z\big)\Big).$$

We conclude that $\circ_{\alpha,\beta}$ defines a BiHom-pre-Lie superalgebra structure on $\mathcal{A}$. □

Proposition 1.3. Let $(\mathcal{A},\circ,\alpha,\beta)$ be a regular BiHom-pre-Lie superalgebra . Define $[\ ,\ ]_C : \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ by $[x,y]_C = x\circ y-(-1)^{|x||y|}(\alpha^{-1}\beta(y))\circ(\alpha\beta^{-1}(x))$. Then $(\mathcal{A},[\ ,\ ]_C,\alpha,\beta)$ defines a BiHom-Lie superalgebra which is denoted by $\mathcal{A}^C$ and called the sub-adjacent BiHom-Lie superalgebra of $\mathcal{A}$ and $\mathcal{A}$ is also called a compatible BiHom-pre-Lie superalgebra structure on the BiHom-Lie superalgebra.

Definition 1.7. Let V be a $\mathbb{Z}_2$-graded vector space, $l,r:\mathcal{A}\longrightarrow End(V)$ be two even linear maps and $\alpha_V,\beta_V:V\longrightarrow V$ be two even linear maps. The quintuple (V,l,r,α_V,β_V) is said to be an $\mathcal{A}$-bimodule of $(\mathcal{A},\circ,\alpha,\beta)$ if, for $x,y\in\mathcal{H}(\mathcal{A})$ and $u\in\mathcal{H}(V)$,

$$\alpha_V l(x)(u)=l(\alpha(x)\alpha_V(u),\quad \beta_V l(x)(u)=l(\beta(x))\beta_V(u), \tag{1.16}$$

$$\alpha_V r(x)(u)=r(\alpha(x))\alpha_V(u), \tag{1.17}$$

$$r(\beta(x))\beta_V(u)=\beta_V r(y)(u), \tag{1.18}$$

$$\begin{aligned}&\Big(l(\beta(x)\circ\alpha(y))\beta_V-l(\alpha\beta(x))l(\alpha(y))\Big)(u)\\ &=(-1)^{|x||y|}\Big(l(\beta(y)\circ\alpha(x))\beta_V-l(\alpha\beta(y))l(\alpha(x))\Big)(u),\end{aligned} \tag{1.19}$$

$$\begin{aligned}&\Big(r(\beta(x))r(\alpha(y))\beta_V-r(\alpha(y)\circ x)\alpha_V\beta_V\Big)(u)\\ &(-1)^{|y||u|}\Big(r(\beta(x))l(\beta(y)\alpha_V-l(\alpha\beta(y))r(x)\alpha_V\Big)(u),\end{aligned} \tag{1.20}$$

Moreover, the tuple $(V,\circ_V,l,r,\alpha_V,\beta_V)$ is said to be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra if (V,l,r,α_V,β_V) is an $\mathcal{A}$-bimodule compatible with the multiplication $\circ_V$ on V, that is, for $x\in\mathcal{H}(\mathcal{A})$ and $u,v\in\mathcal{H}(V)$,

$$\begin{aligned}&l(\beta(x))(\alpha_V(u))\circ_V(\beta_V(v))-l(\alpha\beta(x))(\alpha_V(u)\circ_V v)\\ &=(-1)^{|x||u|}\Big(r(\alpha(x))(\beta_V(u))\circ_V\beta_V(v)-\alpha_V\beta_V(u)\circ_V l(\alpha(x))(v)\Big),\\ &\ r(\beta(x))(\beta_V(u)\circ_V\alpha_V(v))-\alpha_V\beta_V(u)\circ_V(r(z)(\alpha_V(v)))\\ &=(-1)^{|u||v|}\Big(r(\beta(x))(\beta_V(v)\circ_V\alpha_V(u))-\alpha_V\beta_V(v)\circ_V(r(x)(\alpha_V(u)))\Big).\end{aligned}$$

Remark 1.1. Let (V,l,r,α_V,β_V) be a representation of BiHom-pre-Lie superalgebra $(\mathcal{A},\circ,\alpha,\beta)$ then (V,l,α_V,β_V) is a representation of the BiHom-Lie superalgebra $(\mathcal{A},[\ ,\ ]_C,\alpha,\beta)$.

Let $(V,\circ_V,l,r,\alpha_V,\beta_V)$ be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra where α_V,β_V are invertible. An even linear map $T:V\longrightarrow\mathcal{A}$ is called an $\mathcal{O}$-operator of weight $\lambda\in\mathbb{K}$ associated to $(V,\circ_V,l,r,\alpha_V,\beta_V)$ if it satisfies for all $u,v\in\mathcal{H}(V)$:

$$\alpha\circ T=T\circ\alpha_V \ \ and \ \ \beta\circ T=T\circ\beta_V, \tag{1.21}$$

$$T(u)\circ T(v)=T\Big(l(T(u))v+(-1)^{|u||v|}r(T(\alpha_V^{-1}\beta_V(v))\alpha_V\beta_V^{-1}(u))+\lambda u\circ_V v\Big). \tag{1.22}$$

In particular, an $\mathcal{O}$-operator of weight $\lambda\in\mathbb{K}$ associated to the $\mathcal{A}$-bimodule $(\mathcal{A},L_\circ,R_\circ,\alpha,\beta)$ is called a Rota-Baxter operator of weight λ on $(\mathcal{A},\circ,\alpha,\beta)$, that is, R satisfies

$$R(x)\circ R(y)=R\Big(R(x)\circ y+x\circ R(y)+\lambda x\circ y\Big) \tag{1.23}$$

for all x,y in $\mathcal{H}(\mathcal{A})$.

Proposition 1.4. Let $(\mathcal{A},\circ,\alpha,\beta)$ be a BiHom-pre-Lie superalgebra and (V,l,r,α_V,β_V) be an $\mathcal{A}$-bimodule. Let $(\mathcal{A},[\ ,\ ]_C,\alpha,\beta)$ be the subadjacent BiHom-Lie superalgebra. If T is an $\mathcal{O}$-operator associated to (V,l,r,α_V,β_V), then T is an $\mathcal{O}$-operator of $(\mathcal{A},[\ ,\ ]_C,\alpha,\beta)$ associated to (V,l,α_V,β_V).

Theorem 1.1. Let $\mathcal{A}_1 = (\mathcal{A}, \circ, R, \alpha, \beta)$ be a Rota-Baxter regular BiHom-pre-Lie superalgebra of weight zero. Then $\mathcal{A}_2 = (\mathcal{A}, *, R, \alpha, \beta)$ is a Rota-Baxter regular BiHom-pre-Lie superalgebra of weight zero, where the even binary operation "$*$" is defined by

$$x * y \quad = \quad R(x) \circ y - (-1)^{|x||y|}\alpha^{-1}\beta(y) \circ R(\alpha\beta^{-1}(x)).$$

Proof. Since $\mathcal{A}_1 = (\mathcal{A}, \circ, \alpha, \beta)$ is a BiHom-pre-Lie superalgebra then we easily deduce that:

$$\alpha(x * y) = \alpha(x) * \alpha(y) \text{ and } \beta(x * y) = \beta(x) * \beta(y).$$

Let x, y and z be a homogeneous elements in $\mathcal{A}$. Then we have

$$\begin{aligned}
& \alpha\beta(x) * (\alpha(y) * z) \\
= \;& R(\alpha\beta(x)) \circ (\alpha(y) * z) - (-1)^{|x|(|y|+|z|)}(\beta(y) * \alpha^{-1}\beta(z)) \circ R(\alpha^2(x)) \\
= \;& R(\alpha\beta(x)) \circ (R(\alpha(y)) \circ z) - (-1)^{|y||z|}R(\alpha\beta(x) \circ (\alpha^{-1}\beta(z) \circ R(\alpha^2\beta^{-1}(y))) \\
- \;& (-1)^{|x|(|y|+|z|)}(R(\beta(y)) \circ \alpha^{-1}\beta(z)) \circ R(\alpha^2(x)) \\
+ \;& (-1)^{|x|(|y|+|z|)+|y||z|}(\alpha^{-2}\beta^2(z) \circ R(\alpha(y))) \circ R(\alpha^2(x)),
\end{aligned}$$

and

$$\begin{aligned}
& (\beta(x) * \alpha(y)) * \beta(z) \\
= \;& R(\beta(x) * \alpha(y)) \circ \beta(z) - (-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z) \circ R(\alpha(x) * \alpha^2\beta^{-1}(y)) \\
= \;& R(R(\beta(x)) \circ \alpha(y)) \circ \beta(z) - (-1)^{|x||y|}R(\beta(y) \circ R(\alpha(x))) \circ \beta(z) \\
- \;& (-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z) \circ R(R(\alpha(x)) \circ \alpha^2\beta^{-1}(y)) \\
+ \;& (-1)^{|z|(|x|+|y|)+|x||y|}\alpha^{-1}\beta^2(z) \circ R(\alpha(y) \circ R(\alpha^2\beta^{-1}(x))).
\end{aligned}$$

Subtracting the above terms, switching x and y, applying the fact that R is Rota Baxter operator on $(\mathcal{A}_1, \circ, \alpha, \beta)$, and then subtracting the result yields:

$$\begin{aligned}
& ass_{\mathcal{A}_2}(x, y, z) - (-1)^{|x||y|}ass_{\mathcal{A}_2}(y, x, z) \\
& = \Big(\big(R(\beta(x)) \circ R(\alpha(y))\big) \circ \beta(z) - R(\alpha\beta(x)) \circ \big(R(\alpha(y)) \circ z\big) \\
& -(-1)^{|x||y|}\Big(R\big(\beta(y) \circ R(\alpha(x))\big) \circ \beta(z) + R(\alpha\beta(y)) \circ \big(\alpha(x) \circ z\big)\Big) \\
& +(-1)^{|z|(|x|+|y|)}\Big[\Big(- \alpha^{-1}\beta^2(z) \circ \big(R(\alpha(x)) \circ R(\alpha^2\beta^{-1}(y))\big) + \big(\alpha^{-2}\beta^2(z) \circ R(\alpha(x))\big) \circ R(\alpha^2(y))\Big) \\
& +(-1)^{|x||z|}\Big(R(\alpha\beta(x)) \circ \big(\alpha^{-1}\beta(z) \circ R(\alpha^2\beta^{-1}(y))\big) - \big(R(\beta(x)) \circ R(\alpha^{-1}\beta(z))\big) \circ R(\alpha^2(y))\Big)\Big] \\
& (-1)^{|z|(|x|+|y|)+|x||y|}\Big[\Big(- \alpha^{-1}\beta^2(z) \circ \big(R(\alpha(y)) \circ R(\alpha^2\beta^{-1}(x))\big) + \big(\alpha^{-2}\beta^2(z) \circ R(\alpha(y))\big) \circ R(\alpha^2(x))\Big) \\
& +(-1)^{|y||z|}\Big(R(\alpha\beta(y)) \circ \big(\alpha^{-1}\beta(z) \circ R(\alpha^2\beta^{-1}(x))\big) - \big(R(\beta(y)) \circ R(\alpha^{-1}\beta(z))\big) \circ R(\alpha^2(x))\Big)\Big] \\
& = \Big(ass_{\mathcal{A}_1}\big(R(x), R(y), z\big) - (-1)^{|x||y|}ass_{\mathcal{A}_1}\big(R(y), R(x), z\big)\Big) \\
& +(-1)^{|z|(|x|+|y|)}\Big(ass_{\mathcal{A}_1}\big(\alpha^{-2}\beta(z), R(x), R(\alpha^2\beta^{-1}(y))\big) \\
& -(-1)^{|x||z|}ass_{\mathcal{A}_1}\big(R(x), \alpha^{-2}\beta(z), R(\alpha^2\beta^{-1}(y))\big)\Big) \\
& -(-1)^{|z|(|x|+|y|)+|x||y|}\Big(ass_{\mathcal{A}_1}\big(\alpha^{-2}\beta(z), R(y), R(\alpha^2\beta^{-1}(x))\big) \\
& -(-1)^{|y||z|}ass_{\mathcal{A}_1}\big(R(y), \alpha^{-2}\beta(z), R(\alpha^2\beta^{-1}(x))\big)\Big) \\
& = 0.
\end{aligned}$$

Then $\mathcal{A}_2$ is a BiHom-pre-Lie superalgebra, which ends the proof. $\square$

Now, we construct BiHom-pre-Lie superalgebras using $\mathcal{O}$-operators on BiHom-Lie superalgebras.

Proposition 1.5. Let $(\mathcal{A},[\ ,\ ],\alpha,\beta)$ be a BiHom-Lie superalgebra and $(V,\rho,\alpha_V,\beta_V)$ be a representation of $\mathcal{A}$. Suppose that $T: V \longrightarrow \mathcal{A}$ is an $\mathcal{O}$-operator of weight zero associated to $(V,\rho,\alpha_V,\beta_V)$. Then, the even bilinear map

$$u \circ v = \rho(T(u))v, \quad \forall \ \ u,v \in \mathcal{H}(V)$$

defines a BiHom-pre-Lie superalgebra structure on V.

Proof. Let u,v and w be in $\mathcal{H}(V)$. We have

$$\begin{aligned}
& (\beta_V(u)\circ\alpha_V(v))\circ\beta_V(w) - \alpha_V\beta_V(u)\circ(\alpha_V(v)\circ w)\\
= \quad & \rho\Big(T\big(\rho(T(\beta_V(u)))\alpha_V(v)\big)\Big)\beta_V(w)\\
- \quad & \rho\Big(T\big(\alpha_V\beta_V(u)\big)\Big)\rho\Big(T\big(\alpha_V(v)\big)\Big)w,
\end{aligned}$$

$$\begin{aligned}
& (-1)^{|u||v|}\Big((\beta_V(v)\circ\alpha_V(u))\circ\beta_V(w) - \alpha_V\beta_V(v)\circ(\alpha_V(u)\circ w)\Big)\\
= \quad & (-1)^{|u||v|}\Big[\rho\Big(T\big(\rho(T(\beta_V(v)))\alpha_V(u)\big)\Big)\beta_V(w)\\
- \quad & \rho\Big(T\big(\alpha_V\beta_V(v)\big)\Big)\rho\Big(T\big(\alpha_V(u)\big)\Big)w\Big]
\end{aligned}$$

Hence

$$\begin{aligned}
& (\beta_V(u)\circ\alpha_V(v))\circ\beta_V(w) - \alpha_V\beta_V(u)\circ(\alpha_V(v)\circ w)\\
& -(-1)^{|u||v|}\Big((\beta_V(v)\circ\alpha_V(u))\circ\beta_V(w) - \alpha_V\beta_V(v)\circ(\alpha_V(u)\circ w)\Big)\\
= \quad & \rho T\Big(\rho(T(\beta_V(u)))\alpha_V(v) - (-1)^{|u||v|}\rho(T(\beta_V(v)))\alpha_V(u)\Big)\beta_V(w)\\
& -\rho\Big(T\big(\alpha_V\beta_V(u)\big)\Big)\rho\Big(T\big(\alpha_V(v)\big)\Big)w + (-1)^{|u||v|}\rho\Big(T\big(\alpha_V\beta_V(v)\big)\Big)\rho\Big(T\big(\alpha_V(u)\big)\Big)w\\
= \quad & \rho(\underbrace{[T(\beta_V(u)),T(\alpha_V(v))]}_{=[\beta(T(u)),T(\alpha_V(v))]})\beta_V(w) - \rho\Big(T\big(\alpha_V\beta_V(u)\big)\Big)\rho\Big(T\big(\alpha_V(v)\big)\Big)w\\
& +(-1)^{|u||v|}\rho\Big(T\big(\alpha_V\beta_V(v)\big)\Big)\rho\Big(T\big(\alpha_V(u)\big)\Big)w\\
= \quad & \rho\Big(\alpha\beta(T(u))\Big)\rho\Big(T\big(\alpha_V(v)\big)\Big)w - (-1)^{|u||v|}\rho\Big(\beta(T\big(\alpha_V(v)\big)\Big)\rho\Big(\alpha(T(u))\Big)w\\
& \ -\rho\Big(T\big(\alpha_V\beta_V(u)\big)\Big)\rho\Big(T\big(\alpha_V(v)\big)\Big)w + (-1)^{|u||v|}\rho\Big(T\big(\alpha_V\beta_V(v)\big)\Big)\rho\Big(T\big(\alpha_V(u)\big)\Big)w\\
& \ = 0.
\end{aligned}$$

Since (1.6) - (1.8), $(\mathcal{A},\circ,\alpha_V,\beta_V)$ is a BiHom pre-Lie superalgebra. □

Remark 1.2. Let $(\mathcal{A},[\ ,\ ],\alpha,\beta)$ be a BiHom-Lie superalgebra and R be a Rota-Baxter operator of weight zero on $\mathcal{A}$. Then the even binary operation given by $x\circ y = [R(x),y]$, for all $x,y \in \mathcal{H}(\mathcal{A})$, defines a BiHom pre-Lie superalgebra structure on $\mathcal{A}$.

Example 1.1. Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 =< e_1,e_2,e_3 >$ be a 3-dimensional superspace such that $\mathcal{A}_0 =< e_1,e_2 >$ and $\mathcal{A}_1 =< e_3 >$. Define the bracket $[\ ,\ ] : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ by

$$[e_1,e_2] = e_1 \quad and \quad [e_1,e_3] = [e_2,e_3] = [e_3,e_3] = 0.$$

Let $\gamma,\ \lambda$ be two nonzero scalars in $\mathbb{K}$. Consider the maps $\alpha,\ \beta : \mathcal{A} \longrightarrow \mathcal{A}$ defined on the basis elements by

$$\begin{aligned}
\alpha(e_1) = \gamma e_1,\ \alpha(e_2) = e_2,\ \alpha(e_3) = \lambda e_3,\\
\beta(e_1) = \gamma e_1,\ \beta(e_2) = e_2,\ \beta(e_3) = -\lambda e_3.
\end{aligned}$$

It is easy to see that α, β defines two BiHom-Lie superalgebra homomorphisms and $\alpha \circ \beta = \beta \circ \alpha$. Also one may check that the bracket product $[\ ,\ ]$ and the structure maps α, β satisfy Eq. (1.4) and Eq. (1.5), then $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$ is a BiHom-Lie superalgebra.

The Rota-Baxter operators of weight zero on the BiHom-Lie superalgebras $(\mathcal{A}, [\ ,\ ], \alpha, \beta)$ with respect to the homogeneous basis $\{e_1, e_2, e_3\}$ are :

$$R_1(e_1) = 0\ ;\ R_1(e_2) = a_1e_1 + a_2e_2\ ;\ R_1(e_3) = c_1e_3$$

$$R_2(e_1) = b_1e_1 + b_2e_2\ ;\ R_2(e_2) = -\frac{b_1^2}{b_2}e_1 - b_1e_2\ ;\ R_2(e_3) = c_2e_3.$$

The constants a_i, b_i, c_i are parameters.

The operations $\circ_{R_i}$ where $i = 1, 2$ defined on the basis elements of $\mathcal{A}$ by

$$e_k \circ_{R_i} e_l = [R_i(e_k), e_l],\ k, l = 1, 2, 3,$$

define two families of BiHom-pre-Lie superalgebras structures on $\mathcal{A}$.

The Lie-admissible algebras were studied by A. A. Albert in 1948 and M. Goze and E. Remm, in 2004, they introduced the notion of G-associative algebras where G is a subgroup of the permutation group S_3 (see [35]). The graded case was studied by F. Ammar and A. Makhlouf in 2010, (see [6] for more details). In [1], the authors has been construct a functor from a full subcategory of the category of Rota-Baxter Lie-admissible (or associative) superalgebras to the category of pre-Lie superalgebras. In this part, we study this construction in the BiHom case.

Definition 1.8 ([46])**.** A BiHom-Lie admissible superalgebra is a regular BiHom superalgebra $(\mathcal{A}, \mu, \alpha, \beta)$ in which the super-commutator bracket, defined for all homogeneous x, y in $\mathcal{A}$ by

$$[x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(\alpha^{-1}\beta(y), \alpha\beta^{-1}(x)),$$

satisfies the BiHom super-Jacobi identity (1.5).

Theorem 1.2. Let $(\mathcal{A}, \cdot, R, \alpha, \beta)$ be a Rota-Baxter BiHom-Lie admissible superalgebra of weight zero. The even binary operation "$*$" was defined, for any homogeneous element $x, y \in \mathcal{A}$, by

$$x * y = [R(x), y].$$

Then $\mathcal{A}_L = (\mathcal{A}, *, \alpha, \beta)$ is a BiHom-pre-Lie superalgebras.

Proof. A direct consequence of Remark (1.2), since a Rota-Baxter operator on a BiHom-Lie admissible superalgebra is also a Rota-Baxter operator of its supercommutator BiHom-lie superalgebra. □

Theorem 1.3. Let $(\mathcal{A}, \cdot, R, \alpha, \beta)$ be a Rota-Baxter regular BiHom-associative superalgebra of weight $\lambda = -1$. Define the even binary operation "$\circ$" on any homogeneous element $x, y \in \mathcal{A}$ by

$$x \circ y = R(x) \cdot y - (-1)^{|x||y|}\alpha^{-1}\beta(y) \cdot R(\alpha\beta^{-1}(x)) - x \cdot y. \tag{1.24}$$

Then $\mathcal{A}_L = (\mathcal{A}, \circ, \alpha, \beta)$ is a BiHom-pre-Lie superalgebra.

Proof. For x, y, z in $\mathcal{H}(\mathcal{A})$, we have

$$\begin{aligned}
&(\beta(x) \circ \alpha(y)) \circ \beta(z)\\
=\ & R(R(\beta(x)) \cdot \alpha(y)) \cdot \beta(z) - (-1)^{|x||y|}R(\beta(y) \cdot R(\alpha(x))) \cdot \beta(z) - R(\beta(x) \cdot \alpha(y)) \cdot \beta(z)\\
-\ & (-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z) \cdot R(R(\alpha(x)) \cdot \alpha^2\beta^{-1}(y))\\
+\ & (-1)^{|z|(|x|+|y|)+|x||y|}\alpha^{-1}\beta^2(z) \cdot R(\alpha(y) \cdot R(\alpha^2\beta^{-1}(x)))
\end{aligned}$$

$$\begin{aligned}
&+ \quad (-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z)\cdot R\big(\alpha(x)\cdot\alpha^2\beta^{-1}(y)\big)\\
&- \quad \big(R(\beta(x))\cdot\alpha(y)\big)\cdot\beta(z)+(-1)^{|x||y|}\big(\beta(y)\cdot R(\alpha(x))\big)\cdot\beta(z)+\big(\beta(x)\cdot\alpha(y)\big)\cdot\beta(z).
\end{aligned}$$

and

$$\begin{aligned}
& \alpha\beta(x)\circ(\alpha(y)\circ z)\\
= \;& R\big(\alpha\beta(x)\big)\cdot\big(R(\alpha(y))\cdot z\big)-(-1)^{|y||z|}R\big(\alpha\beta(x)\big)\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(y))\big)\\
- \;& R\big(\alpha\beta(x)\big)\cdot\big(\alpha(y)\cdot z\big)-(-1)^{|x|(|y|+|z|)}\big(R(\beta(y))\cdot\alpha^{-1}\beta(z)\big)R\big(\alpha^2(x)\big)\\
+ \;& (-1)^{|x|(|y|+|z|)+|y||z|}\big(\alpha^{-2}\beta^2(z)\cdot R(\alpha(y))\big)\cdot R\big(\alpha^2(x)\big)\\
+ \;& (-1)^{|x|(|y|+|z|)}\big(\beta(y)\cdot\alpha^{-1}\beta(z)\big)\cdot R\big(\alpha^2(x)\big)-\alpha\beta(x)\cdot\big(R(\alpha(y))\cdot z\big)\\
+ \;& (-1)^{|y||z|}\alpha\beta(x)\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(y))\big)+\alpha\beta(x)\cdot\big(\alpha(y)\cdot z\big).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
& as_{\mathcal{A}_L}(x,y,z)-(-1)^{|x||y|}as_{\mathcal{A}_L}(y,x,z)\\
&=(\beta(x)\circ\alpha(y))\circ\beta(z)-\alpha\beta(x)\circ(\alpha(y)\circ z)-(-1)^{|x||y|}(\beta(y)\circ\alpha(x))\circ\beta(z)\\
&+(-1)^{|x||y|}\alpha\beta(y)\circ(\alpha(x)\circ z)\\
&=R\big(R(\beta(x))\cdot\alpha(y)\big)\cdot\beta(z)-(-1)^{|x||y|}R\big(\beta(y)\cdot R(\alpha(x))\big)\cdot\beta(z)-R\big(\beta(x)\cdot\alpha(y)\big)\cdot\beta(z)\\
&-(-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z)\cdot R\big(R(\alpha(x))\cdot\alpha^2\beta^{-1}(y)\big)+(-1)^{|z|(|x|+|y|)+|x||y|}\alpha^{-1}\beta^2(z)\cdot R\big(\alpha(y)\cdot R(\alpha^2\beta^{-1}(x))\big)\\
&+(-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z)\cdot R\big(\alpha(x)\cdot\alpha^2\beta^{-1}(y)\big)-\big(R(\beta(x))\cdot\alpha(y)\big)\cdot\beta(z)+(-1)^{|x||y|}\big(\beta(y)\cdot R(\alpha(x))\big)\cdot\beta(z)\\
&+\big(\beta(x)\cdot\alpha(y)\big)\cdot\beta(z)-R\big(\alpha\beta(x)\big)\cdot\big(R(\alpha(y))\cdot z\big)+(-1)^{|y||z|}R\big(\alpha\beta(x)\big)\cdot\big(\alpha^{-1}\beta(z)\cdot R\big(\alpha^2\beta^{-1}(y)\big)\big)\\
&+R\big(\alpha\beta(x)\big)\cdot\big(\alpha(y)\cdot z\big)+(-1)^{|x|(|y|+|z|)}\big(R(\beta(y))\cdot\alpha^{-1}\beta(z)\big)R(\alpha^2(x))\\
&-(-1)^{|x|(|y|+|z|)+|y||z|}\big(\alpha^{-2}\beta^2(z)\cdot R(\alpha(y))\big)\cdot R\big(\alpha^2(x)\big)-(-1)^{|x|(|y|+|z|)}\big(\beta(y)\cdot\alpha^{-1}\beta(z)\big)\cdot R\big(\alpha^2(x)\big)\\
&+\alpha\beta(x)\cdot\big(R(\alpha(y))\cdot z\big)-(-1)^{|y||z|}\alpha\beta(x)\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(y))\big)-\alpha\beta(x)\cdot\big(\alpha(y)\cdot z\big)\\
&-(-1)^{|x||y|}R\big(R(\beta(y))\cdot\alpha(x)\big)\cdot\beta(z)+R\big(\beta(x)\cdot R(\alpha(y))\big)\cdot\beta(z)-R\big(\beta(y)\cdot\alpha(x)\big)\cdot\beta(z)\\
&+(-1)^{|z|(|x|+|y|)+|x||y|}\alpha^{-1}\beta^2(z)\cdot R\big(R(\alpha(y))\cdot\alpha^2\beta^{-1}(x)\big)-(-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z)\cdot R\big(\alpha(x)\cdot R(\alpha^2\beta^{-1}(y))\big)\\
&-(-1)^{|z|(|x|+|y|)+|x||y|}\alpha^{-1}\beta^2(z)\cdot R\big(\alpha(y)\cdot\alpha^2\beta^{-1}(x)\big)+(-1)^{|x||y|}\big(R(\beta(y))\cdot\alpha(x)\big)\cdot\beta(z)\\
&-\big(\beta(x)\cdot R(\alpha(y))\big)\cdot\beta(z)-(-1)^{|x||y|}\big(\beta(y)\cdot\alpha(x)\big)\cdot\beta(z)+(-1)^{|x||y|}R\big(\alpha\beta(y)\big)\cdot\big(R(\alpha(x))\cdot z\big)\\
&-(-1)^{|y|(|x|+|z|)}R\big(\alpha\beta(y)\big)\cdot\big(\alpha^{-1}\beta(z)\cdot R\big(\alpha^2\beta^{-1}(x)\big)\big)-(-1)^{|x||y|}R\big(\alpha\beta(y)\big)\cdot\big(\alpha(x)\cdot z\big)\\
&-(-1)^{|x||z|}\big(R\big(\beta(x)\big)\cdot\alpha^{-1}\beta(z)\big)R(\alpha^2(y))+(-1)^{|z|(|x|+|y|)}\big(\alpha^{-2}\beta^2(z)\cdot R(\alpha(x))\big)\cdot R\big(\alpha^2(y)\big)\\
&+(-1)^{|x||z|}\big(\beta(x)\cdot\alpha^{-1}\beta(z)\big)\cdot R\big(\alpha^2(y)\big)-(-1)^{|x||y|}\alpha\beta(y)\cdot\big(R(\alpha(x))\cdot z\big)\\
&-(-1)^{|y||z|}\alpha\beta(y)\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(x))\big)+(-1)^{|x||y|}\alpha\beta(y)\cdot\big(\alpha(x)\cdot z\big)\\
&=\underbrace{\underbrace{\Big[R\big(R(\beta(x))\cdot\alpha(y)\big)+R\big(\beta(x)\cdot R(\alpha(y))\big)-R(\beta(x)\cdot\alpha(y))\Big]\cdot\beta(z)}_{=\big(R(\beta(x))\cdot R(\alpha(y))\big)\cdot\beta(z)}-R\big(\alpha\beta(x)\big)\cdot\big(R(\alpha(y))\cdot z\big)}_{=0}\\
&-(-1)^{|x||y|}\Big[R\big(R(\beta(y))\cdot\alpha(x)\big)+R\big(\beta(y)\cdot R(\alpha(x))\big)-R(\beta(y)\cdot\alpha(x))\Big]\cdot\beta(z)\\
&-(-1)^{|z|(|x|+|y|)}\alpha^{-1}\beta^2(z)\cdot\Big[R\big(R(\alpha(x))\cdot\alpha^2\beta^{-1}(y)\big)+R\big(\alpha(x)\cdot R(\alpha^2\beta^{-1}(y))\big)-R(\alpha(x)\cdot\alpha^2\beta^{-1}(y))\Big]\\
&+(-1)^{|x||y|}R\Big(\big(\alpha\beta(y)\big)\cdot\big(R(\alpha(x))\cdot z\big)\Big)+(-1)^{|z|(|x|+|y|)}\Big(\big(\alpha^{-2}\beta^2(z)\cdot R(\alpha(x))\big)\cdot R(\alpha^2(y))\Big)\\
&+(-1)^{|z|(|x|+|y|)+|x||y|}\alpha^{-1}\beta^2(z)\cdot\Big[R\big(R(\alpha(y))\cdot\alpha^2\beta^{-1}(x)\big)+R\big(\alpha(y)\cdot R(\alpha^2\beta^{-1}(x))\big)-R(\alpha(y)\cdot\alpha^2\beta^{-1}(x))\Big]\\
&-(-1)^{|z|(|x|+|y|)+|x||y|}\Big(\big(\alpha^{-2}\beta^2(z)\cdot R(\alpha(y))\big)\cdot R(\alpha^2(x))\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \Big[\big(R(\beta(x))\cdot\alpha(y)\big)\cdot\beta(z) - r(\alpha\beta(x))\cdot\big(\alpha(y)\cdot z\big)\Big]\\
&+ (-1)^{|x||y|}\Big[\big(R(\beta(y))\cdot\alpha(x)\big)\cdot\beta(z) - r(\alpha\beta(y))\cdot\big(\alpha(x)\cdot z\big)\Big]\\
&+ \Big[\alpha\beta(x)\cdot\big(R(\alpha(y))\cdot z\big) - \big(\beta(x)\cdot R(\alpha(y))\big)\cdot\beta(z)\Big]\\
&+ (-1)^{|x||y|}\Big[\big(\beta(y)\cdot R(\alpha(x))\big)\cdot\beta(z) - \alpha\beta(y)\cdot\big(R(\alpha(x))\cdot z\big)\Big]\\
&+ \Big[\big(\beta(x)\cdot\alpha(y)\big)\cdot\beta(z) - \alpha\beta(x)\cdot\big(\alpha(y)\cdot z\big)\Big] - (-1)^{|x||y|}\Big[\big(\beta(y)\cdot\alpha(x)\big)\cdot\beta(z) - \alpha\beta(y)\cdot\big(\alpha(x)\cdot z\big)\Big]\\
&- (-1)^{|y||z|}\Big[R(\alpha\beta(x))\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(y))\big) - \big(R(\beta(x))\cdot\alpha^{-1}\beta(z)\big)\cdot R(\alpha^2(y))\Big]\\
&- (-1)^{|x|(|y|+|z|)}\Big[R(\alpha\beta(y))\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(x))\big) - \big(R(\beta(y))\cdot\alpha^{-1}\beta(z)\big)\cdot R(\alpha^2(x))\Big]\\
&- (-1)^{|y||z|}\Big[\alpha\beta(x)\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(y))\big) - \big(\beta(x)\cdot\alpha^{-1}\beta(z)\big)\cdot R(\alpha^2(y))\Big]\\
&- (-1)^{|x|(|y|+|z|)}\Big[\alpha\beta(y)\cdot\big(\alpha^{-1}\beta(z)\cdot R(\alpha^2\beta^{-1}(x))\big) - \big(\beta(y)\cdot\alpha^{-1}\beta(z)\big)\cdot R(\alpha^2(x))\Big]\\
&= 0.
\end{aligned}
$$

Since the BiHom-associativity and the identity of Rota-Baxter operator (1.2) with $\lambda = -1$, the last equation holds. □

Corollary 1.1. Let $(\mathcal{A}, \cdot, R, \alpha, \beta)$ be a Rota-Baxter BiHom-associative superalgebra of weight $\lambda = -1$. Then R is still a Rota-Baxter operator of weight $\lambda = -1$ on the BiHom-pre-Lie superalgebra $(\mathcal{A}, \circ, \alpha, \beta)$ defined in (1.24).

As a consequence of Theorem 1.3 and Corollary 1.1, we have

Proposition 1.6. Let $(\mathcal{A}, \cdot, R, \alpha, \beta)$ be a Rota-Baxter regular BiHom-associative superalgebra of weight $\lambda = -1$. Then the binary operation defined, for any homogeneous elements $x,\ y$ in $\mathcal{A}$, by

$$
\begin{aligned}
[x,y] &= R(x)\cdot y - (-1)^{|x||y|}\alpha^{-1}\beta(y)\cdot R(\alpha\beta^{-1}(x)) - x\cdot y + x\cdot R(y)\\
&- (-1)^{|x||y|}R(\alpha^{-1}\beta(y))\cdot\alpha\beta^{-1}(x) + (1)^{|x||y|}\alpha^{-1}\beta(y)\cdot\alpha\beta^{-1}(x),
\end{aligned}
$$

defines a Rota-Baxter BiHom-Lie superalgebra structures on $(\mathcal{A}, [\ ,\ ], R, \alpha, \beta)$ of weight $\lambda = -1$.

Proof. Straightforward

□

2. BiHom-L-Dendriform superalgebras

The notion of L-dendriform algebra was introduced by C. Bai, L. Liu and X. Ni in 2010 (see [15]). The Hom case was introduced by Ibrahima Bakayoko (see [17]). In this section, we extend this notion and introduce BiHom-L-dendriform superalgebras. Then we study relationships between BiHom-associative superalgebras, BiHom-L-dendriform superalgebras and BiHom-pre-Lie superalgebras. Moreover, we introduce the notion of Rota-Baxter operator (of weight zero) on the $\mathcal{A}$-bimodule and we provide construction of associative bimodules from bimodules over BiHom-L-dendriform superalgebras.

2.1. BiHom-L-Dendriform superalgebras and BiHom-associative superalgebras

Definition 2.1. A BiHom-L-dendriform superalgebra is a quintuple $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$, two even linear maps $\rhd, \lhd : \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}$ and two commuting even linear maps $\alpha, \beta : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the following conditions (for all $x, y, z \in \mathcal{H}(\mathcal{A})$):

$$\alpha(x \rhd y) = \alpha(x) \rhd \alpha(y)\ ,\ \alpha(x \lhd y) = \alpha(x) \lhd \alpha(y), \tag{2.1}$$

$$\beta(x \rhd y) = \beta(x) \rhd \beta(y)\ ,\ \beta(x \lhd y) = \beta(x) \lhd \beta(y), \tag{2.2}$$

$$\begin{aligned} \alpha\beta(x) \rhd (\alpha(y) \rhd z) &= (\beta(x) \rhd \alpha(y)) \rhd \beta(z) + (\beta(x) \lhd \alpha(y)) \rhd \beta(z) \\ &+ (-1)^{|x||y|}\alpha\beta(y) \rhd (\alpha(x) \rhd z) - (-1)^{|x||y|}(\beta(y) \lhd \alpha(x)) \rhd \beta(z) \\ &- (-1)^{|x||y|}(\beta(y) \rhd \alpha(x)) \rhd \beta(z), \end{aligned} \tag{2.3}$$

$$\begin{aligned} \alpha\beta(x) \rhd (\alpha(y) \lhd z) &= (\beta(x) \rhd \alpha(y)) \lhd \beta(z) + (-1)^{|x||y|}\alpha\beta(y) \lhd (\alpha(x) \rhd z) \\ &+ (-1)^{|x||y|}\alpha\beta(y) \lhd (\alpha(x) \lhd z) - (-1)^{|x||y|}(\beta(y) \lhd \alpha(x)) \lhd \beta(z). \end{aligned} \tag{2.4}$$

Remark 2.1. If $\alpha = \beta = Id$, then $(\mathcal{A}, \rhd, \lhd)$ is an L-dendriform superalgebra.

Proposition 2.1. Let $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ be a regular BiHom-L-dendriform superalgebra (i.e.α, β are invertible).

(1) The even binary operation $\circ : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ given, for all $x, y \in \mathcal{H}(\mathcal{A})$, by

$$x \circ y \quad = \quad x \rhd y - (-1)^{|x||y|}\alpha^{-1}\beta(y) \lhd \alpha\beta^{-1}(x), \tag{2.5}$$

defines a BiHom-pre-Lie superalgebra $(\mathcal{A}, \circ, \alpha, \beta)$ which is called the associated vertical BiHom-pre-Lie superalgebra of $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ and $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ is called a compatible BiHom-L-dendriform superalgebra structure on the BiHom-pre-Lie superalgebra $(\mathcal{A}, \circ, \alpha, \beta)$.

(2) The even binary operation $\bullet : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ given by

$$x \bullet y = x \rhd y + x \lhd y, \quad \forall \;\; x, y \in \mathcal{H}(\mathcal{A}) \tag{2.6}$$

defines a BiHom-pre-Lie superalgebra $(\mathcal{A}, \bullet, \alpha, \beta)$ which is called the associated horizontal BiHom-pre-Lie superalgebra of $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ and $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ is called a compatible BiHom-L-dendriform superalgebra structure on the BiHom-pre-Lie superalgebra $(\mathcal{A}, \bullet, \alpha, \beta)$.

Corollary 2.1. Let $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ be a regular BiHom-L-dendriform superalgebra. Then the bracket

$$\begin{aligned} [x, y] &= x \circ y - (-1)^{|x||y|}\alpha^{-1}\beta(y) \circ \alpha\beta^{-1}(x) \\ &= x \rhd y - (-1)^{|x||y|}\alpha^{-1}\beta(y) \lhd \alpha\beta^{-1}(x) - (-1)^{|x||y|}\alpha^{-1}\beta(y) \rhd \alpha\beta^{-1}(x) + x \lhd y, \end{aligned} \tag{2.7}$$

defines a BiHom-Lie superalgebra structure on $\mathcal{A}$.

Remark 2.2.

$$\begin{aligned} \{x, y\} &= x \bullet y - (-1)^{|x||y|}\alpha^{-1}\beta(y) \bullet \alpha\beta^{-1}(x) \\ &= x \rhd y - (-1)^{|x||y|}\alpha^{-1}\beta(y) \lhd \alpha\beta^{-1}(x) - (-1)^{|x||y|}\alpha^{-1}\beta(y) \rhd \alpha\beta^{-1}(x) + x \lhd y \\ &= [x, y]. \end{aligned}$$

Then $\{,\}$ defines also a BiHom-Lie superalgebra structure on $\mathcal{A}$.

The below result allows us to obtain a BiHom-L-dendriform superalgebra from a given one by transposition.

Proposition 2.2. Let $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ be a regular BiHom-L-dendriform superalgebra. Define two even binary operations $\rhd^t, \lhd^t : A \otimes A \longrightarrow A$ by

$$x \rhd^t y := x \rhd y, \quad x \lhd^t y := -(-1)^{|x||y|}\alpha^{-1}\beta(y) \lhd \alpha\beta^{-1}(x) \tag{2.8}$$

Then $(A, \rhd^t, \lhd^t, \alpha, \beta)$ is a BiHom-L-dendriform superalgebra, and $\circ^t = \bullet$ and $\bullet^t = \circ$. The BiHom-L-dendriform superalgebra $(A, \rhd^t, \lhd^t, \alpha, \beta)$ is called the transpose of $(A, \rhd, \lhd, \alpha, \beta)$.

Proof. Straightforward □

Definition 2.2.

(1) Let $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ be a BiHom-L-dendriform superalgebra, V be a $\mathbb{Z}_2$-graded vector space, $l_\rhd, r_\rhd, l_\lhd, r_\lhd : \mathcal{A} \longrightarrow End(V)$ be four even linear maps and $\alpha_V, \beta_V : V \longrightarrow V$ be two even linear maps. The tuple $(V, l_\rhd, r_\rhd, l_\lhd, r_\lhd, \alpha_V, \beta_V)$ is an $\mathcal{A}$-bimodule if for any homogeneous elements $x, y \in \mathcal{A}$, the following identities are satisfied

$$l_\rhd([\beta(x), \alpha(y)])\beta_V = l_\rhd(\alpha\beta(x))l_\rhd(\alpha(y)) - (-1)^{|x||y|}l_\rhd(\alpha\beta(y))l_\rhd(\alpha(x)), \tag{2.9}$$

$$\begin{aligned} l_\lhd(\beta(x) \circ \alpha(y))\beta_V &= l_\rhd(\alpha\beta(x))l_\lhd(\alpha(y)) - (-1)^{|x||y|}l_\lhd(\alpha\beta(y))l_\lhd(\alpha(x)) \\ &\quad - (-1)^{|x||y|}l_\lhd(\alpha\beta(y))l_\rhd(\alpha(x)), \end{aligned} \tag{2.10}$$

$$\begin{aligned} r_\rhd(\alpha(x) \rhd y)\alpha_V\beta_V &= r_\rhd(\beta(y))r_\rhd(\alpha(x))\beta_V + r_\rhd(\beta(y))r_\lhd(\alpha(x))\beta_V \\ &\quad + (-1)^{|x||y|}l_\rhd(\alpha\beta(x))r_\rhd(y)\alpha_V - (-1)^{|x||y|}r_\rhd(\beta(y))l_\rhd(\beta(x))\alpha_V \\ &\quad - (-1)^{|x||y|}r_\rhd(\beta(y))l_\lhd(\beta(x))\alpha_V, \end{aligned} \tag{2.11}$$

$$\begin{aligned} r_\rhd(\alpha(x) \lhd y)\alpha_V\beta_V &= r_\lhd(\beta(y))r_\rhd(\alpha(x))\beta_V + (-1)^{|x||y|}l_\lhd(\alpha\beta(x))r_\rhd(y)\alpha_V \\ &\quad + (-1)^{|x||y|}l_\lhd(\alpha\beta(x))r_\lhd(y)\alpha_V - (-1)^{|x||y|}r_\lhd(\beta(y))l_\lhd(\beta(x))\alpha_V, \end{aligned} \tag{2.12}$$

$$\begin{aligned} r_\lhd(\alpha(x) \bullet y)\alpha_V\beta_V &= r_\lhd(\beta(y))r_\lhd(\alpha(x))\beta_V - (-1)^{|x||y|}r_\lhd(\beta(y))l_\rhd(\beta(x))\alpha_V \\ &\quad + (-1)^{|x||y|}l_\rhd(\alpha\beta(x))r_\lhd(y)\alpha_V. \end{aligned} \tag{2.13}$$

Moreover, The tuple $(V, \rhd_V, \lhd_V, l_\rhd, r_\rhd, l_\lhd, r_\lhd, \alpha_V, \beta_V)$ is an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra if the following identities are satisfied

$$\begin{aligned} &\alpha_V\beta_V(u) \rhd_V l_\rhd(\alpha(x))(v) - (-1)^{|x||u|}l_\rhd(\alpha\beta(x))(\alpha_V(u) \rhd_V v) \\ &= r_\rhd(\alpha(x))(\beta_V(u)) \rhd_V \beta_V(v) + r_\lhd(\alpha(x))(\beta_V(u)) \rhd_V \beta_V(v) \\ &- (-1)^{|x||u|}l_\lhd(\beta(x))(\alpha_V(u) \rhd_V \beta_V(v)) - (-1)^{|x||u|}l_\rhd(\beta(x))(\alpha_V(u) \rhd_V \beta_V(v)), \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\alpha_V\beta_V(u) \rhd_V r_\rhd(x)(\alpha_V(v)) - (-1)^{|x||u|}\alpha_V\beta_V(v) \rhd_V r_\rhd(x)(\alpha_V(u)) = r_\rhd(\beta(x))(\beta_V(u) \bullet_V \alpha_V(v)) \\ &- (-1)^{|x||u|}r_\rhd(\beta(x))(\beta_V(v) \bullet_V \alpha_V(u)), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &l_\lhd(\alpha\beta(x))(\alpha_V(u) \lhd_v v) = l_\rhd(\beta(x))(\alpha_V(u)) \lhd_V \beta_V(v) + (-1)^{|x||u|}\alpha_V\beta_V(u) \lhd_V l_\rhd(\alpha(x))(v) \\ &+ (-1)^{|x||u|}\alpha_V\beta_V(u) \lhd_V l_\lhd(\alpha(x))(v) - (-1)^{|x||u|}r_\lhd(\alpha(x))(\beta_V(u)) \lhd_V \beta_V(v). \end{aligned} \tag{2.16}$$

$$\begin{aligned} &\alpha_V\beta_V(u) \lhd_V l_\lhd(\alpha(x))(v) = r_\rhd(\alpha(x))(\beta_V(u)) \lhd_V \beta_V(v) + (-1)^{|x||u|}l_\lhd(\alpha\beta(x))(\alpha_V(u) \rhd_V v) \\ &+ (-1)^{|x||u|}l_\lhd(\alpha\beta(x))(\alpha_V(u) \lhd_v v) - (-1)^{|x||u|}l_\lhd(\beta(x))(\alpha_V(u)) \lhd_V \beta_V(v). \end{aligned} \tag{2.17}$$

(2) Let $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ be a BiHom-L-dendriform superalgebra and $(V, \rhd_V, \lhd_V, l_\rhd, r_\rhd, l_\lhd, r_\lhd, \alpha_V, \beta_V)$ be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra. An even linear map $T : V \longrightarrow \mathcal{A}$ is called an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated to $(V, \rhd_V, \lhd_V, l_\rhd, r_\rhd, l_\lhd, r_\lhd, \alpha_V, \beta_V)$ if T satisfies for any homogeneous elements u, v in V

$$\begin{aligned} \alpha \circ T &= T \circ \alpha_V \quad \text{and} \quad \beta \circ T = T \circ \beta_V, \\ T(u) \rhd T(v) &= T\Big(l_\rhd(T(u))v + (-1)^{|u||v|}r_\rhd(T(v))u + \lambda u \rhd_V v\Big), \\ T(u) \lhd T(v) &= T\Big(l_\lhd(T(u))v + (-1)^{|u||v|}r_\lhd(T(v))u + \lambda u \lhd_V v\Big). \end{aligned}$$

In particular, an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ of the BiHom-L-dendriform superalgebra $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ associated to the bimodule $(\mathcal{A}, l_\rhd, r_\rhd, l_\lhd, r_\lhd, \alpha_V, \beta_V)$ is called a Rota-Baxter operator (of weight λ) on $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$, that is, R satisfies for any homogeneous elements x, y in $\mathcal{A}$

$$\begin{aligned} \alpha \circ R &= R \circ \alpha \quad \textit{and} \quad \beta \circ R = R \circ \beta, \\ R(x) \rhd R(y) &= R\Big(R(x) \rhd y + x \rhd R(y) + \lambda x \rhd y\Big), \\ R(x) \lhd R(y) &= R\Big(R(x) \lhd y + x \lhd R(y) + \lambda x \lhd y\Big). \end{aligned}$$

The following theorem provides a construction of BiHom-L-dendriform superalgebras using $\mathcal{O}$-operators of BiHom-associative superalgebras.

Theorem 2.1. Let $(\mathcal{A}, \mu, \alpha, \beta)$ be a BiHom-associative superalgebra and $(V, l, r, \alpha_V, \beta_V)$ be a $\mathcal{A}$-bimodule. If T is an $\mathcal{O}$-operator of weight zero associated with $(V, l, r, \alpha_V, \beta_V)$, then there exists a BiHom-L-dendriform superalgebra structure on V defined by

$$u \rhd v = l(T(u))v, \quad u \lhd v = (-1)^{|u||v|} r(T(v))u, \quad \forall \;\; u, v \in \mathcal{H}(V). \tag{2.18}$$

Proof. By a direct computation, for any homogeneous elements u, v and w in V, we have

$$\begin{aligned} &(\beta_V(u) \rhd \alpha_V(v)) \rhd \beta_V(w) + (\beta_V(u) \lhd \alpha_V(v)) \rhd \beta_V(w) - \alpha_V\beta_V(u) \rhd (\alpha_V(v) \rhd w) \\ &+(-1)^{|u||v|}\alpha_V\beta_V(v) \rhd (\alpha_V(u) \rhd w) - (-1)^{|u||v|}(\beta_V(v) \lhd \alpha_V(u)) \rhd \beta_V(w) \\ &-(-1)^{|u||v|}(\beta_V(v) \rhd \alpha_V(u)) \rhd \beta_V(w) \\ &= l\Big(T(l(T(\beta_V(u))))\alpha_V(v)\Big)\beta_V(w) + (-1)^{|u||v|} l\Big(T(r(T(\alpha_V(v))))\beta_V(u)\Big)\beta_V(w) \\ &-l\Big(T(\alpha_V\beta_V(u))\Big)l\Big(T(\alpha_V(v))\Big)w + (-1)^{|u||v|} l\Big(T(\alpha_V\beta_V(v))\Big)l\Big(T(\alpha_V(u))\Big)w \\ &-l\Big(T(r(T(\alpha_V(u))))\beta_V(v)\Big)\beta_V(w) - (-)^{|u||v|} l\Big(T(l(T(\beta_V(v))))\alpha_V(u)\Big)\beta_V(w) \\ &= 0, \end{aligned}$$

and similarly, we have

$$\begin{aligned} &\alpha\beta(x) \rhd (\alpha(y) \lhd z) - (\beta(x) \rhd \alpha(y)) \lhd \beta(z) - (-1)^{|x||y|}\alpha\beta(y) \lhd (\alpha(x) \rhd z) \\ &-(-1)^{|x||y|}\alpha\beta(y) \lhd (\alpha(x) \lhd z) + (-1)^{|x||y|}(\beta(y) \lhd \alpha(x)) \lhd \beta(z) = 0. \end{aligned}$$

Therefore $(V, \rhd, \lhd, \alpha_V, \beta_V)$ is a BiHom-L-dendriform superalgebra. ☐

A direct consequence of Theorem 2.1 is the following construction of a BiHom-L-dendriform superalgebra from a Rota-Baxter operator (of weight zero) of a BiHom-associative superalgebra.

Corollary 2.2. Let $(\mathcal{A}, \mu, \alpha, \beta, R)$ be a Rota-Baxter BiHom-associative superalgebra of weight zero. Then, the even binary operations given by

$$x \rhd y = \mu(R(x), y), \qquad x \lhd y = \mu(x, R(y)), \qquad \forall \;\; x, y \in \mathcal{H}(\mathcal{A})$$

defines a BiHom-L-dendriform superalgebra structure on $\mathcal{A}$.

Definition 2.3. Let $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$ be a BiHom-L-dendriform superalgebra and $R : \mathcal{A} \longrightarrow \mathcal{A}$ be a Rota-Baxter operator of weight zero. A Rota-Baxter operator on $\mathcal{A}$-bimodule V (relative to R) is an even map $R_V : V \longrightarrow V$ such that for all homogeneous elements x in $\mathcal{A}$ and v in V

$$\begin{aligned} R(x) \rhd R_V(v) &= R_V\Big(R(x) \rhd v + x \rhd R_V(v)\Big), \\ R_V(v) \rhd R(x) &= R_V\Big(R_V(v) \rhd x + v \rhd R(x)\Big), \\ R(x) \lhd R_V(v) &= R_V\Big(R(x) \lhd v + x \lhd R_V(v)\Big), \end{aligned}$$

$$R_V(v) \lhd R(x) \quad = \quad R_V\Big(R_V(v) \lhd x + v \lhd R(x)\Big).$$

Proposition 2.3. Let $(\mathcal{A}, \mu, \alpha, \beta)$ be a BiHom-associative superalgebra, $R : \mathcal{A} \longrightarrow \mathcal{A}$ a Rota-Baxter operator on $\mathcal{A}$, V an $\mathcal{A}$-bimodule and R_V a Rota-Baxter operator on V. Define four actions of $\mathcal{A}$ on V by

$$x \rhd v = \mu(R(x), v), \quad v \rhd x = \mu(R_V(v), x), \quad x \lhd v = \mu(x, R_V(v)), \quad v \lhd x = \mu(v, R(x)).$$

Equipped with these actions, V becomes an $\mathcal{A}$-bimodule over the associated BiHom-L-dendriform superalgebra.

Corollary 2.3. Let $(V, l_\rhd, r_\rhd, l_\lhd, r_\lhd, \alpha_V, \beta_V)$ be an $\mathcal{A}$-bimodule of a BiHom-L-dendriform superalgebra $(\mathcal{A}, \rhd, \lhd, \alpha, \beta)$. Let $(\mathcal{A}, \mu, \alpha, \beta)$ be the associated BiHom-associative superalgebra. If T is an $\mathcal{O}$-operator associated to $(V, l_\rhd, r_\rhd, l_\lhd, r_\lhd, \alpha_V, \beta_V)$, then T is an $\mathcal{O}$-operator of $(\mathcal{A}, \mu, \alpha, \beta)$ associated to $(V, l_\rhd + l_\lhd, r_\rhd + r_\lhd, \alpha_V, \beta_V)$.

2.2. BiHom-L-dendriform superalgebras and BiHom-pre-Lie superalgebras

Conversely, we can construct BiHom-L-dendriform superalgebras from $\mathcal{O}$-operators of BiHom-pre-Lie superalgebras.

Theorem 2.2. Let $(\mathcal{A}, \circ, \alpha, \beta)$ be a BiHom-pre-Lie superalgebra and $(V, l, r, \alpha_V, \beta_V)$ be an $\mathcal{A}$-bimodule, where α, β are invertible. If T is an $\mathcal{O}$-operator of weight zero associated to $(V, l, r, \alpha_V, \beta_V)$, then there exists a BiHom-L-dendriform superalgebra structure on V defined by

$$u \rhd v = l(T(u))v, \quad u \lhd v = -r(T(u))v, \quad \forall \;\; u, v \in \mathcal{H}(V). \tag{2.19}$$

Therefore, there is a BiHom-pre-Lie superalgebra structure on V defined by

$$u \circ v = u \rhd v - (-1)^{|u||v|}\alpha_V^{-1}\beta_V(v) \lhd \alpha_V\beta_V^{-1}(u), \quad \forall \;\; u, v \in \mathcal{H}(V) \tag{2.20}$$

as the associated vertical BiHom-pre-Lie superalgebra of $(V, \rhd, \lhd, \alpha_V, \beta_V)$ and T is a homomorphism of BiHom-pre-Lie superalgebra.

Furthermore, $T(V) = \{T(v) \;\; / \;\; v \in V\} \subset \mathcal{A}$ is a BiHom-pre-Lie subsuperalgebra of $(\mathcal{A}, \circ, \alpha, \beta)$ and there is a BiHom-L-dendriform superalgebra structure on $T(V)$ given by

$$T(u) \rhd T(v) = T(u \rhd v), \quad T(u) \lhd T(v) = T(u \lhd v), \quad \forall \;\; u, v \in \mathcal{H}(V). \tag{2.21}$$

Moreover, the corresponding associated vertical BiHom-pre-Lie superalgebra structure on $T(V)$ is a BiHom-pre-Lie subsuperalgebra of $(\mathcal{A}, \circ, \alpha, \beta)$ and T is an homomorphism of BiHom-L-dendriform superalgebra.

Proof. For any homogeneous elements u, v and w in V, we have

$$\alpha_V\beta_V(u) \rhd (\alpha_V(v) \rhd w) = l(T(\alpha_V\beta_V(u)))l(T(\alpha_V(v)))w,$$
$$(\beta_V(u) \rhd \alpha_V(v)) \rhd \beta_V(w) = l\Big(T(l(T(\beta_V(u)))\alpha_V(v))\Big)\beta_V(w),$$
$$(\beta_V(u) \lhd \alpha_V(v)) \rhd \beta_V(w) = -l\Big(T(r(T(\beta_V(u)))\alpha_V(v))\Big)\beta_V(w),$$
$$(-1)^{|u||v|}\alpha_V\beta_V(v) \rhd (\alpha_V(u) \rhd w) = (-1)^{|u||v|}l(T(\alpha_V\beta_V(v)))l(T(\alpha_V(u)))w,$$
$$(-1)^{|u||v|}(\beta_V(v) \lhd \alpha_V(u)) \rhd \beta_V(w) = -(-1)^{|u||v|}l\Big(T(r(T(\beta_V(v)))\alpha_V(u))\Big)\beta_V(w),$$
$$(-1)^{|u||v|}(\beta_V(v) \rhd \alpha_V(u)) \rhd \beta_V(w) = (-1)^{|u||v|}l\Big(T(l(T(\beta_V(v)))\alpha_V(u))\Big)\beta_V(w).$$

Hence

$$\alpha_V\beta_V(u) \rhd (\alpha_V(v) \rhd w) - (\beta_V(u) \rhd \alpha_V(v)) \rhd \beta_V(w) - (\beta_V(u) \lhd \alpha_V(v)) \rhd \beta_V(w)$$
$$- (-1)^{|u||v|}\alpha_V\beta_V(v) \rhd (\alpha_V(u) \rhd w) + (-1)^{|u||v|}(\beta_V(u) \lhd \alpha_V(v)) \rhd \beta_V(w)$$

$$
\begin{aligned}
&+(-1)^{|u||v|}(\beta_V(v)\rhd\alpha_V(u))\rhd\beta_V(w)\\
&=l(T(\alpha_V\beta_V(u)))l(T(\alpha_V(v)))w-(-1)^{|u||v|}l(T(\alpha_V\beta_V(v)))l(T(\alpha_V(u)))w\\
&-l\Big(T(l(T(\beta_V(u)))\alpha_V(v))\Big)\beta_V(w)-(-1)^{|u||v|}l\Big(T(r(T(\beta_V(v)))\alpha_V(u))\Big)\beta_V(w)\\
&+(-1)^{|u||v|}l\Big(T(l(T(\beta_V(v)))\alpha_V(u))\Big)\beta_V(w)+l\Big(T(r(T(\beta_V(u)))\alpha_V(v))\Big)\beta_V(w)\\
&=l(T(\alpha_V\beta_V(u)))l(T(\alpha_V(v)))w-(-1)^{|u||v|}l(T(\alpha_V\beta_V(v)))l(T(\alpha_V(u)))w\\
&-l\Big(T(\beta_V(u))\circ T(\alpha_V(v))\Big)\beta_V(w)+(-1)^{|u||v|}l\Big(T(\beta_V(v))\circ T(\alpha_V(u))\Big)\beta_V(w)\\
&=0.
\end{aligned}
$$

The last equation follows from the equation (1.19). By the same way, we show that

$$
\begin{aligned}
&\alpha\beta(x)\rhd(\alpha(y)\lhd z)-(\beta(x)\rhd\alpha(y))\lhd\beta(z)-(-1)^{|x||y|}\alpha\beta(y)\lhd(\alpha(x)\rhd z)\\
&-(-1)^{|x||y|}\alpha\beta(y)\lhd(\alpha(x)\lhd z)+(-1)^{|x||y|}(\beta(y)\lhd\alpha(x))\lhd\beta(z)\\
&=0.
\end{aligned}
$$

Therefore, $(V,\rhd,\lhd,\alpha_V,\beta_V)$ is a BiHom-L-dendriform superalgebra. The other conditions follow easily. $\square$

A direct consequence of Theorem 2.2 is the following construction of a BiHom-L-dendriform superalgebra from a Rota-Baxter operator (of weight zero) of a BiHom-pre-Lie superalgebra.

Corollary 2.4. Let $(\mathcal{A},\circ,\alpha,\beta)$ be a regular BiHom-pre-Lie superalgebra and R be a Rota-Baxter operator on $\mathcal{A}$ (of weight zero). We define the even binary operations "$\rhd$" and "$\lhd$" on $\mathcal{A}$ by

$$x\rhd y=R(x)\circ y,\qquad x\lhd y=-(-1)^{|x||y|}y\circ R(x). \tag{2.22}$$

Then $(\mathcal{A},\rhd,\lhd,\alpha,\beta)$ is a BiHom-L-dendriform superalgebra.

Corollary 2.5. Let $(\mathcal{A},\circ,\alpha,\beta)$ be a BiHom-pre-Lie superalgebra. Then there exists a compatible BiHom-L-dendriform superalgebra structure on $(\mathcal{A},\circ,\alpha,\beta)$ such that $(\mathcal{A},\circ,\alpha,\beta)$ is the associated vertical BiHom-pre-Lie superalgebra if and only if there exists an invertible super $\mathcal{O}$-operator (of weight zero) of $(\mathcal{A},\circ,\alpha,\beta)$.

Proof. The first implication is obvious when we take the $\mathcal{O}$-operator identity.

Reciprocally, suppose that there exist an invertible $\mathcal{O}$-operator (of weight zero) of $(\mathcal{A},\circ,\alpha,\beta)$ associated to (V,l,r,α_V,β_V). By eqs. (2.19) and (2.21) of theorem (2.2), we define two even binary operations on $T(V)=\mathcal{A}$ given by

$$x\rhd_{\mathcal{A}} y=T(u)\rhd T(v)=T(u\rhd v)=T(l(T(u))v)=T(l(x)T^{-1}(y)),$$

and

$$x\lhd_{\mathcal{A}} y=T(u)\lhd T(v)=T(u\lhd v)=T(-r(T(u))v)=-T(r(x)T^{-1}(y)).$$

$\forall\ x=T(u),y=T(v)\in\mathcal{A}$.

$(\mathcal{A},\rhd_{\mathcal{A}},\lhd_{\mathcal{A}},\alpha,\beta)$ is a BiHom-L-dendriform superalgebra and $(\mathcal{A},\circ,\alpha,\beta)$ is the associated vertical BiHom-pre-Lie superalgebra. $\square$

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