International Journal of Informatics and Applied Mathematics
e-ISSN:2667-6990 Vol. 4, No. 2, 53-59

# An Infeasible Interior-point Algorithm for Monotone Linear Complementarity Problems

Welid Grimes and Mohamed Achache

Laboratoire de Mathématiques Fondamentales et Numériques. Sétif 1. Sétif 19000. Algérie. e-mail: welid.grimes@univ-setif.dz
welid.grimes@univ-setif.dz, achache_m@univ-setif.dz

**Abstract.** In this study, we implement a variant of infeasible interior-point algorithm for solving monotone linear complementarity problems (LCP). We first reformulate the monotone LCP as an minimization problem. Then a descent iterative method is applied to the latter. The descent direction is computed via the Newton method. However, for maintaining the positivity of iterates, a novel and efficient strategy is proposed. Some numerical results are reported to show the efficiency of our proposed approach.

**Keywords:** Monotone Linear Complementarity Problems · Newton Method · Logarithmic Penalty Approach

## 1 Introduction

Consider the following linear complementarity problem (LCP): find $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ such that

$$y = Mx + q,\, x \geq 0,\, y \geq 0,\, x^T y = 0 \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$. Here $x,\, y \geq 0$ says that $x$ and $y$ are nonnegative vectors in $\in \mathbb{R}^n$.

The importance of the LCP is due to its broad range of applications in the fields of economic, engineering such as game theory and mathematical programming (linear and quadratic programming).

There are a variety of solution approaches for LCP which have been studied intensively. Besides the simplicial Lemeke's method, the interior-point algorithms (IPM) gained more attention than others. Primal-dual path-following algorithms are the most attractive of IPM [10, 11]. We distinguish two type of primal-dual IPMs such as feasible [3, 5, 7] and infeasible [1, 4]. As is known, in all feasible interior-point algorithms, a crucial numerical problem is to find an initial strictly feasible for their starting. Therefore an infeasible-interior-point algorithm is suggested to remedy this drawback. The advantage of this latter is that they start with any positive initial point which is not necessarily strictly feasible. To do so, we use the so-called logarithmic penalty method to transform the LCP into an equivalent minimization problem. The first order optimality condition gives a perturbed nonlinear system of equations and the latter is then solved by using a descent Newton method. For maintaining the positivity of iterates during the algorithm process a novel and a practical strategy for computing the step-length is proposed.

The rest of the paper is built as follows. In section 2, we give a through description of the approach. In section 3, the prototype algorithm is described. In section 4, numerical results are reported. Finally, a conclusion is drawn the last section of the paper.

Throughout this paper the following notations are used. Given $x, y \in \mathbb{R}^n$, $x^T y = \sum_{i=1}^{n} x_i y_i$ is their usual scalar product whereas $xy = (x_i y_i)_{1\leq i\leq n}$ denotes their coordinate-wise product and the same as for the vectors $x/y = (x_i/y_i)_{1\leq i\leq n}$ and $x^{-1} = (1/x_i)_{1\leq i\leq n}$. The identity and the vector of all ones are denoted respectively by $I$ and $e$. Moreover, $\mathrm{diag}(x)$ is a diagonal matrix, which contains on his main diagonal the components of $x$ in the original order.

## 2 Description of the penalty method

The feasible set and the optimal solution set of the LCP, are denoted, respectively, by:

$$\mathcal{F} = \{(x, y) \in \mathbb{R}^{2n} : Mx + q = y,\ x \geq 0,\ y \geq 0\},$$

and

$$\mathrm{Sol(LCP)} = \{(x, y) \in \mathcal{F}\, :\, x^T y = 0\}.$$

Throughout the paper, we assume the following assumptions.
**Assumption 1.** There exists a couple of vectors $(x^0, y^0)$ such that:

$$y^0 = Mx^0 + q,\ x^0 > 0,\ y^0 > 0.$$

This assumption is often used to develop the interior-point method.
**Assumption 2.** The matrix $M$ is positive semi-definite, i.e., for all $v \in \mathbb{R}^n$, $v^T M v \geq 0$ then the LCP is called monotone LCP.
Under these assumptions, the Sol(LCP) is nonempty convex and compact set. We reformulate the LCP into the following equivalent minimization problem given by:

$$\min_{(x,\,y)} \left[ x^T y \quad s.t. \quad (x, y) \in \mathcal{F} \right], \tag{2}$$

in the sense that if $(x, y)$ is a complementarity solution, then it is a global minimizer to (2) with the objective value is zero.
Applying the logarithmic penalty (barrier) function to system (2), we get the following penalized problem associated to it:

$$\min_{(x,\,y)} \left[ x^T y - \mu \sum_{i=1}^{n} \ln x_i - \mu \sum_{i=1}^{n} \ln y_i \right] s.t.\ (x, y) \in \mathcal{F}. \tag{3}$$

It known that (3) has a unique solution for any $\mu > 0$. Moreover if $\mu$ goes to zero, we get an optimal solution (1) (see [10]). The role of the logarithmic penalty function is to keep $x$ and $y$ both positive, and if we denote by $\mathcal{L}(x, y, z, \mu)$ the Lagrangian for (3), we have that

$$\mathcal{L}(x, y, z, \mu) = x^T y - \mu \sum_{i=1}^{n} \ln x_i - \mu \sum_{i=1}^{n} \ln y_i - z^T (Mx + q - y) \tag{4}$$

The first order conditions for (4) yield the system of non linear equations

$$y - \mu X^{-1} e - M^T z = 0, \tag{5}$$

$$x - \mu Y^{-1} e + z = 0, \tag{6}$$

$$Mx + q - y = 0, \tag{7}$$

where $z \in \mathbb{R}^n$ , $X = \text{diag}(x)$ and $Y = \text{diag}(y)$ and under our assumptions, the matrix $XM^T + Y$ is non singular [6]. Using the equations in (5-7), we obtain the following system

$$\begin{pmatrix} Mx + q - y \\ XYe \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e \end{pmatrix}. \tag{8}$$

Hence, solving system (3) is equivalent to solving system (8).

### 2.1 Computation of Newton search direction

Now, by a direct application of Newton's method to (8), we get the following system of equations:

$$\begin{pmatrix} M & -I \\ Y & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} y - Mx - q \\ \mu e - xy \end{pmatrix}, \tag{9}$$

Under our hypothesis system (9) has a unique solution $(\Delta x, \Delta y)$, since the bloc matrix is nonsingular (Proposition 3.1 in [12]).

### 2.2 A novel strategy for determining the step-length

In this subsection, we are interested to determine a step-length along the Newton direction for keeping iterates positives. Let define $\alpha_-, \alpha_+, \beta_-$ and $\beta_+$ as:

$$\alpha_+ = \begin{cases} +\infty & \text{if} \quad (\Delta x)_i \leq 0, \ \forall\ i, \\ \min_i \dfrac{x_i}{(\Delta x)_i} & \text{if not;} \end{cases}$$

$$\alpha_- = \begin{cases} -\infty & \text{if} \quad (\Delta x)_i \geq 0, \ \forall\ i, \\ \max_i \dfrac{x_i}{(\Delta x)_i} & \text{if not;} \end{cases}$$

$$\beta_+ = \begin{cases} +\infty & \text{if} \quad (\Delta y)_i \leq 0, \ \forall\ i, \\ \min_i \dfrac{y_i}{(\Delta y)_i} & \text{if not;} \end{cases}$$

$$\beta_- = \begin{cases} -\infty & \text{if} \quad (\Delta y)_i \geq 0 \ \forall\ i, \\ \max_i \dfrac{y_i}{(\Delta y)_i} & \text{if not.} \end{cases}$$

In this work we choose $(\alpha, \beta)$ as the optimal solution of the linear program:

$$\Delta = \max\{\alpha\langle y, \Delta x\rangle + \beta\langle x, \Delta y\rangle \ : \ \rho\alpha_- < \alpha < \rho\alpha_+ \text{ and } \rho\beta_- < \beta < \rho\beta_+\},$$

where $\rho \in (0, 1)$.
Then it is easily seen that $\alpha$ and $\beta$ are as follows:

$$\alpha = \begin{cases} \rho\alpha_+ \text{ if } \langle y, \Delta x\rangle > 0 \\ \rho\alpha_- \text{ if } \langle y, \Delta x\rangle \leq 0, \end{cases} \tag{10}$$

and

$$\beta = \begin{cases} \rho\beta_+ \text{ if } \langle x, \Delta y\rangle > 0 \\ \rho\beta_- \text{ if } \langle x, \Delta y\rangle \leq 0. \end{cases} \tag{11}$$

By taking a step-length along this direction, we construct a new ordered pair $(x_+, y_+)$ with

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \beta\Delta y.$$

## 3 The prototype algorithm for monotone LCPs

In this section, we describe the Damped-Newton step IPMs for monotone LCP in Figure 1. First, we use an accuracy parameter $\varepsilon > 0$ and a barrier default $\hat{\mu} > 0$. The algorithm starts by a positive point $(x^0 > 0, y^0 > 0)$ not necessarily strictly feasible. Using the obtained search directions $(\Delta x, \Delta y)$ from (9) and the step-length $(\alpha, \beta)$ from (10) and (11) and we take a Damped-Newton step, the algorithm produces a new iterate $(x_+, y_+) = (x+\alpha\Delta x, y+\beta\Delta y)$. Then, it updates the barrier parameter $\mu$ to $\theta\mu$ where $0 < \theta < 1$ and target a new $\mu$-center and so on. This procedure is repeated until the stopping criterion $\max(\|z\|, \|s\|) \leq \varepsilon$ is satisfied where $z = X^{-1}\Delta x$, $s = Y^{-1}\Delta y$. If $\alpha = \beta = 1$, we get the full-Newton steps IPMs for monotone LCPs.

An accuracy parameter $\varepsilon > 0$ and a barrier default $\hat{\mu} > 0$ are given.

1. An initial positive point $(x^0, y^0) > 0$ and $\mu > \hat{\mu} > 0$.
2. Compute $(\Delta x, \Delta y)$ from (9), $z = X^{-1}\Delta x$ and $s = Y^{-1}\Delta y$.
3. If $\max(\|z\|, \|s\|) > \varepsilon$. Then determine $(\alpha, \beta)$ following (10) and (11) . Set $x := x + \alpha\Delta x$ and $y := y + \beta\Delta y$ and return to 2.
4. If $\max(\|z\|, \|s\|) \leq \varepsilon$ we have a good approximation.
   (a) If $\mu \geq \hat{\mu}$, decrease $\mu$ ($\mu := \theta\mu$) with $0 < \theta < 1$ and go to 2.
   (b) If $\mu < \hat{\mu}$, STOP. We have obtained a good approximation of the optimal solution for LCP.

**Fig 1. The prototype algorithm for monotone LCPs**

## 4 Numerical results

In this section, we test our algorithm on some examples of monotones LCPs. We implemented the algorithm on software **MATLAB 7.9** and run on a **PC** with **CPU 2.13 GHz** and **2G RAM** memory and double precision format. In the implementation, our accuracy is set to $\varepsilon = 10^{-8}$. Also in view of the influence of barrier parameter $\mu$ and $\theta$, different values of them are used in order to improve the performances of algorithm. Also in the numerical tables, we display the following notations: "Iter" and "CPU" to denote the number of iterations and the elapsed times, respectively.

**Problem 1.** The data of the following monotone LCP problem is given by

$$M = \begin{pmatrix} 7 & 13 & 6 & 8 & 7 \\ 13 & 34 & 19 & 20 & 24 \\ 6 & 19 & 12 & 10 & 15 \\ 8 & 20 & 10 & 14 & 13 \\ 7 & 24 & 15 & 13 & 22 \end{pmatrix}, \; q = \begin{pmatrix} -3.2 \\ -17 \\ -7.4 \\ -8 \\ -11.2 \end{pmatrix}.$$

An exact solution of Problem 1, is:

$$x_{\text{exact}} = (0,\, 0.5,\, 0,\, 0,\, 0)^T,\; y_{\text{exact}} = (3.3,\, 0,\, 2.1,\, 2,\, 0.8)^T.$$

The numerical results with different values of $\mu$ and $\theta$ are shown in Table 1.

**Table 1.** Numerical results for Problem 1. where $\rho = 1$

| $\mu_0 \rightarrow$ | 0.5 | | 0.05 | | 0.005 | | 0.0005 | | 0.00005 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\theta \downarrow$ | Iter | CPU | Iter | CPU | Iter | CPU | Iter | CPU | Iter | CPU |
| 0.9 | 153 | 0.3774 | 132 | 0.3282 | 110 | 0.2799 | 88 | 0.2362 | 66 | 0.1780 |
| 0.7 | 46 | 0.1642 | 39 | 0.1121 | 33 | 0.0930 | 26 | 0.0742 | 20 | 0.0537 |
| 0.5 | 24 | 0.0585 | 20 | 0.0551 | 17 | 0.0403 | 14 | 0.0384 | 10 | 0.0302 |
| 0.3 | 14 | 0.0338 | 12 | 0.0215 | 10 | 0.0194 | 8 | 0.0173 | 6 | 0.0158 |
| 0.1 | 8 | 0.0173 | 7 | 0.0164 | 6 | 0.0158 | 5 | 0.0143 | 4 | 0.0035 |

**Problem 2.** The monotone LCP problem in this example is given by

$$M = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 \\ -1 & 4 & -2 & \ddots & 0 \\ 0 & -1 & 4 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix},\; q = \begin{pmatrix} -1 \\ 1 \\ \vdots \\ 1 \\ -1 \end{pmatrix}.$$

An exact solution of Problem 1, is:

$$x_{\text{exact}} = (0.25,\, 0,\, \cdots,\, 0,\, 0.25)^T,\; y_{\text{exact}} = (0,\, 0.75,\, 1,\, \cdots,\, 1,\, 0.5,\, 0)^T.$$

The numerical results with different values of $n$ and $\mu$ are shown in Table 2.

**Table 2.** Numerical results for Problem 2. where $\theta = 0.1$ and $\rho = 0.9$

| $\mu_0 \rightarrow$ | 0.5 | | 0.05 | | 0.005 | | 0.0005 | | 0.00005 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n \downarrow$ | Iter | CPU | Iter | CPU | Iter | CPU | Iter | CPU | Iter | CPU |
| 5 | 8 | 0.0073 | 7 | 0.0065 | 6 | 0.0059 | 5 | 0.0037 | 4 | 0.0030 |
| 10 | 8 | 0.0093 | 7 | 0.0081 | 6 | 0.0073 | 5 | 0.0062 | 4 | 0.0038 |
| 25 | 8 | 0.0232 | 7 | 0.0214 | 6 | 0.0181 | 5 | 0.0168 | 4 | 0.0091 |
| 50 | 8 | 0.0472 | 7 | 0.0384 | 6 | 0.0327 | 5 | 0.0285 | 4 | 0.0224 |
| 100 | 8 | 0.4225 | 7 | 0.3993 | 6 | 0.3453 | 5 | 0.2789 | 4 | 0.1468 |
| 500 | 8 | 8.0997 | 6 | 6.4651 | 6 | 5.6456 | 5 | 4.7452 | 4 | 4.0035 |
| 1000 | 8 | 241.0415 | 7 | 215.1133 | 6 | 189.6889 | 5 | 155.1797 | 4 | 123.2585 |

## 5 Conclusion

In this study, we presented a variant of an infeasible interior-point algorithm for solving monotone LCP. For its numerical implementation an efficient step length is suggested. Moreover, our preliminary numerical results are very encouraging. Future work, we may extended this new idea for the class of sufficient LCPs.

## Acknowledgment

This work has been supported by: La Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT-MESRS). Algérie.

## References

1. Achache, M.: A numerical implementation of an infeasible interior-point algorithm for linear complementarity problem. Fast East Journal of Applied Mathematics **13**(2), 117–126 (2003)
2. Achache, M.: A weighted-path-following method for the linear complementarity problem. Studia Univ. Babeş-Bolyai. Ser. Informatica **49**(1), 61–73 (2004)
3. Achache, M.: A new primal-dual path-following method for convex quadratic programming. Comput. Appl. Math. **25**(1), 97–110 (2006)
4. Achache, M., Roumili, H., Keraghel, A.: A numerical study of an infeasible primal-dual path-following algorithm for linear programming. Applied Mathematics and Computation **186**(2), 1472–1479 (2007)
5. Achache, M.: Complexity analysis and numerical implementation of a short-step primal-dual algorithm for linear complementarity problems. Appl. Math. Comput. **216**(7), 1889–1895 (2010)
6. Cottle, R.W., Pang, J.S., and Stone, R.E.: The Linear Complementarity Problem. Academic Press, San Diego (1992)
7. Grimes, W.: Path-following interior-point algorithm for monotone linear complementarity problems. Asian European Journal of Mathematics, (2021)
8. Haddou, M., Migot, T., Omer, J.: A generalized direction in interior-point method for monotone linear complementarity problems. Optimization Letters **13**, 35–53 (2019)
9. Mansouri, H., Pirhaji, M.: A Polynomial Interior-Point Algorithm for Monotone Linear Complementarity Problems. J. Optim. Theory Appl. **157**, 451–461 (2013)
10. Wright, S.J.: Primal-Dual Interior Point Methods. SIAM, Philadelphia, USA (1997)
11. Ye Y.: Interior-point algorithms. Theory and Analysis. Jhon-Wiley. Sons, Chichester, UK (1997)
12. Zhang, L., Xu, Y.: A full-Newton step primal dual interior-point algorithm for linear complementarity problem. J. Inf. Comput. Sci **5**(4), 305–313 (2010)