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EIGENVALUE PROBLEMS FOR A CLASS OF STURM-LIOUVILLE OPERATORS ON TWO DIFFERENT TIME SCALES

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Abstract. In this study, we consider a boundary value problem generated by the Sturm-Liouville equation with a frozen argument and with non-separated boundary conditions on a time scale. Firstly, we present some solutions and the characteristic function of the problem on an arbitrary bounded time scale. Secondly, we prove some properties of eigenvalues and obtain a formulation for the eigenvalues-number on a finite time scale. Finally, we give an asymptotic formula for eigenvalues of the problem on another special time scale: $\mathbb{T} = [\alpha, \delta_1] \cup [\delta_2, \beta]$.

1. Introduction

A Sturm-Liouville equation with a frozen argument has the form

$$-y''(t) + q(t)y(a) = \lambda y(t),$$

where $q(t)$ is the potential function, a is the frozen argument and λ is the complex spectral parameter. The spectral analysis of boundary value problems generated with this equation is studied in several publications [3], [15], [16], [26], [33] and references therein. This kind problems are related strongly to non-local boundary value problems and appear in various applications [4], [12], [31] and [38].

A Sturm-Liouville equation with a frozen argument on a time scale $\mathbb{T}$ can be given as

$$-y^{\Delta\Delta}(t) + q(t)y(a) = \lambda y^{\sigma}(t), \ t \in \mathbb{T}^{\kappa^2} \tag{1}$$

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where $y^{\Delta\Delta}$ and σ denote the second order Δ-derivative of y and forward jump operator on $\mathbb{T}$, respectively, $q(t)$ is a real-valued continuous function, $a \in \mathbb{T}^{\kappa} := \mathbb{T}\backslash\left(\rho\left(\sup\mathbb{T}\right), \sup\mathbb{T}\right]$, $y^{\sigma}(t) = y(\sigma(t))$ and $\mathbb{T}^{\kappa^2} = \left(\mathbb{T}^{\kappa}\right)^{\kappa}$.

Spectral properties the classical Sturm-Liouville problem on time scales were given in various publications (see e.g. [1], [2], [5]- [9], [11], [17]- [25], [27]- [30], [34]- [37], [39] and references therein). However, there is no any publication about the Sturm-Liouville equation with a frozen argument on an arbitrary time scale.

In the present paper, we consider a boundary value problem which is generated by equation (1) and the following boundary conditions

$$U(y) \quad : \quad = a_{11}y\left(\alpha\right) + a_{12}y^{\Delta}\left(\alpha\right) + a_{21}y\left(\beta\right) + a_{22}y^{\Delta}\left(\beta\right) \tag{2}$$

$$V\left(y\right) \quad : \quad = b_{11}y\left(\alpha\right) + b_{12}y^{\Delta}\left(\alpha\right) + b_{21}y\left(\beta\right) + b_{22}y^{\Delta}\left(\beta\right) \tag{3}$$

where $\alpha = \inf\mathbb{T}$, $\beta = \rho(\sup\mathbb{T})$, $\alpha \neq \beta$ and a_{ij}, $b_{ij} \in \mathbb{R}$ for $i, j = 1, 2$. We aim to give some properties of some solutions and eigenvalues of (1)-(3) for two different cases of $\mathbb{T}$

For the basic notation and terminology of time scales theory, we recommend to see [10], [13], [14] and [32].

2. Preliminaries

Let $S(t,\lambda)$ and $C(t,\lambda)$ be the solutions of (1) under the initial conditions

$$S(a,\lambda) \quad = \quad 0,\ S^{\Delta}(a,\lambda) = 1, \tag{4}$$

$$C(a,\lambda) \quad = \quad 1,\ C^{\Delta}(a,\lambda) = 0, \tag{5}$$

respectively. Clearly, $S(t,\lambda)$ and $C(t,\lambda)$ satisfy

$$\begin{aligned} S^{\Delta\Delta}(t,\lambda) + \lambda S^{\sigma}(t,\lambda) &= 0 \\ C^{\Delta\Delta}(t,\lambda) + \lambda C^{\sigma}(t,\lambda) &= q(t), \end{aligned}$$

respectively and so these functions and their Δ-derivatives are entire on λ for each fixed t (see [34]).

Lemma 1. *Let $\varphi(t,\lambda)$ be the solution of (1) under the initial conditions $\varphi(a,\lambda) = \delta_1$, $\varphi^{\Delta}(a,\lambda) = \delta_2$ for given numbers δ_1, δ_2. Then $\varphi(t,\lambda) = \delta_1 C(t,\lambda) + \delta_2 S(t,\lambda)$ is valid on $\mathbb{T}$.*

Proof. It is clear that the function $y(t,\lambda) = \delta_1 C(t,\lambda) + \delta_2 S(t,\lambda)$ is the solution of the initial value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + \lambda y^{\sigma}(t) &= q(t)\delta_1 \\ y(a,\lambda) &= \delta_1 \\ y^{\Delta}(a,\lambda) &= \delta_2. \end{aligned}$$

We obtain by taking into account uniqueness of the solution of an initial value problem that $y(t,\lambda) = \varphi(t,\lambda)$. □

Consider the function

$$\Delta(\lambda) : \det\begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix}. \tag{6}$$

It is obvious $\Delta(\lambda)$ is also entire.

Theorem 1. *The zeros of the function $\Delta(\lambda)$ coincide with the eigenvalues of the problem (1)-(3).*

Proof. Let λ_0 be an eigenvalue and $y(t,\lambda_0) = \delta_1 C(t,\lambda_0) + \delta_2 S(t,\lambda_0)$ is the corresponding eigenfunction, then $y(t,\lambda_0)$ satisfies (2) and (3). Therefore,

$$\delta_1 U(C(t,\lambda_0)) + \delta_2 U(S(t,\lambda_0)) = 0,$$
$$\delta_1 V(C(t,\lambda_0)) + \delta_2 V(S(t,\lambda_0)) = 0.$$

It is obvious that $y(t,\lambda_0) \neq 0$ iff the coefficients-determinant of the above system vanishes, i.e., $\Delta(\lambda_0) = 0$. $\square$

Since $\Delta(\lambda)$ is an entire function, eigenvalues of the problem (1)-(3) are discrete.

3. Eigenvalues of (1)-(3) on a Finite Time Scale

Let $\mathbb{T}$ be a finite time scale such that there are m (or r) many elements which are larger (or smaller) than a in $\mathbb{T}$. Assume $m \geq 1$, $r \geq 0$ and $r + m \geq 2$. It is clear that the number of elements of $\mathbb{T}$ is $n = m + r + 1$. We can write $\mathbb{T}$ as follows

$$\mathbb{T} = \left\{\rho^r(a), \rho^{r-1}(a), ..., \rho^2(a), \rho(a), a, \sigma(a), \sigma^2(a), ..., \sigma^{m-1}(a), \sigma^m(a)\right\},$$

where $\sigma^j = \sigma^{j-1} \circ \sigma$, $\rho^j = \rho^{j-1} \circ \rho$ for $j \geq 2$, $\rho^r(a) = \alpha$ and $\sigma^{m-1}(\alpha) = \beta$.

Lemma 2. *i) If $r \geq 3$ and $m \geq 2$, the following equalities hold for all λ*

$$S(\alpha,\lambda) = (-1)^r \mu^\rho(a)\left[\mu^{\rho^2}(a)\mu^{\rho^3}(a)...\mu^{\rho^r}(a)\right]^2 \lambda^{r-1} + O\left(\lambda^{r-2}\right)$$
$$S^\sigma(\alpha,\lambda) = (-1)^{r-1} \mu^\rho(a)\left[\mu^{\rho^2}(a)\mu^{\rho^3}(a)...\mu^{\rho^{r-1}}(a)\right]^2 \lambda^{r-2} + O\left(\lambda^{r-3}\right)$$
$$S(\beta,\lambda) = S^{\sigma^{m-1}}(a,\lambda) = (-1)^m\left[\mu(a)\mu^\sigma(a)...\mu^{\sigma^{m-3}}(a)\right]^2 \lambda^{m-2}\mu^{\sigma^{m-2}}(a) + O\left(\lambda^{m-3}\right)$$
$$S^\sigma(\beta,\lambda) = S^{\sigma^m}(a,\lambda) = (-1)^{m+1}\left[\mu(a)\mu^\sigma(a)...\mu^{\sigma^{m-2}}(a)\right]^2 \lambda^{m-1}\mu^{\sigma^{m-1}}(a) + O\left(\lambda^{m-2}\right)$$
$$C(\alpha,\lambda) = (-1)^r\left[\mu^\rho(a)\mu^{\rho^2}(a)...\mu^{\rho^r}(a)\right]^2 \lambda^r + O\left(\lambda^{r-1}\right)$$
$$C^\sigma(\alpha,\lambda) = (-1)^{r-1}\left[\mu^\rho(a)\mu^{\rho^2}(a)...\mu^{\rho^{r-1}}(a)\right]^2 \lambda^{r-1} + O\left(\lambda^{r-2}\right)$$

$$C(\beta,\lambda) = C^{\sigma^{m-1}}(a,\lambda) = (-1)^m\mu(a)\left[\mu^\sigma(a)\mu^{\sigma^2}(a)...\mu^{\sigma^{m-3}}(a)\right]^2\mu^{\sigma^{m-2}}(a)\lambda^{m-2} + O\left(\lambda^{m-3}\right)$$
$$C^\sigma(\beta,\lambda) = C^{\sigma^m}(a,\lambda) = (-1)^{m+1}\mu(a)\left[\mu^\sigma(a)\mu^{\sigma^2}(a)...\mu^{\sigma^{m-2}}(a)\right]^2\mu^{\sigma^{m-1}}(a)\lambda^{m-1} + O\left(\lambda^{m-2}\right),$$

where $O(\lambda^l)$ *denotes a polynomial whose degree is* l.

ii) If $r \in \{0,1,2\}$ *or* $m \in \{0,1\}$, *degrees of all above functions are vanish.*

Proof. It is clear from $f^{\sigma}(t) = f(t)+\mu(t)f^{\Delta}(t)$ that $S^{\sigma}(a,\lambda) = \mu(a)$ and $C^{\sigma}(a,\lambda) = 1$. On the other hand, since $S(t,\lambda)$ and $C(t,\lambda)$ satisfy (1) then the following equalities hold for each $t \in \mathbb{T}^{\kappa}$ and for all λ.

$$\begin{aligned} S^{\sigma^2}(t,\lambda) &= \left(1+\frac{\mu(t)}{\mu^{\sigma}(t)} - \lambda\mu(t)\mu^{\sigma}(t)\right)S^{\sigma}(t,\lambda) \\ &\quad -\frac{\mu^{\sigma}(t)}{\mu(t)}S(t,\lambda) \end{aligned} \tag{7}$$

$$\begin{aligned} C^{\sigma^2}(t,\lambda) &= \left(-\mu(t)\mu^{\sigma}(t)\lambda + 1 + \frac{\mu(t)}{\mu^{\sigma}(t)}\right)C^{\sigma}(t,\lambda) \\ &\quad -\frac{\mu^{\sigma}(t)}{\mu(t)}C(t,\lambda) + \mu(t)\mu^{\sigma}(t)q(t) \end{aligned} \tag{8}$$

It can be calculated from (7) and (8) that

$$\begin{aligned} S^{\sigma^j}(a,\lambda) &= (-1)^{j+1}\left(\mu(a)\mu^{\sigma}(a)...\mu^{\sigma^{j-2}}(a)\right)^2 \mu^{\sigma^{j-1}}(a)\lambda^{j-1} \\ &\quad +O\left(\lambda^{j-2}\right) \end{aligned} \tag{9}$$

$$\begin{aligned} S^{\rho^j}(a,\lambda) &= (-1)^j \mu^{\rho}(a)\left(\mu^{\rho^2}(a)\mu^{\rho^3}(a)...\mu^{\rho^j}(a)\right)^2 \lambda^{j-1} \\ &\quad +O\left(\lambda^{j-2}\right) \end{aligned} \tag{10}$$

$$\begin{aligned} C^{\sigma^k}(a,\lambda) &= (-1)^{k+1}\mu(a)\left(\mu^{\sigma}(a)\mu^{\sigma^2}(a)...\mu^{\sigma^{k-2}}(a)\right)^2 \mu^{\sigma^{k-1}}(a)\lambda^{k-1} \\ &\quad +O\left(\lambda^{k-2}\right) \end{aligned} \tag{11}$$

$$\begin{aligned} C^{\rho^k}(a,\lambda) &= (-1)^k\left(\mu^{\rho}(a)\mu^{\rho^2}(a)...\mu^{\rho^k}(a)\right)^2 \lambda^k \\ &\quad +O\left(\lambda^{k-1}\right) \end{aligned} \tag{12}$$

for $j = 2,3,...m$ and $k = 2,3,...,r$. Using (9)-(12) and taking into account $\alpha = \rho^r(a)$ and $\beta = \sigma^{m-1}(\alpha)$ we have our desired relations. □

Corollary 1. $\deg C(\alpha,\lambda)S^{\sigma}(\beta,\lambda) = \begin{cases} r+m-1, & r>0 \text{ and } m>1 \\ 1, & \text{the other cases} \end{cases}$.

Lemma 3. *The following equlaties hold for all* $\lambda \in \mathbb{C}$.

$$\begin{aligned} S^{\sigma}(\alpha,\lambda)C(\alpha,\lambda) - S(\alpha,\lambda)C^{\sigma}(\alpha,\lambda) &= A\lambda^{\delta} + O\left(\lambda^{\delta-1}\right) \\ S^{\sigma}(\beta,\lambda)C(\beta,\lambda) - S(\beta,\lambda)C^{\sigma}(\beta,\lambda) &= B\lambda^{\gamma} + O\left(\lambda^{\gamma-1}\right) \end{aligned}$$

where $A = (-1)^r \mu(\alpha) \mu^{\rho}(a) \left[\mu^{\rho^2}(a) ... \mu^{\rho^{r-1}}(a)\right]^2 \mu^{\rho^r}(a) q(\alpha)$,

$B = (-1)^{m-1} \mu(\beta) \left[\mu(a) \mu^{\sigma}(a) ... \mu^{\sigma^{m-2}}(a)\right]^2 q(\rho(\beta))$,

$\delta = \begin{cases} r-2, & r \geq 3 \\ 0, & r < 3 \end{cases}$ *and* $\gamma = \begin{cases} m-2, & m \geq 3 \\ 0, & m < 3. \end{cases}$

Proof. Consider the function

$$\varphi(t,\lambda) := \frac{1}{\mu(t)} \left[S^{\sigma}(t,\lambda) C(t,\lambda) - S(t,\lambda) C^{\sigma}(t,\lambda)\right] \tag{13}$$

It is clear that

$$\varphi(t,\lambda) := \left[S^{\Delta}(t,\lambda) C(t,\lambda) - S(t,\lambda) C^{\Delta}(t,\lambda)\right] = W\left[C(t,\lambda), S(t,\lambda)\right]$$

and it is the solution of initial value problem

$$\begin{aligned} \varphi^{\Delta}(t) &= -q(t) S^{\sigma}(t,\lambda) \\ \varphi(a) &= 1 \end{aligned}$$

Therefore, we can obtain the following relations

$$\varphi^{\sigma}(t,\lambda) = \varphi(t,\lambda) - \mu(t) q(t) S^{\sigma}(t,\lambda), \tag{14}$$

$$\varphi^{\rho}(t,\lambda) = \varphi(t,\lambda) + \mu^{\rho}(t) q(\rho(t)) S(t,\lambda). \tag{15}$$

By using (9), (10), (14) and (15), the proof is completed. □

Corollary 2. *i)* $\deg\left(S^{\sigma}(\alpha,\lambda) C(\alpha,\lambda) - S(\alpha,\lambda) C^{\sigma}(\alpha,\lambda)\right) < \deg C(\alpha,\lambda) S^{\sigma}(\beta,\lambda)$,

ii) $\deg\left(S^{\sigma}(\beta,\lambda) C(\beta,\lambda) - S(\beta,\lambda) C^{\sigma}(\beta,\lambda)\right) < \deg C(\alpha,\lambda) S^{\sigma}(\beta,\lambda)$.

The next theorem gives the number of eigenvalues of the problem (1)-(3) on $\mathbb{T}$. Recall $n = m + r + 1$ denotes the number of elements of $\mathbb{T}$ and put
$A = \begin{pmatrix} a_{11}\mu(\alpha) - a_{12} & b_{11}\mu(\alpha) - b_{12} \\ a_{22} & b_{22} \end{pmatrix}$.

Theorem 2. *If* $\det A \neq 0$*, the problem (1)-(3) has exactly* $n-2$ *many eigenvalues with multiplications, otherwise the eigenvalues-number of (1)-(3) is least than* $n-2$.

Proof. Since $\mathbb{T}$ is finite, $\Delta(\lambda)$ is a polinomial and its degree gives the number eigenvalues of the problem. It can be calculated from (6)-(14) that

$$\begin{aligned}\Delta(\lambda) &= \frac{1}{\mu(\alpha)\mu(\beta)} \det \begin{pmatrix} a_{11}\mu(\alpha) - a_{12} & b_{11}\mu(\alpha) - b_{12} \\ a_{22} & b_{22} \end{pmatrix} C(\alpha,\lambda) S^{\sigma}(\beta,\lambda) \\ &+ \frac{1}{\mu(\alpha)} \det \begin{pmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{pmatrix} (S^{\sigma}(\alpha,\lambda) C(\alpha,\lambda) - S(\alpha,\lambda) C^{\sigma}(\alpha,\lambda)) \\ &+ \frac{1}{\mu(\beta)} \det \begin{pmatrix} a_{21} & a_{22} \\ b_{21} & b_{22} \end{pmatrix} (S^{\sigma}(\beta,\lambda) C(\beta,\lambda) - S(\beta,\lambda) C^{\sigma}(\beta,\lambda)) \\ &+ O(\lambda^{n+m-2}).\end{aligned}$$

According to Corollary 1 and Corollary 2, if $\det A \neq 0$,
$\deg \Delta(\lambda) = \deg C(\alpha,\lambda) S^{\sigma}(\beta,\lambda) = m + r - 1 = n - 2.$ □

Corollary 3. *i) The eigenvalues-number of (1)-(3) depends only on the elements-number of $\mathbb{T}$ and the coefficients of the boundary conditions (2) and (3). On the other hand, it does not depend on $q(t)$ and a (neither value nor location of a on $\mathbb{T}$). ii) If $\det A \neq 0$, the eigenvalues-number of (1)-(3) and the elements-number of $\mathbb{T}$ determine uniquely each other.*

Remark 1. *As is known, all eigenvalues of the classical Sturm-Liouville problem with separated boundary conditions on time scales are real and algebraicly simple [2]. However, the Sturm-Liouville problem with the frozen argument may have non-real or non-simple eigenvalues even if it is equipped with separated boundary conditions.*

We end this section with two example problems that have non-real or non-simple eigenvalues.

Example 1. *Consider the following problem on $\mathbb{T} = \{0,1,2,3,4,5\}$.*

$$L_1 : \begin{cases} -y^{\Delta\Delta}(t) + q_1(t) y(3) = \lambda y^{\sigma}(t), \ t \in \{0,1,2,3\} \\ y^{\Delta}(0) = 0 \\ y^{\Delta}(4) + y(4) = 0, \end{cases}$$

where $q_1(t) = \begin{cases} 0 & t = 0 \\ 1 & t = 1 \\ 0 & t = 2 \\ 2 & t = 3 \end{cases}$. *Eigenvalues of L_1 are $\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$,*
$\lambda_3 = \frac{3}{2} + \frac{1}{2}\sqrt{5}$, $\lambda_4 = \frac{3}{2} - \frac{1}{2}\sqrt{5}$.

Example 2. *Consider the following problem on $\mathbb{T} = \{0,1,2,3,4,5\}$.*

$$L_2 : \begin{cases} -y^{\Delta\Delta}(t) + q_2(t) y(3) = \lambda y^{\sigma}(t), \ t \in \{0,1,2,3\} \\ y^{\Delta}(0) + 2y(0) = 0 \\ y^{\Delta}(4) + y(4) = 0, \end{cases}$$

where $q_2(t)=\begin{cases} -1 & t=0 \\ 2 & t=1 \\ 0 & t=2 \\ 1 & t=3 \end{cases}$ *. Eigenvalues of* L_2 *are* $\lambda_1=\lambda_2=\lambda_3=2,\ \lambda_4=3.$

4. Eigenvalues of (1)-(3) on the Time Scale $\mathbb{T}=[\alpha,\delta_1]\cup[\delta_2,\beta]$

In this section, we investigate eigenvalues of the problem (1)-(3) on another special time scale: $\mathbb{T}=[\alpha,\delta_1]\cup[\delta_2,\beta]$, where $\alpha<a<\delta_1<\delta_2<\beta$. We assume that $a\in(\alpha,\delta_1)$. The similar results can be obtained in the case when $a\in(\delta_2,\beta)$.

The following relations are valid on $[\alpha,\delta_1]$ (see [15]).

$$\begin{aligned} S(t,\lambda) &= \frac{\sin\sqrt{\lambda}\,(t-a)}{\sqrt{\lambda}} \\ C(t,\lambda) &= \cos\sqrt{\lambda}\,(t-a)+\int_a^t \frac{\sin\sqrt{\lambda}\,(t-\xi)}{\sqrt{\lambda}}q(\xi)d\xi \end{aligned}$$

The following asymptotic relations for the solutions $S(t,\lambda)$ and $C(t,\lambda)$ can be proved by using a method similar to that in [35].

$$S(t,\lambda)=\begin{cases} \dfrac{\sin\sqrt{\lambda}\,(t-a)}{\sqrt{\lambda}}, & t\in[\alpha,\delta_1], \\ \delta^2\sqrt{\lambda}\cos\sqrt{\lambda}\,(\delta_1-a)\sin\sqrt{\lambda}(\delta_2-t)+O\left(\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{16}$$

$$S^{\Delta}(t,\lambda)=\begin{cases} \cos\sqrt{\lambda}\,(t-a), & t\in[\alpha,\delta_1), \\ -\delta^2\lambda\cos\sqrt{\lambda}\,(\delta_1-a)\cos\sqrt{\lambda}(\delta_2-t)+O\left(\sqrt{\lambda}\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{17}$$

$$C(t,\lambda)=\begin{cases} \cos\sqrt{\lambda}\,(t-a)+O\left(\dfrac{1}{\sqrt{\lambda}}\exp|\tau|\,|t-a|\right), & t\in[\alpha,\delta_1], \\ -\delta^2\lambda\sin\sqrt{\lambda}\,(\delta_1-a)\sin\sqrt{\lambda}(\delta_2-t)+O\left(\sqrt{\lambda}\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{18}$$

$$C^{\Delta}(t,\lambda)=\begin{cases} -\sqrt{\lambda}\sin\sqrt{\lambda}\,(t-a)+O\left(\exp|\tau|\,|t-a|\right), & t\in[\alpha,\delta_1), \\ \delta^2\lambda^{3/2}\sin\sqrt{\lambda}\,(\delta_1-a)\cos\sqrt{\lambda}(\delta_2-t)+O\left(\lambda\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{19}$$

where $\delta=\delta_2-\delta_1$, $\tau=\mathrm{Im}\sqrt{\lambda}$ and O denotes Landau's symbol.

Lemma 4. *The following equlaties hold for all* $\lambda\in\mathbb{C}$ *and* $t\in\mathbb{T}$.

$$C^{\Delta}(t,\lambda)S\,(t,\lambda)-C(t,\lambda)S^{\Delta}\,(t,\lambda)=O\left(\sqrt{\lambda}\exp|\tau|\,(\beta-\alpha-\delta)\right)$$

Proof. It is clear the function

$$\varphi(t,\lambda) := C^{\Delta}(t,\lambda)S(t,\lambda) - C(t,\lambda)S^{\Delta}(t,\lambda)$$

satisfies initial value problem

$$\begin{aligned} \varphi^{\Delta}(t) &= q(t)\,S^{\sigma}(t,\lambda),\ t\in[\alpha,\delta_1] \\ \varphi(a) &= 1 \end{aligned}$$

and

$$\begin{aligned} \varphi^{\Delta}(t) &= q(t)\,S^{\sigma}(t,\lambda),\ t\in[\delta_2,\beta] \\ \varphi(\delta_2) &= \varphi(\delta_1) + \delta q(\delta_1)S(\delta_2,\lambda). \end{aligned}$$

Hence, we get proof by using (16). □

Theorem 3. *i) The problem (1)-(3) on* $\mathbb{T} = [\alpha,\delta_1]\cup[\delta_2,\beta]$ *has countable many eigenvalues such as* $\{\lambda_n\}_{n\geq 0}$.
ii) The numbers $\{\lambda_n\}_{n\geq 0}$ *are real for sufficiently large* n.
iii) If $a_{22}b_{12} - a_{12}b_{22} \neq 0$ *and* $\beta - \delta_2 = \delta_1 - \alpha$*, the following asymptotic formula holds for* $n\to\infty$.

$$\sqrt{\lambda_n} = \frac{(n-1)\pi}{2(\beta-\delta_2)} + O\left(\frac{1}{n}\right) \tag{20}$$

Proof. The proof of (i) is obvious, since $\Delta(\lambda)$ is entire on λ.

By calculating directly, we get

$$\begin{aligned} \Delta(\lambda) &= \det\begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix} \\ &= (a_{22}b_{12} - a_{12}b_{22})\left[C^{\Delta}(\beta,\lambda)S^{\Delta}(\alpha,\lambda) - C^{\Delta}(\alpha,\lambda)S^{\Delta}(\beta,\lambda)\right] + \\ &\quad +(a_{22}b_{21} - a_{21}b_{22})\left[C^{\Delta}(\beta,\lambda)S(\beta,\lambda) - C(\beta,\lambda)S^{\Delta}(\beta,\lambda)\right] + \\ &\quad +(a_{12}b_{11} - a_{11}b_{12})\left[C^{\Delta}(\alpha,\lambda)S(\alpha,\lambda) - C(\alpha,\lambda)S^{\Delta}(\alpha,\lambda)\right] \\ &\quad +O\left(\lambda\exp|\tau|(\beta-\alpha-\delta)\right). \end{aligned}$$

It follows from (16)-(19) and Lemma 4 that

$$\begin{aligned} \Delta(\lambda) &= (a_{22}b_{12} - a_{12}b_{22})\delta^2\lambda^{3/2}\sin\sqrt{\lambda}(\delta_1-\alpha)\cos\sqrt{\lambda}(\beta-\delta_2) \\ &\quad +O\left(\lambda\exp|\tau|(\beta-\alpha-\delta)\right) \end{aligned}$$

is valid for $|\lambda|\to\infty$. Thus, we obtain the proof of (ii).

Since $a_{22}b_{12} - a_{12}b_{22} \neq 0$ and $\beta-\delta_2 = \delta_1-\alpha$, the numbers $\{\lambda_n\}_{n\geq 0}$ are roots of

$$\lambda^2\frac{\sin 2\sqrt{\lambda}(\beta-\delta_2)}{\sqrt{\lambda}} + O\left(\lambda\exp 2|\tau|(\beta-\delta_2)\right) = 0. \tag{21}$$

Now, we consider the region

$$G_n := \{\lambda\in\mathbb{C} : \lambda = \rho^2, |\rho| < \frac{n\pi}{2(\beta-\delta_2)} + \varepsilon\}$$

where ε is sufficiently small number. There exist some positive constants C_ε such that, $\left|\lambda^2 \frac{\sin 2\sqrt{\lambda}(\beta-\delta_2)}{\sqrt{\lambda}}\right| \geq C_\varepsilon |\lambda|^{3/2} \exp 2|\tau|(\beta-\delta_2)$ for sufficiently large $\lambda \in \partial G_n$. Therefore, by applying Rouche's theorem to (21) on G_n, we can show that (20) holds for sufficiently large n. □

Remark 2. *Since $\mu(\alpha) = 0$ in the considered time scale, the term $a_{22}b_{12} - a_{12}b_{22}$ is not another than $detA$ in section 3.*

5. Conclusion

In this paper, we give some spectral properties of a boundary value problem generated by the Sturm-Liouville equation with a frozen argument and with non-separated boundary conditions on time scales. We focus on two different time scales: a finite set and a union of two discrete closed intervals. On the finite set, we obtain a formulation for some solutions, characteristic function and the eigenvalues-number of the problem. On the other time scale, we give some properties and an asymptotic formula for eigenvalues.

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