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ON THE ZEROS OF R-BONACCI POLYNOMIALS AND THEIR DERIVATIVES

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Abstract. The purpose of the present paper is to examine the zeros of R-Bonacci polynomials and their derivatives. We obtain new characterizations for the zeros of these polynomials. Our results generalize the ones obtained for the special case $r = 2$. Furthermore, we find explicit formulas of the roots of derivatives of R-Bonacci polynomials in some special cases. Our formulas are substantially simple and useful.

1. Introduction

The problem finding a convenient method to determine the zeros of a polynomial has a long history that dates back to the work of Cauchy [14]. Zeros of polynomials, which can be real or complex conjugate, have been perhaps among the most popular topics of study for centuries. When the historical development of polynomial studies have been examined, in 2000 BC, the ancient Babylon Tribe living in Mesopotamia stands out. This tribe knowing how to calculate positive roots is perhaps the best example. Some recent applications of the theory of polynomials with symmetric zeros can be found in [21]. This is a short review on the polynomials whose zeros are symmetric either to the real line or to the unit circle. These kind polynomials are very important in mathematics and physics (for more details see [21] and the references therein). On the other hand, the open problem of determining the exact number of zeros of a given polynomial on the unit circle was studied in [22]. Several classes of polynomials with symmetric zeros are also discussed in detail.

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Fibonacci polynomials, a broad class of polynomials, were first described by Belgian mathematician Eugene Charles Catalan (1814-1894), German mathematician E. Jacobsthal and Lucas polynomials in 1970 by M. Bicknell. The starting point of this polynomial class is based on well-known Golden Ratio and Fibonacci numbers, which are still of great interest in the world of modern applied sciences and whose new applications are still found (see, for instance, [1]- [16] and [18]- [20]). For any positive real number x, the Fibonacci polynomials are defined by

$$F_{n+2}\left(x\right) = xF_{n+1}\left(x\right) + F_{n}\left(x\right),$$

with initial values $F_0(x) = 0,\ F_1(x) = 1$. In [10], V. E. Hoggat and M. Bicknell are found explicitly the zeros of these polynomials using hyperbolic trigonometric functions. The symmetric polynomials of the zeros of Fibonacci polynomials were found by M. X. He, D. Simon and P. E. Ricci in [7]. Furthermore, in [8], the location and distribution of the zeros of the Fibonacci polynomials were determined. Fibonacci polynomials and their different properties have been examined (see, for example, [3], [24], [25], and the references therein).

In this paper our aim is to examine the zeros of R-Bonacci polynomials and their derivatives. R-Bonacci polynomials $R_n(x)$ are defined by the following recursive equation in [9] for any integer n and $r \geq 2$:

$$R_{n+r}(x) = x^{r-1}R_{n+r-1}\left(x\right) + x^{r-2}R_{n+r-2}\left(x\right) + \cdots + R_{n}\left(x\right), \tag{1}$$

with the initial values $R_{-k}(x) = 0,\ k = 0, 1, \cdots, r-2,\ R_1(x) = 1$. For $r = 2, 3$ in the recurrence relation (1), R-Bonacci polynomials become the so called Fibonacci and Tribonacci polynomials, respectively. Although, there are a large number of publications regarding to Fibonacci polynomials and their generalizations (see [7]-[9], [11] and [13]), the open expressions have not been found for the zeros of Tribonacci polynomials and their derivatives yet. Instead, numerical studies have been done more intensively in recent years. Zero attractors of these polynomials were obtained by W. Goh, M. X. He and P. E. Ricci in [6]. In [15], the number of the real roots of Tribonacci-coefficient polynomials were found. Recently, the smallest disc or annulus containing the zeros of Tribonacci polynomials have been examined by Ö. Öztunç Kaymak and an algorithm has given to use in other boundary problems in [12].

In this study, in order to determine the distribution of the zeros of R-Bonacci polynomials, we examine some properties of R-bonacci polynomials, a more general class of Fibonacci and Tribonacci polynomials. In Section 2, we consider some classes of R-Bonacci polynomials. We find the symmetric polynomials which are made up of the r^{th} order of the zeros of R-Bonacci polynomials. Using these symmetric polynomials, we determine the reference roots for the polynomials $R_{rn+p}(x)$ for $p = 0, 1$ and $n = 1$. So, we have generalized the results obtained for the special case $r = 2$ in [10].

On the other hand, there are several papers on the derivatives of the Fibonacci polynomials (see [4], [5], [17], [23] and the references therein). In Section 3, we study the roots of the derivatives of R-Bonacci polynomials. We obtain the most general symmetric polynomials which are made up of the r^{th} order of the zeros of derivatives of R-Bonacci polynomials. Using these symmetric polynomials, we find some formulas for the zeros of derivatives of R-Bonacci polynomials for some special values of t.

2. Zeros of Some Classes of R-Bonacci Polynomials

The general representations for R-Bonacci polynomials was given in [9] as

$$R_n(x) = \sum_{j=0}^{\left[\frac{(r-1)(n-1)}{r}\right]} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj}. \tag{2}$$

Here $r_{n,j} = \binom{n}{j}_r$ denotes the r-nomial coefficient and $[.]$ denotes the greatest integer function. In this section, we obtain the symmetric polynomials including the zeros of R-Bonacci polynomials. Before finding symmetric polynomial of the zeros of R-Bonacci polynomials, the following observation based on 2:

Observation 1. *The zeros of $R_n(x)$ and $R_n(xe^{\frac{2\pi}{r}i})$ are identical.*

To see the above observation, the following result is obtained by writing $xe^{\frac{2\pi}{r}i}$ instead of x in 2:

$$R_n(xe^{\frac{2\pi}{r}i}) = \sum_{j=0}^{\left[\frac{(r-1)(n-1)}{r}\right]} r_{n,j} \left(xe^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)-rj}. \tag{3}$$

Then, the desired result is easily seen by taking a parenthesis $\left(e^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)}$ and we have

$$\begin{aligned} R_n(xe^{\frac{2\pi}{r}i}) &= \left(e^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)} \left(\begin{array}{c} r_{n,0}\ x^{(r-1)(n-1)} + r_{n,1}\ x^{(r-1)(n-1)-r} \\ + \cdots + r_{n,\left[\frac{(r-1)(n-1)}{r}\right]}\ x \end{array} \right) \\ &= \left(e^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)} R_n(x). \end{aligned}$$

By this observation, we can simply state that the zeros of R-Bonacci polynomials can be created by rotating the angle of $\frac{2\pi}{r}$ degrees in the complex plane. The zeros of $R_n(x)$ are same as $R_n(xe^{\frac{2\pi i}{r}})$, as they are with $R_n(xe^{-\frac{2\pi i}{r}})$. Thus, the zeros of $R_n(x)$ can be divided into r sets: $\{x_i\}$, $\left\{x_i e^{\frac{2\pi i}{r}}\right\}, \cdots, \left\{x_i e^{\frac{2\pi i}{r}}\right\}$. Here we refer to this set $\{x_i\}$ as a set of reference zeros. The zeros of the 20^{th} Tribonacci polynomial are seen in Figure 1. Notice that the zeros of this polynomial can be generated at an angle of 120 degrees with reference to the set $\{x_i\}$.

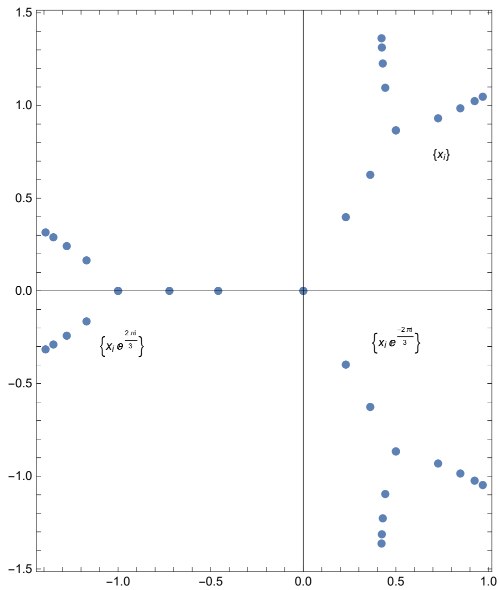

FIGURE 1. The zeros of $T_{20}(x)$

Our theorems are coincide with the ones obtained in [7] for $R = 2, 3$. Actually, Theorem 1 and Theorem 2 are the most generalized versions of the results obtained for Tribonacci and Fibonacci polynomials. For the definition of a symmetric polynomial one can see [7].

Theorem 1. *The most general form of the j^{th} symmetric polynomials consisting of over the r^{th} zeros of $R_{rn}(x)$ is as follows*:

$$\sigma_j\left(x_1^r, \cdots, x_{(r-1)n-1}^r\right) = (-1)^j \binom{rn-j-1}{j}_r. \tag{4}$$

Proof. It is known that the zeros of R-Bonacci polynomials lie in the argument $\frac{2\pi}{r}$ and hence the polynomial $R_{rn}(x)$ can be factorized as

$$R_{rn}(x) = x \prod_{k=1}^{(r-1)n-1} (x - x_k)\left(x - x_k e^{\frac{2\pi i}{r}}\right) \cdots \left(x - x_k e^{-\frac{2\pi i}{r}}\right).$$

If we rearrange this equation, we obtain

$$\begin{aligned} R_{rn}(x) &= x\{x^{r^2n-rn-r} - \\ &\quad x^{r^2n-rn-2r} \sum_{k=1}^{(r-1)n-1} x_k^r + x^{r^2n-rn-3r} \sum_{j \neq k} x_j^r x_k^r \end{aligned}$$

$$-x^{r^2n-rn-4r}\sum_{j\neq k\neq l}x_j^r x_k^r x_l^r+\cdots-\prod_{k=1}^{(r-1)n-1}x_k^r\}$$

$$=\left\{\sum_{j=0}^{(r-1)n-1}(-1)^j x^{(r-1)(rn-1)-rj}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j}x_{l_i}^r\right\}\right\}$$

$$=\sum_{j=0}^{(r-1)n-1}(-1)^j\sigma_j\left(x_1^r,x_2^r,\cdots,x_{(r-1)n-1}^r\right)x^{(r-1)(rn-1)-rj}. \quad (5)$$

On the other hand by (2) we can write

$$R_{rn}(x)=\sum_{j=0}^{(r-1)n-1}\binom{rn-j-1}{j}_r x^{(r-1)(rn-1)-rj}. \quad (6)$$

Since the equations (5) and (6) are equal, we obtain the desired result (4). □

Corollary 1. *The following equations are satisfied by the zeros of* $R_{rn}(x)$ *:*

$$\sum_{k=1}^{(r-1)n-1}x_k^r=-\binom{rn-2}{1}_r. \quad (7)$$

Proof. By setting $j=1$ in the equation (4) desired result is obtained. □

Theorem 2. *The most general form of the* j^{th} *symmetric polynomials consisting of the* r^{th} *zeros of* $R_{rn+1}(x)$ *is as follows :*

$$\sigma_j\left(x_1^r,\cdots,x_{(r-1)n}^r\right)=(-1)^j\binom{rn-j}{j}_r. \quad (8)$$

Proof. By a similar way used in the proof of Theorem 1, we can write

$$R_{rn+1}(x)=\prod_{k=1}^{(r-1)n}(x-x_k)\left(x-x_k e^{\frac{2\pi i}{r}}\right)\cdots\left(x-x_k e^{-\frac{2\pi i}{r}}\right).$$

Then we get

$$\begin{aligned}R_{rn+1}(x) &= \{x^{r^2n-rn}-\\ &\quad x^{r^2n-rn-r}\sum_{k=1}^{(r-1)n}x_k^r+x^{r^2n-rn-2r}\sum_{j\neq k}x_j^r x_k^r\\ &\quad -x^{r^2n-rn-3r}\sum_{j\neq k\neq l}x_j^r x_k^r x_l^r+\cdots-\prod_{k=1}^{(r-1)n}x_k^r\}\\ &= \left\{\sum_{j=0}^{(r-1)n}(-1)^j x^{rn(r-1)-rj}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j}x_{l_i}^r\right\}\right\}\end{aligned}$$

$$= \sum_{j=0}^{(r-1)n} (-1)^j \ \sigma_j \left(x_1^r, x_2^r, \cdots, x_{(r-1)n}^r\right) x^{rn(r-1)-rj}. \tag{9}$$

By putting $rn+1$ instead of n in (2), we find

$$R_{rn+1}(x) = \sum_{j=0}^{n(r-1)} \binom{rn-j}{j}_r x^{(r-1)rn-rj}. \tag{10}$$

It follows from the comparison (9) and (10), it is possible to write the desired result (8). □

Corollary 2. *The following equations are satisfied by the zeros of $R_{rn+1}(x)$:*

$$\sum_{k=1}^{(r-1)n} x_k^r = -\binom{rn-1}{1}_r. \tag{11}$$

Proof. If we set $j=1$ in the equation (8) then we get the equation (11). □

Now, using these symmetric polynomials, we obtain the reference roots of $R_{rn+p}(x)$ for $p = 0, 1$.

Theorem 3. *For $p = 0, 1$ and $n = 1$, let $x_j (1 \leq j \leq r)$ be the reference zeros of $R_{rn+p}(x)$. Then we have*

$$x_j^r = -1. \tag{12}$$

Proof. Let $p = 0$ or $p = 1$ and let the set of the reference zeros of $R_{rn+p}(x)$ be $\{x_1, \cdots, x_r\}$. The other zeros of the polynomial $R_{rn+p}(x)$ will be generated by the argument $\frac{2\pi}{r}$ except the root $x = 0$. For a fixed j, using the equations (11) and (7), we have

$$\sum_{k=1}^{r-1} x_k^r = x_1^r + x_2^r + \cdots + x_{r-1}^r$$

$$= x_j^r + \left(x_j e^{\frac{2\pi i}{r}}\right)^r + \left(x_j e^{\frac{4\pi i}{r}}\right)^r + \cdots + \left(x_j e^{\frac{2(r-2)\pi i}{r}}\right)^r = -(r-1)$$

and

$$\sum_{k=1}^{r-2} x_k^r = x_1^r + x_2^r + \cdots + x_{r-1}^r$$

$$= x_j^r + \left(x_j \ e^{\frac{2\pi i}{r}}\right)^r + \left(x_j \ e^{\frac{4\pi i}{r}}\right)^r + \cdots + \left(x_j e^{\frac{2(r-3)\pi i}{r}}\right)^r = -(r-2),$$

respectively. Rearranging the above equations, it can be easily seen that the reference roots of $R_{rn+p}(x)$ as in the equation (12). □

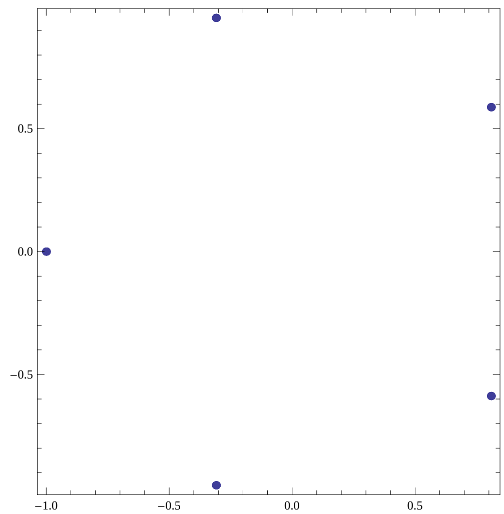

FIGURE 2. The zeros of $B_6(x)$

Example 1. *Let us consider the following* 5*-Bonacci polynomial*

$$B_6(x) = (x^5+1)^4.$$

Using (*12*)*, if we solve the equation* $x_j^5 = -1 (1 \leq j \leq 5)$*, the reference roots of the polynomial* $B_6(x)$ *are found as follows* (*see Figure 2*) :

$$x_1 = (-1), x_2 = (-1)^{\frac{1}{5}}, x_3 = -(-1)^{\frac{2}{5}}, x_4 = (-1)^{\frac{3}{5}}, x_5 = -(-1)^{\frac{4}{5}}.$$

3. ZEROS OF DERIVATIVES OF R-BONACCI POLYNOMIALS

Before we find the symmetric polynomials which are made up of the r^{th} order of the zeros of the derivatives of R-Bonacci polynomials $R_n^{(t)}(x)$, we write the algebraic representations of them. For any fixed n, using the equation (2), the algebraic representation of the derivative polynomial $R_n^{(t)}(x)$ is obtained as follows:

$$R_n^{(t)}(x) = \sum_{j=0}^{\left[\frac{(r-1)(n-1)}{rt}\right]} \binom{n-j-1}{j}_r ((r-1)(n-1)-rj)...((r-1)(n-1)-rj-t+1)x^{(r-1)(n-1)-rj-t}. \tag{13}$$

Now, we determine the symmetric polynomials for $R_{rn+p}^{(t)}(x)$ for special values of t. We give the following theorem.

Theorem 4. *Let* $k \in \mathbb{N}^+$*,* $p \in \{0, 1, \cdots, r-1\}$*. If we consider*

$$t = rk - (1-p)(r-1), \tag{14}$$

$$\mu = ((r-1)(rn+p-1))\cdots(rn(r-1)-t+(p-1)r+(2-p)) \tag{15}$$

and

$$\eta = (r-1)n - \left(\frac{t+(1-p)(r-1)}{r}\right), \tag{16}$$

then the most general form of the symmetric polynomials consisting of the zeros of $R^{(t)}_{rn+p}(x)$ *is as follows*:

$$\sigma\left(x_1^r,...,x_\eta^r\right) = \tag{17}$$

$$\frac{(-1)^j((r-1)(rn+p-1)-rj)...((r-1)(rn+p-1)-rj-t+1)}{\mu}\binom{rn+p-j-1}{j}_r.$$

Proof. It can be easily seen that

$$R^{(t)}_{rn+p}(x) = \mu\prod_{k=1}^{\eta}(x-x_k)\left(x-x_ke^{\frac{2\pi i}{r}}\right)\cdots\left(x-x_ke^{-\frac{2\pi i}{r}}\right),$$

where μ is a constant. Then we have

$$\begin{aligned} R^{(t)}_{rn+p}(x) &= \mu\{x^{r^2n-rn-(t+(1-p)(r-1))} - \\ &\quad x^{r^2n-rn-(t+(1-p)(r-1))-r}\sum_{k=1}^{\eta}x_k^r + x^{r^2n-rn-(t+(1-p)(r-1))-2r}\sum_{j\neq k}x_j^r\,x_k^r \\ &\quad -x^{r^2n-rn-(t+(1-p)(r-1))-3r}\sum_{j\neq k\neq l}x_j^rx_k^rx_l^r+\cdots-\prod_{k=1}^{\eta}x_k^r\} \\ &= \mu\left\{\sum_{j=0}^{\eta}(-1)^jx^{r^2n-rn-(t+(1-p)(r-1))-rj}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j}x_{l_i}^r\right\}\right\} \\ &= \mu\sum_{j=0}^{\eta}(-1)^j\sigma_j\left(x_1^r,x_2^r,\cdots,x_\eta^r\right)x^{(r-1)(rn+p-1)-rj-t}. \end{aligned} \tag{18}$$

By using the equation (13) and taking $rn+p$ instead of n we can write

$$R^{(t)}_{rn+p}(x) = \sum_{j=0}^{\left[\frac{(r-1)(rn+p-1)}{rt}\right]}\binom{rn+p-j-1}{j}_r\times$$

$$((r-1)(rn+p-1)-rj)...((r-1)(rn+p-1)-rj-t+1)x^{(r-1)(rn+p-1)-rj-t}. \tag{19}$$

Since the equations (18) and (19) are equal, then the proof follows. □

Corollary 3. *Let* t *and* η *be as in the equations (14) and (16), respectively. For* $k\in\mathbb{N}^+$ *and* $p\in\{0,1,\cdots,r-1\}$*, the following equations are satisfied by the zeros* *of* $R_{rn+p}^{(t)}(x)$:

$$(i)\prod_{k=1}^{\eta}x_k^r = \tag{20}$$

$$\frac{(-1)^{\eta}t\ (t-1)...(1)}{((r-1)(rn+p-1))...(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-\eta-1}{\eta}_r.$$

and

$$(ii)\sum_{k=1}^{\eta}x_k^r = \tag{21}$$

$$-\frac{((r-1)(rn+p-1)-r)...((r-1)(rn+p-1)-r-t+1)}{((r-1)(rn+p-1))...(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-2}{1}_r.$$

Proof. In the equation (4), if we put $j=\eta$ and $j=1$ we obtain the desired results, respectively. □

Let

$$\upsilon_{\eta} = \tag{22}$$

$$\frac{(-1)^{\eta}t\ (t-1)\cdots(1)}{((r-1)(rn+p-1))\cdots(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-\eta-1}{\eta}_r$$

and

$$\psi_{\eta} = \tag{23}$$

$$-\frac{((r-1)(rn+p-1)-r)\cdots((r-1)(rn+p-1)-r-t+1)}{((r-1)(rn+p-1))\cdots(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-2}{1}_r.$$

Then we can give the following theorem.

Theorem 5. *For* $t=r(r-1)n-2r-(1-p)(r-1)$, $R_{rn+p}^{(t)}(x)$ *has* $r\left((r-1)n-\left(\frac{t+(1-p)(r-1)}{r}\right)\right)$ *roots and these roots are*

$$x_k=\left(\frac{\psi_2\pm\sqrt{\psi_2^2-4\upsilon_2}}{2}\right)^{\frac{1}{r}}e^{\frac{2k\pi i}{r}},(k=0,1,\cdots,r-1), \tag{24}$$

where υ_2 *and* ψ_2 *are defined by the equations (3) and (3), respectively.*

Proof. Since $R_{rn+p}^{(r(r-1)n-2r-(1-p)(r-1))}(x)$ is a polynomial of $r\left((r-1)n-\left(\frac{t+(1-p)(r-1)}{r}\right)\right)$-th degree then by using the equations (3) and (3) we have

$$\prod_{k=1}^{2}x_k^r=x_1^rx_2^r=\upsilon_2 \tag{25}$$

and

$$\sum_{k=1}^{2} x_k^r = x_1^r + x_2^r = \psi_2. \tag{26}$$

Since we know that $x_1^r = \frac{\upsilon_2}{x_2^r}$ it can be easily seen that

$$x_2^{2r} - \psi_2\ x_2^r + \upsilon_2 = 0.$$

Solving this last equation of the second degree, the roots can be easily found. So the roots of $R_{rn+p}^{(t)}(x)$ must be as in the equation (24). □

Since we have Fibonacci and Tribonacci polynomials for $r = 2$ and $r = 3$, respectively, we can give the following corollaries.

Corollary 4. *Let $p \in \{0, 1\}$ and $t = 2n - 5 + p$. The zeros of the polynomial $F_{2n+p}^{(t)}(x)$ can be formulized as follows*:

$$x_k = \left(\frac{\psi_2 \pm \sqrt{\psi_2^2 - 4\upsilon_2}}{2} \right)^{\frac{1}{2}} e^{k\pi i}, (k = 0, 1)$$

where υ_2 and ψ_2 are defined by the equations (3) and (3), respectively.

In [23], J. Wang proved the following equation for any fixed n

$$L_n^{(t)}(x) = nF_n^{(t-1)}(x), n \geq 1, \tag{27}$$

where $L_n(x)$ are Lucas polynomials. Hence the zeros of $L_n^{(t+1)}(x)$ and $F_n^{(t)}(x)$ are identical.

Corollary 5. *Let $p \in \{0, 1, 2\}$ and $t = 6n - 8 + 2p$. The zeros of the polynomial $T_{3n+p}^{(t)}(x)$ are*

$$x_k = \left(\frac{\psi_2 \pm \sqrt{\psi_2^2 - 4\upsilon_2}}{2} \right)^{\frac{1}{3}} e^{\frac{2k\pi i}{3}} (k = 0, 1, 2), \tag{28}$$

where υ_2 and ψ_2 are defined by the equations (3) and (3), respectively.

Now we give some examples.

Example 2. *Consider the zeros of the polynomial*

$$T_6^{(iv)}(x) = 5040x^6 + 3360x^3 + 144.$$

In the equation (28), writing $\psi_2 = 2/3$, $\upsilon_2 = 1/35$, we find the zeros of this polynomial as

$$x_k = \sqrt[3]{\frac{2/3 \pm \sqrt{(2/3)^2 - 4/35}}{2}} e^{\frac{2k\pi i}{3}}, (k = 0, 1, 2)$$

(*see Figure 3*).

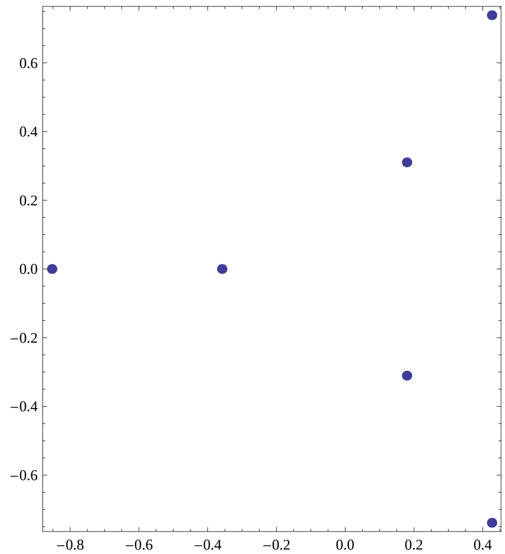

FIGURE 3. The roots of $T_6^{(iv)}(x)$.

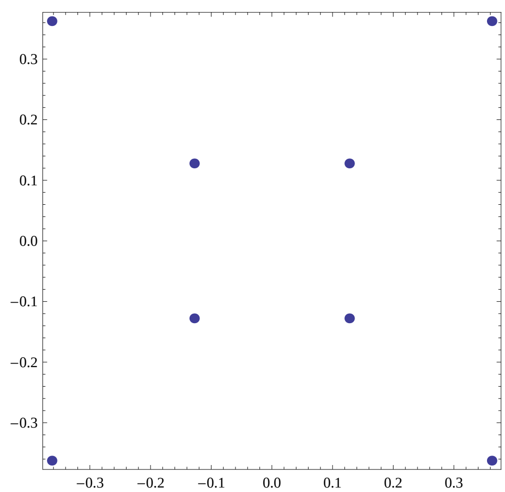

FIGURE 4. The roots of $Q_8^{(13)}(x)$

Example 3. *For $p = 0$, $n = 2$ and $r = 4$, let us consider the polynomial*

$$Q_8^{(13)}(x) = 93405312000 + 88921857024000x^4 + 1267136462592000x^8.$$

Using the equations (3) and (3) we have

$$\prod_{k=1}^{2} x_k^4 = \frac{1}{13566} = \upsilon_2$$

and

$$\sum_{k=1}^{2} x_k^4 = -\frac{4}{57} = \psi_2.$$

Then the roots of $Q_8^{(13)}(x)$ *are generated by* x_k $(k = 0, 1, 2, 3)$. *By (24), the roots of the polynomial* $Q_8^{(13)}(x)$ *are obtained as*

$$x_1 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} = 0.127788 + 0.127788i,$$

and

$$x_2 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} = 0.36255 + 0.36255i$$

for $k = 0$,

$$x_3 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{\pi i}{2}} = -0.36255 + 0.36255i$$

and

$$x_4 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{\pi i}{2}} = -0.127788 + 0.127788i$$

for $k = 1$,

$$x_5 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\pi i} = -0.127788 - 0.127788i$$

and

$$x_6 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\pi i} = -0.36255 - 0.36255i,$$

for $k = 2$,

$$x_7 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{3\pi i}{2}} = 0.127788 - 0.127788i$$

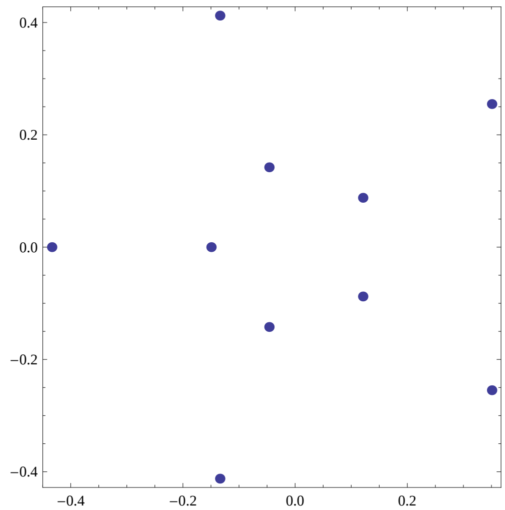

FIGURE 5. The roots of $B_8^{(18)}(x)$

and

$$x_8 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{3\pi i}{2}} = 0.36255 - 0.36255i,$$

for $k = 3$ (*see Figure 4*).

Example 4. *Let us consider the* 5*-Bonacci polynomials* $B_8^{18}(x)$*. In this case, we have* $p = 3$, $n = 1$, $r = 5$ *and we obtain*

$$B_8^{18}(x) = 96035605585920000 + 129260083694424883200 0x^5$$
$$+840190544013761740800 00x^{10}.$$

The roots of this polynomial are found as follows (*see Figure 5*) :

$$x_k = \sqrt[5]{\frac{\psi_2 \pm \sqrt{\psi_2^2 - 4\upsilon_2}}{2}} e^{\frac{2k\pi i}{5}}, k = 0, 1, 2, 3, 4.$$

4. Conclusion and Future Work

In this paper, in order to obtain new formulas for the zeros of R-Bonacci polynomials and their derivatives, the most general form of the j^{th}symmetric polynomials consisting of over the r^{th} zeros of $R_n(x)$ and $R_{rn+p}^{(t)}(x)$ are given. Using some consequences of these symmetric polynomials, some explicit formulas for the zeros of these polynomials, which have been given in (12) and (24), are found. Although these formulas are simple, they are valuable because they formulate the zero values of many R-Bonacci polynomials, which is the most general form of the Fibonacci polynomials, and their derivatives.

Given the future studies on this topic, the zeros of the remaining R-Bonacci polynomials can be formulated using different methods. For this reason, it is thought that formulating the zeros of a R-Bonacci polynomial will increase the applicability of this problem in different engineering applications. In addition, this study is also thought to be a guide for formulating the zero locations of polynomials with unknown zero locations. Because this method is applicable for all polynomial classes.

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