On sum annihilator ideals in Ore extensions

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Abstract

A ring $R$ is called a left Ikeda-Nakayama ring (left IN-ring) if the right annihilator of the intersection of any two left ideals is the sum of the two right annihilators. As a generalization of left IN-rings, a ring $R$ is called a right SA-ring if the sum of right annihilators of two ideals is a right annihilator of an ideal of $R$. It would be interesting to find conditions under which an Ore extension $R[x; \alpha, \delta]$ is IN and SA. In this paper, we will present some necessary and sufficient conditions for the Ore extension $R[x; \alpha, \delta]$ to be left IN or right SA. In addition, for an $(\alpha, \delta)$-compatible ring $R$, it is shown that: (i) If $S = R[x; \alpha, \delta]$ is a left IN-ring with $\text{Idm}(R) = \text{Idm}(R[x; \alpha, \delta])$, then $R$ is left McCoy. (ii) Every reduced left IN-ring with finitely many minimal prime ideals is a semiprime left Goldie ring. (iii) If $R$ is a commutative principal ideal ring, then $R$ and $R[x]$ are IN. (iv) If $R$ is a reduced ring and $n$ is a positive integer, then $R$ is right SA if and only if $R[x]/(x^n+1)$ is right SA.

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1. Introduction and preliminary definitions

According to [5], a ring $R$ is called a left Ikeda-Nakayama ring (left IN-ring) if $r_R(I \cap J) = r_R(I) + r_R(J)$ for all left ideals $I, J$ of $R$. For example, all left self-injective rings, all left uniserial rings and all left uniform domains are left IN-ring. Kaplansky [13] introduced Dual rings as rings which every right or left ideal of them is an annihilator. Hajarnavis and Norton [7] proved that every dual ring is a right (and left) IN-ring. Wisbauer et al. [19] extended the notion of an Ikeda-Nakayama ring to bimodules and derived various characterizations and properties for modules with this property.

As a generalization of IN-rings, Birkenmeier et al. [3, 4] introduced SA-rings. A ring $R$ is called a right SA-ring, if for any ideals $I$ and $J$ of $R$, there is an ideal $K$ of $R$ such that $r_R(I) + r_R(J) = r_R(K)$. They showed that this class of rings is exactly the class of rings for which the lattice of right annihilator ideals is a sub-lattice of the lattice of ideals. The class of right SA-rings includes all quasi-Baer (hence all Baer) rings and all right IN-rings (hence all right self-injective rings). Also they showed that this class is closed under direct products, full and upper triangular matrix rings and certain classes of polynomial rings.

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Throughout this paper, $R$ denotes an associative ring with unity, $\alpha : R \to R$ is an endomorphism, and $\delta$ is an $\alpha$-derivation of $R$ (i.e., $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$). We denote by $S = R[x; \alpha, \delta]$ the Ore extension whose elements are the polynomials over $R$, where addition is defined as usual and multiplication by $xb = \alpha(b)x + \delta(b)$ for any $b \in R$. For a subset $A \subseteq R$, we denote the right annihilator of $A$ in $R$ by $\ell_R(A)$ and $\ell_R(A)$, respectively. The set of all right zero divisors of $R$ is denoted by $Z_r(R)$.

It is natural to ask if these properties (IN and SA) can be extended from $R$ to $R[x; \alpha, \delta]$. The purpose of the present paper is to study Ore extensions over IN-rings and SA-rings. In this note we show that some portions of the results in [18] can be generalized to the Ore extension $R[x; \alpha, \delta]$, where the base coefficient ring $R$ is an $(\alpha, \delta)$-compatible ring. In addition, in Section 2, we show that if $R[x; \alpha, \delta]$ is a left IN-ring with $\text{Id}(R[x; \alpha, \delta]) = \text{Id}(R)$, then $\ell_{R[x; \alpha, \delta]}(g) \cap R \neq \{0\}$, for each $g \in Z_r(R[x; \alpha, \delta])$. Furthermore, it is proved that every reduced left IN-ring $R$ with finitely many minimal prime ideals is a semiprime left Goldie ring and $R[x; \alpha, \delta]$ is a left IN-ring. Finally, for a commutative principal ideal ring, it is shown that the IN property is inherited by polynomial extensions. In the third section, we investigate Ore extensions over SA-rings. For example, it is proved that if $R[x; \alpha, \delta]$ is a right SA-ring, then so is $R$, and the reverse is true when $R$ satisfy SQA1 condition. In addition, it is shown that for a reduced ring $R$ and a positive integer $n$, $R$ is right SA if and only if $R[x]/(x^{n+1})$ is right SA. Moreover, each section contains some examples to show that the “$(\alpha, \delta)$-compatible” assumption on $R$ is not superfluous. Also, examples of non-reduced IN-ring $R$ such that $R[x]$ is left IN-ring are provided.

2. Skew polynomials over IN-rings

In this section, we will present some necessary and sufficient conditions for the Ore extension $R[x; \alpha, \delta]$ to be an IN ring. To fulfill this plan, we shall need to find a McCoy-like property of an IN Ore extension. The aim of our first result in this section is to state and prove it.

According to [8], an ideal $I$ is called an $\alpha$-compatible ideal if for each $a, b \in R$, $ab \in I \iff a\alpha(b) \in I$. In addition, $I$ is said to be a $\delta$-compatible ideal if for each $a, b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If $I$ is both $\alpha$-compatible and $\delta$-compatible, we say that $I$ is an $(\alpha, \delta)$-compatible ideal. If $I = 0$ is $\alpha$-compatible (resp., $\delta$-compatible), then the ring $R$ is called $\alpha$-compatible (resp., $\delta$-compatible). Also, if $R$ is both $\alpha$-compatible and $\delta$-compatible, then $R$ is said to be $(\alpha, \delta)$-compatible. The concept of $\alpha$-compatible rings were defined in [9], as a common generalization of $\alpha$-rigid rings. It was proved [9, Lemma 2.2] that $R$ is $\alpha$-rigid if and only if $R$ is $\alpha$-compatible and reduced. Clearly, each compatible endomorphism is a monomorphism.

We begin this section with the following essential lemmas.

**Lemma 2.1.** [10, Lemma 2.1] Let $R$ be an $(\alpha, \delta)$-compatible ring and $a, b \in R$. Then we have the following:

1. If $ab = 0$, then $a\alpha^n(b) = 0 = \alpha^n(a)b$ for each non-negative integer $n$.
2. If $\alpha^k(a)b = 0$ for some non-negative integer $k$, then $ab = 0$.
3. If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for any non-negative integers $m, n$.
4. If $ab = 0$, then $\alpha(a)\alpha(b) = 0 = \delta(a)\delta(b)$.
5. If $ab = 0$, then $ax^m b = 0$ in $R[x; \alpha, \delta]$, for each $m \geq 0$.
6. If $ax^m b = 0$ in $R[x; \alpha, \delta]$, for some $m \geq 0$, then $ab = 0$.

**Lemma 2.2.** [9, Lemma 2.3] Let $R$ be an $(\alpha, \delta)$-compatible ring. If $f = a_0 + a_1x + \cdots + a_\nu x^n \in R[x; \alpha, \delta]$, $r \in R$ and $fr = 0$, then $a_ir = 0$ for each $i$.

We denote the set of all idempotent elements of $R$ by $\text{Id}(R)$.
Proposition 2.3. Let $R$ be an $(\alpha, \delta)$-compatible ring. Also, let $f = a_0 + a_1x + \cdots + a_nx^n$ and $g = b_0 + b_1x + \cdots + b_mx^m$ be non-zero elements of $R[x; \alpha, \delta]$ such that $fg = 0$. If $S = R[x; \alpha, \delta]$ is a left IN-ring with $\text{Idm}(R) = \text{Idm}(R[x; \alpha, \delta])$, then $f = a_0$ or there exists $r \in R$ such that $0 \neq ra_n$ and $ra_ng = 0$.

**Proof.** Since $fg = 0$, then by Lemma 2.1, $a_nb_m = 0$. Also, since $S = R[x; \alpha, \delta]$ is left IN, we have $r_S(f) + r_S(a_n) = r_S(Sf \cap Sa_n)$. Now, we consider the following two cases:

**Case 1:** Assume that $Sf \cap Sa_n = \{0\}$. Then there exists an idempotent $e \in R$, such that $Sf \subseteq Sf e$ and $Sa_n \subseteq S(1-\epsilon) e$, by [5, Corollary 4]. Then $f = fe$ and $a_n = a_n(1-\epsilon)$. Hence $a_n = a_n\alpha^n(e)$, and since $R$ is $\alpha$-compatible, we have $a_n = a_ne$. Therefore, $a_n = 0$, which implies that $f = a_0$.

**Case 2:** Assume that $Sf \cap Sa_n \neq \{0\}$. Let $\gamma(1), \beta(1) \in S$ such that $0 \neq \gamma(1)f = \beta(1)a_n$. Assume that $\beta(1)a_n = \beta_10 + \beta_11x + \cdots + \beta_{t1}t^{t1}(n)$, with $\beta_{t1} \neq 0$. Clearly, $\beta_1t_1 = r_1\alpha^{t1}(a_n)$, for some $r_1 \in R$. Since $a_nb_m = 0$, hence by Lemma 2.1, $\beta_1b_m = 0$, for each $0 \leq i \leq t_1$. Then $(\gamma(1)f)g_1 = (\beta(1)a_n)g_1 = 0$, where $g_1 = b_0 + b_1x + \cdots + b_{m-1}x^{m-1}$. Hence $\beta_{t1}t_{m-1} = 0$, since $R$ is $\alpha$-compatible. Since $S$ is left IN, we have $r_S(\beta(1)a_n) + r_S(\beta_{t1}t) = r_S((S\beta(1)a_n) \cap (S\beta_{t1}t))$. If $(S\beta(1)a_n) \cap (S\beta_{t1}t) = \{0\}$, then by Case 1, $\beta(1)a_n = \beta_{t1}$. Since $\beta_{t1}t_{m-1} = 0$, hence by Lemma 2.1, $\beta_{t1}t_{m-1} = 0$, for each $0 \leq i \leq t_2$. Then $(\gamma(1)f)g_2 = (\gamma(1)f)g_2 = (\beta(1)a_n)g_2 = (\beta_1t_{m-1})g_2 = 0$, where $g_2 = b_0 + b_1x + \cdots + b_{m-2}x^{m-2}$.

By continuing this process we can find a non-zero element $\beta_{m-1}t_{m-1} \in R$ such that $\beta_{m-1}t_{m-1}g = 0$ and $r_S(\beta_{m-1}t_{m-1}) = r_S(\beta_{m-1}t_{m-1}) = r_S(\beta_{m-1}t_{m-1}) = \cdots = r_S(\beta_{m-1}t_{m-1}) = 0$, for each $0 \leq i \leq n$ and $0 \leq j \leq m$.

As an immediate consequence of Proposition 2.3, we get the following result.

**Corollary 2.4.** Let $R$ be an $(\alpha, \delta)$-compatible ring. Let $f = a_0 + a_1x + \cdots + a_nx^n$ and $g = b_0 + b_1x + \cdots + b_mx^m$ be non-zero elements of $R[x; \alpha, \delta]$ satisfy $fg = 0$. If $S = R[x; \alpha, \delta]$ is a left IN-ring with $\text{Idm}(R) = \text{Idm}(R[x; \alpha, \delta])$, then there exists $r \in R$ such that $0 \neq rf$ and $ra_jb_j = 0$, for each $0 \leq i \leq n$ and $0 \leq j \leq m$.

It is often taught in an elementary algebra course that if $R$ is a commutative ring, and $f(x)$ is a zero-divisor in $R[x]$, then there is a non-zero element $r \in R$ with $f(x)r = 0$. This was first proved by McCoy [16, Theorem 2]. Recall from [17] that a ring $R$ is called left McCoy when the equation $f(x)g(x) = 0$ over $R[x]$, where $f(x), g(x) \neq 0$, implies there exists a non-zero $r \in R$ with $rg(x) = 0$.

Taking $\alpha = id_R$ and $\delta = 0$, the following result is immediate from Proposition 2.3.

**Corollary 2.5.** Let $S = R[x]$ be a left IN-ring with $\text{Idm}(R) = \text{Idm}(R[x])$. Then $R$ is left McCoy.

Now, we give some classes of rings $R$, such that $\text{Idm}(R) = \text{Idm}(R[x; \alpha, \delta])$. Recall that a ring $R$ is called abelian if all idempotent elements of $R$ are central.

**Example 2.6.** (i) Let $R$ be an $(\alpha, \delta)$-compatible ring. If $R[x; \alpha, \delta]$ is an abelian ring, then $\text{Idm}(R) = \text{Idm}(R[x; \alpha, \delta])$.

(ii) Let $R$ be an abelian $\alpha$-compatible ring. Then $\text{Idm}(R) = \text{Idm}(R[x; \alpha])$. 
(2.1)
\[
\begin{align*}
\delta(e_0) &= 0; \\
\alpha(e_0) + \delta(e_1) &= e_0; \\
\alpha(e_1) + \delta(e_2) &= e_1; \\
&\vdots \\
\alpha(e_{n-1}) + \delta(e_n) &= e_{n-1}; \\
\alpha(e_n) &= e_n.
\end{align*}
\]

Since \(e^2 = e\), then \(e_0^2 + \delta(e_0) + \cdots + e_n\delta^n(e_0) = e_0 \) and \(e_n\alpha^n(e_n) = 0\). Then by using (2.1), we have \(e_0^2 = e_0\). Now, by the abelian assumption on \(R[x; \alpha, \delta]\) and by using [12, Theorem 3.13], we obtain \(e \in \text{Idm}(R)\).

(ii) By a similar argument as used in the proof of (i), one can show that \(\text{Idm}(R) = \text{Idm}(R[x; \alpha])\).

Corollary 2.7. Let \(R\) be an \((\alpha, \delta)\)-compatible ring and \(g \in Z_r(R[x; \alpha, \delta])\). If \(R[x; \alpha, \delta]\) is an abelian left IN-ring, then \(\ell_{R[x; \alpha, \delta]}(g) \cap R \neq \{0\}\).

Corollary 2.8. Let \(R\) be an abelian \((\alpha, \delta)\)-compatible ring and \(g \in Z_r(R[x; \alpha])\). If \(R[x; \alpha]\) is a left IN-ring, then \(\ell_{R[x; \alpha]}(g) \cap R \neq \{0\}\).

Question 1: Let \(R\) be an \((\alpha, \delta)\)-compatible ring and \(S = R[x; \alpha, \delta]\) be a left IN-ring. Let \(f = a_0 + a_1x + \cdots + a_nx^n\), \(g = b_0 + b_1x + \cdots + b_mx^m\) be non-zero elements of \(R[x; \alpha, \delta]\) satisfy \(fg = 0\). Can we conclude \(a_ib_j = 0\), for each \(i, j\)?

Let \(\alpha\) be an endomorphism and \(\delta\) an \(\alpha\)-derivation on a ring \(R\). Recall that an ideal \(I\) of \(R\) is called \(\alpha\)-ideal if \(\alpha(I) \subseteq I\); \(I\) is called a \(\delta\)-ideal if \(\delta(I) \subseteq I\); \(I\) is called an \((\alpha, \delta)\)-ideal if it is both \(\alpha\)- and \(\delta\)-ideal. Clearly, if \(K\) is an \((\alpha, \delta)\)-ideal of \(R\), then \(K[x; \alpha, \delta]\) is an ideal of \(R[x; \alpha, \delta]\).

Proposition 2.9. Let \(R\) be an \((\alpha, \delta)\)-compatible ring. If \(S = R[x; \alpha, \delta]\) is a left IN-ring, then for any \((\alpha, \delta)\)-ideals \(I\) and \(J\) of \(R\), \(r_R(I) + r_R(J) = r_R(I \cap J)\).

Proof. Let \(I, J\) be \((\alpha, \delta)\)-ideals of \(R\). Clearly \(r_R(I) + r_R(J) \subseteq r_R(I \cap J)\). To prove the reverse inclusion, let \(t \in r_R(I \cap J)\). Then \(t \in r_S(I \cap J)\). By Lemma 2.2. On the other hand, \(r_S(I \cap J) = r_S(I \cap J)\cap J\), since \(S\) is a left IN-ring. Now, since \(r_S(I \cap J)\), it follows that \(t = h(x) + k(x)\), for some \(h(x) = \sum_{i=0}^{n}h_i x^i \in r_S(I \cap J)\) and \(k(x) = \sum_{i=0}^{m}k_i x^i \in r_S(J[x; \alpha, \delta])\). Then, since \(Ih_0 = 0 = Jk_0\) and \(t = h_0 + k_0\), hence \(t \in r_R(I) + r_R(J)\) and thus \(r_R(I) + r_R(J) = r_R(I \cap J)\) as claimed.

Lemma 2.10. Let \(R\) be a reduced ring and \(\{P_i\}_{i \in I}\) be the set of all distinct minimal prime ideals of \(R\). If \(X\) is a non-zero left ideal of \(R\) contained in \(\cap_{j \neq i} P_j\), for some \(i \in I\), then \(r_R(X) = P_i\).

Proof. This follows from [6, Proposition 7.1].

Proposition 2.11. Let \(R\) be a reduced left IN-ring. If \(R\) has finitely many minimal prime ideals, then \(R\) has a finite left uniform dimension.

Proof. Assume that \(P_1, P_2, \ldots, P_n\) are all of the distinct minimal prime ideals of \(R\). It is easy to see that \(r_R(P_i) = \cap_{j \neq i} P_j\) for each \(1 \leq i \leq n\). Now since \(\cap_{i=1}^n P_i = 0\) and \(R\) is a left IN-ring, we have \(r_R(P_1) + \cdots + r_R(P_n) = r_R(P_1 \cap \cdots \cap P_n) = R\). Therefore, \((\cap_{i \neq 1} P_i) \oplus \cdots \oplus (\cap_{i \neq n} P_i) = R\) and it is sufficient to prove that \(\cap_{j \neq i} P_j\) is a uniform left
ideal of \( R \), for each \( 1 \leq i \leq n \). To see this, suppose that \( X, Y \) are non-zero left ideals of \( R \) contained in \( \bigcap_{j \neq i} P_j \) with \( X \cap Y = 0 \). By using the left IN property of \( R \) and Lemma 2.10, we have
\[
P_j = P_j + r_R(X) + r_R(Y) = r_R(X \cap Y) = R,
\]
which is a contradiction. Therefore \( \bigcap_{j \neq i} P_j \) is a uniform left ideal of \( R \), for each \( 1 \leq i \leq n \).

\[ \square \]

**Corollary 2.12.** Let \( R \) be a reduced left IN-ring. If \( R \) has finitely many minimal prime ideals, then \( R \) is a semiprime left Goldie ring.

**Proof.** It follows from Proposition 2.11 and [15, Theorem 2.15]. \[ \square \]

Recall that an ideal \( P \) of \( R \) is called completely prime whenever \( R/P \) is a domain.

**Theorem 2.13.** Let \( R \) be a reduced \((\alpha, \delta)\)-compatible left IN-ring. If \( R \) has finitely many minimal prime ideals, then \( R[x; \alpha, \delta] \) is a left IN-ring.

**Proof.** Let \( P_1, \ldots, P_n \) be all of the distinct minimal prime ideals of \( R \). By using Lemma 2.10 and the left IN property of \( R \), we have \( P_r + P_s = r_R(\bigcap_{j \neq r} P_j) + r_R(\bigcap_{j \neq s} P_j) = r_R(0) = R \), for each \( r \neq s \). Now, by the Chinese Remainder Theorem, we have \( R = R/P_1 \times \cdots \times R/P_n \). Since \( R \) is a reduced ring, hence \( P_i \) is completely prime and by Corollary 2.12 and [15, Theorem 2.5], \( R/P_i \) is a prime left Goldie ring, for each \( i \). Also, since \( P_i \) is an annihilator ideal of \( R \), hence \( P_i \) is an \((\alpha, \delta)\)-compatible ideal of \( R \), and so \( R/P_i \) is an \((\alpha, \delta)\)-compatible ring, by [8, Proposition 2.1], where \( \alpha : R/P_i \to R/P_i \) is defined by \( \alpha(a + P_i) = \alpha(a) + P_i \) and \( \delta : R/P_i \to R/P_i \) is defined by \( \delta(a + P_i) = \delta(a) + P_i \), for each \( a \in R \). Then, by [14, Corollary 3.5], \( R/P_i[x; \alpha, \delta] \) is a left Ore domain, for each \( i \).

Finally, suppose that \( X, Y \) are left ideals of \( R[x; \alpha, \delta] \). Since \( R[x; \alpha, \delta] \cong R/P_1[x; \alpha, \delta] \times \cdots \times R/P_n[x; \alpha, \delta] \), hence for each \( i \), there exist left ideals \( I_i, J_i \) of \( R/P_i[x; \alpha, \delta] \), such that \( X = I_1 \times \cdots \times I_n \) and \( Y = J_1 \times \cdots \times J_n \). Then it is clear that \( r_{R[x;\alpha,\delta]}(X) = r_{R/P_1[x;\alpha,\delta]}(I_1) \times \cdots \times r_{R/P_n[x;\alpha,\delta]}(I_n) \) and by using the fact that \( R/P_i[x; \alpha, \delta] \) is a left Ore domain for each \( i \), it follows that \( r_{R[x;\alpha,\delta]}(X) + r_{R[x;\alpha,\delta]}(Y) = r_{R[x;\alpha,\delta]}(X \cap Y) \), which implies that \( R[x; \alpha, \delta] \) is a left IN-ring. \[ \square \]

Now, we give an example to show that the “\( \alpha \)-compatible” assumption on \( R \), in Theorem 2.13 is not superfluous.

**Example 2.14.** Let \( \mathbb{Z}_2 \) be the field of integers modulo 2 and \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Clearly \( R \) is a reduced commutative IN-ring. Let \( \alpha : R \to R \) be the endomorphism defined by \( \alpha(a, b) = (b, a) \). Then \( \alpha \) is an automorphism of \( R \), and since \((1,0)(0,1) = 0 \) but \((1,0)\alpha((0,1)) \neq 0 \), hence \( R \) is not \( \alpha \)-compatible. Now let \( p(x) = (1,0) + (1,0)x \) and \( q(x) = (0,1) + (0,1)x \in R[x; \alpha] \). Let \( I \) and \( J \) be the left ideals of \( R[x; \alpha] \) generated by \( p(x) \) and \( q(x) \), respectively. By a simple computation one can show that
\[
I = \{(r_0, 0) + (r_0, s_1)x + \cdots + (r_t, s_{t-1})x^t \mid r_i, s_j \in \mathbb{Z}_2, \ t = 2i\} \cup
\{(r_0, 0) + (r_0, s_1)x + \cdots + (r_{t-1}, s_t)x^t + (0, s_t)x^{t+1} \mid r_i, s_j \in \mathbb{Z}_2, \ t = 2i + 1\}
\]
and
\[
J = \{(0, w_0) + (v_1, w_0)x + \cdots + (v_{k-1}, w_k)x^k \mid (0, w_k)x^{k+1} \mid v_i, w_j \in \mathbb{Z}_2, \ k = 2i\} \cup
\{(0, w_0) + (v_1, w_0)x + \cdots + (v_{k-1}, w_k)x^k + (v_k, 0)x^{k+1} \mid v_i, w_j \in \mathbb{Z}_2, \ k = 2i + 1\}.
\]
Then \( I \cap J = 0 \) and hence \( r_{R[x;\alpha]}(I \cap J) = R[x; \alpha] \). On the other hand, for each \( g = (r_0, s_0) + (r_1, s_1)x + \cdots + (r_n, s_n)x^n \in r_{R[x;\alpha]}(I) \), we have \( r_0 = s_0 = 0 \) and \( r_i + s_{i-1} = 0 \), for each \( 1 \leq i \leq n \). Also, for each \( h(x) = (v_0, w_0) + (v_1, w_1)x + \cdots + (v_m, w_m)x^m \in r_{R[x;\alpha]}(J) \), we have \( w_0 = v_m = 0 \) and \( v_i + w_{i-1} = 0 \), for each \( 1 \leq i \leq m \). Now, one can easily show that \((1,1) \notin r_{R[x;\alpha]}(I) + r_{R[x;\alpha]}(J) \). Therefore, \( r_{R[x;\alpha]}(I) + r_{R[x;\alpha]}(J) \neq R[x; \alpha] \), which implies that \( R[x; \alpha] \) is not a left IN-ring. Thus, the “\( \alpha \)-compatible” assumption on \( R \) in Theorem 2.13 is not superfluous.
The following example shows that we cannot eliminate the “reduced $\delta$-compatible” assumption in Theorem 2.13.

**Example 2.15.** Let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation $\delta$ such that $\delta(t) = 1$ where $\bar{t} = t + (t^2)$ in $R$ and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. It is clear that $R$ is a non-reduced commutative IN-ring. Consider the differential polynomial ring $R[x;\bar{\delta}]$. By [2, Example 11], $R[x;\bar{\delta}] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. Since $\mathbb{Z}_2[y]$ is not a left self-injective ring, hence by [5, Theorem 7], $M_2(\mathbb{Z}_2)[y]$ is not a left IN-ring.

In the following, we construct some classes of commutative non-reduced IN-rings $R$ with the property that $R[x]$ is also IN. However, the reduced condition in Theorem 2.13 plays an important role in the proof, the following examples show that it is not a necessary condition.

For the remainder of this section, $R$ will denote a commutative ring with identity. Following Zariski and Samuel [20, page 22], we say the elements $a, b \in R$ are relatively prime, if $(a, b) = 1$. A principal ideal ring (PIR) is a ring with identity in which every ideal is principal. Any PIR is obviously Noetherian, and the PIR’s may be considered the simplest type of Noetherian rings. By Zariski and Samuel [20, page 245], a PIR is called special if it has only one prime ideal $P \neq R$ and $P$ is nilpotent, that is, $P^n = (0)$ for some positive integer $n$. If we place $P = PR$, and if we denote by $m$ the smallest integer such that $p^m = 0$, then every non-zero element $x$ in $R$ may obviously be written in the form $x = ep^k$, where $0 \leq k \leq m - 1$, and where $e \notin PR$ (i.e., $e$ and $p$ are relatively prime).

Special principal ideal rings are examples of uniserial rings.

A ring is called Armendariz whenever polynomials $f = a_0 + a_1x + \cdots + a_nx^n$ and $g = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $fg = 0$, then $a_0b_j = 0$, for each $i, j$. The name “Armendariz ring” was chosen, because Armendariz had noted that a reduced ring satisfies this condition.

**Proposition 2.16.** Let $R$ be a special principal ideal ring. Then $S = R[x]$ is an IN-ring.

**Proof.** Let $R$ be a special principal ideal ring with maximal ideal $M = mR$ and $n$ be the smallest integer such that $m^n = 0$. For an ideal $K$ of $S$, we denote

$$K_0 = \{a \in R \mid a \in C_f \text{ for some } f \in K\}.$$

Now let $I, J$ be non-zero ideals of $S$. It is clear that $I_0, J_0$ are ideals of $R$. Assume that $I_0 = m^kR, J_0 = m^sR$ such that $0 \leq k \leq s \leq n - 1$. Since $r_R(I_0) = m^{n-k}R, r_R(J_0) = m^{n-s}R$ and $R$ is an Armendariz ring, then we have $r_S(I) = r_S(I_0[x]) = m^{n-k}R[x]$ and $r_S(J) = r_S(J_0[x]) = m^{n-s}R[x]$. Hence $r_S(I) + r_S(J) = r_S(J) = m^{n-s}R[x]$.

Now we claim that $r_S(I \cap J) = r_S((I \cap J_0)[x]) = m^{n-s}R[x]$. Since $m^k \in I_0$, there exists a non-zero element $f \in I$ such that $m^k \in C_f$. Assume that $f = r_0m^{k+i_0} + r_1m^{k+i_1}x + \cdots + r_nm^{k+i_n}x^n$ such that $(r_i, m) = 1$ and $i_j = 0$ for some $0 \leq j \leq n$. Then we have $f = m^k f_1(x)$, where $f_1(x) = r_0m^{i_0} + r_1m^{i_1}x + \cdots + r_nm^{i_n}x^n$ and $i_j = 0$ for some $0 \leq j \leq n$. By a similar argument, we can show that there exists a non-zero element $g \in J$ such that $g = m^n g_1(x)$, where $g_1(x) = r_0m^{i_0} + r_1m^{i_1}x + \cdots + r_nm^{i_n}x^n$, $(r_i, m) = 1$ for all $0 \leq i' \leq n'$ and $i' = 0$ for some $0 \leq j \leq n'$. Thus, $(m, d) = 1$, for some $d \in C_{f_1g_1}$. Therefore $m^d f_1(x)g_1(x) \in I \cap J$ and $m^d \in (I \cap J_0)$ where $m$ and $d$ are relatively prime. Hence $r_{r_P}(I \cap J_0) \subseteq r_{r_P}(m^R) = m^{n-s}R$. Therefore, $r_{r_P}(I \cap J) = r_{r_P}((I \cap J_0)[x]) \subseteq r_S(m^{n-s}R[x]) = m^{n-s}R[x]$. The reverse inclusion is trivial and the proof is completed.

**Theorem 2.17.** [20, Theorem 33] Every principal ideal ring $R$ is the direct sum of principal ideal domains (PID) and special principal ideal rings.

**Theorem 2.18.** Let $R$ be a principal ideal ring (PIR). Then $R[x]$ is an IN-ring.
By Theorem 2.16, $R$ can be written in the form $R_1 \times \cdots \times R_n$, where $R_i$ is either a principal ideal domain or a special principal ideal ring for each $1 \leq i \leq n$. Then we have $R[x] = R_1[x] \times \cdots \times R_n[x]$. Now let $I, J$ be ideals of $R[x]$. Hence, $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$, for some ideals $I_i, J_i$ of $R_i[x]$. Clearly, $r_{R[x]}(I) = r_{R_1[x]}(I_1) \times \cdots \times r_{R_n[x]}(I_n)$. Now, since integral domains are IN-ring, hence by Proposition 2.16, one can easily prove that $r_{R[x]}(I) = r_{R[x]}(I) + r_{R[x]}(J)$. \hfill \Box

**Corollary 2.19.** Every principal ideal ring is an Armendariz IN-ring.

**Example 2.20.** Let $R = F[x]/(x^n)$, where $n \geq 2$, $F$ is a field and $(x^n)$ denotes the ideal of $F[x]$ generated by $x^n$. Then it is clear that $R$ is a principal ideal ring. Thus, $R$ is a non-reduced IN-ring and by Theorem 2.18, $R[y]$ is an IN-ring.

Let $R$ be a commutative ring and $M$ an $R$-module. Recall that $R \oplus M$ with coordinate-wise addition and multiplication given by $(r, m)(r', m') = (rr', rm' + mr')$ is a commutative ring with unity called the *idealization* of $M$ or the *trivial extension of $R$ by $M$*. By Anderson and Camillo [1], a right $R$-module $M$ is called *Armendariz* if $m(x)f = 0$ with $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f = \sum_{i=0}^{k} f_i x^i \in R[x]$, implies $m_i f_j = 0$ for each $i, j$.

**Example 2.21.** (i) Let $R$ be an integral domain and $M$ a torsion-free $R$-module. Then $T = R \oplus M$ is a commutative non-reduced ring. We show that $T$ is an IN-ring. To see this, it suffices to know that for a non-zero ideal $I$ of $T$, either $I$ contains an element $(r, m)$, where $0 \neq r \in R$ and $0 \neq m \in M$, which implies $r_{T}(I) = 0$, or all elements of $I$ has the form $(0, m)$, where $m \in M$, which implies $r_{T}(I) = 0 \oplus M$. Then it is not hard to check that $T$ is an IN-ring.

(ii) Let $R$ be an integral domain and $M$ an Armendariz torsion-free $R$-module. Now, since $M$ is an Armendariz torsion-free module, $M[x]$ is a torsion-free as an $R[x]$-module. Therefore, by (i), $T[x] = R[x] \oplus M[x]$ is an IN-ring.

### 3. Skew polynomials over SA-rings

According to [3, Definition 2.1], a ring $R$ is called a right SA-ring, if for any ideals $I$ and $J$ of $R$ there is an ideal $K$ of $R$ such that $r_{R}(I) + r_{R}(J) = r_{R}(K)$. Since $r_{R}(X) = r_{R}(RX)$ for all right ideal $X$ of $R$, $R$ is a right SA-ring, if for any right ideals $X$ and $Y$ of $R$ there is a right ideal $V$ of $R$ such that $r_{R}(X) + r_{R}(Y) = r_{R}(V)$. In this section, we will present some necessary and sufficient conditions for the Ore extension $R[x; \alpha, \delta]$ to be an SA ring.

For a left (right) ideal $I$ of $R$, we use $I[x; \alpha, \delta]$ to denote the set of all polynomials of $R[x; \alpha, \delta]$ with coefficients in $I$.

**Proposition 3.1.** Let $R$ be an $(\alpha, \delta)$-compatible ring. If $S = R[x; \alpha, \delta]$ is a right SA-ring, then $R$ is a right SA-ring.

**Proof.** Let $I, J$ be right ideals of $R$. It is easy to show that $I[x; \alpha, \delta]$ and $J[x; \alpha, \delta]$ are right ideals of $S$. Since $S$ is a right SA-ring, there exists a right ideal $K$ of $S$ such that $r_{S}(I[x; \alpha, \delta]) + r_{S}(J[x; \alpha, \delta]) = r_{S}(K)$. Now let $K_0$ be the right ideal of $R$ generated by the set $\bigcup_{f \in K} C_f$. We show that $r_{R}(I) + r_{R}(J) = r_{R}(K_0)$. Let $b \in r_{R}(I)$ and $c \in r_{R}(J)$. Then $b \in r_{S}(I[x; \alpha, \delta])$ and $c \in r_{S}(J[x; \alpha, \delta])$, by Lemma 2.1. Thus $b + c \in r_{S}(K)$. Hence $b + c \in r_{R}(K_0)$, by Lemma 2.2. Therefore, $r_{R}(I) + r_{R}(J) \subseteq r_{R}(K_0)$.

Now let $d \in r_{R}(K_0)$. Then $d \in r_{S}(K)$, by Lemma 2.1. Hence there exist $h = \sum_{i=0}^{n} h_i x^i \in r_{S}(I[x; \alpha, \delta])$ and $g = \sum_{i=0}^{m} g_i x^i \in r_{S}(J[x; \alpha, \delta])$ such that $d = h + g$ and so $d = h_0 + g_0$. Since $h_0 \in r_{R}(I)$ and $g_0 \in r_{R}(J)$, we have $d \in r_{R}(I) + r_{R}(J)$. This shows that $r_{R}(K_0) \subseteq r_{R}(I) + r_{R}(J)$ as claimed. \hfill \Box

Authors in [8] introduced the SQA1 condition, which is a skew polynomial version of the quasi-Armendariz rings. Let $\alpha$ be a monomorphism of $R$ and $\delta$ an $\alpha$-derivation. We say $R$ satisfies the SQA1 condition, if whenever $f = a_0 + a_1x + \cdots + a_n x^n$ and
\(g = b_0 + b_1 x + \cdots + b_m x^m \in R[x]\) satisfy \(fR[x; \alpha, \delta]g = 0\), then \(a_i r b_j = 0\), for each \(i, j\) and \(r \in R\). They showed that if \(R\) is an \((\alpha, \delta)\)-compatible quasi-Baer ring, then \(R\) satisfies SQA1 condition [8, Corollary 2.8].

**Proposition 3.2.** Let \(R\) be an \((\alpha, \delta)\)-compatible right SA-ring. If \(R\) satisfies the SQA1 condition, then \(S = R[x; \alpha, \delta]\) is a right SA-ring.

**Proof.** For an ideal \(K\) of \(S\), let \(K_0\) be the right ideal of \(R\) generated by the set \(\bigcup_{f \in K} C_f\).

Assume that \(I, J\) are right ideals of \(R[x; \alpha, \delta]\). By assumption, there is a right ideal \(P\) of \(R\) such that \(r_R(I_0) + r_R(J_0) = r_R(P)\). We claim that \(r_R(I) + r_R(J) = r_R(P[x; \alpha, \delta])\).

To see this, let \(f = a_0 + a_1 x + \cdots + a_n x^n \in r_R(I)\) and \(g = b_0 + b_1 x + \cdots + b_m x^m \in r_R(J)\). For each \(a \in I_0\), there is \(r_i \in R\) and \(c_i \in C_{b_i}\), for some \(h_i \in I\), such that \(a = \sum_{i=1}^{k} c_i r_i\).

Since \(R\) satisfies the SQA1 condition and \(h_i S f = 0\), for each \(1 \leq i \leq k\), hence we have \(c_i r a_j = 0\), for each \(c_i \in C_{b_i}, r \in R, 1 \leq i \leq k\) and \(0 \leq j \leq n\). Thus \(a a_j = 0\), for each \(0 \leq j \leq n\). It follows that \(a_j \in r_R(I_0)\), for each \(0 \leq j \leq m\). By a similar argument, one can show that \(b_i \in r(J_0)\) for each \(0 \leq i \leq m\) and hence \(a_i + b_i \in r_R(P)\). Then by Lemma 2.1, we have \(f + g \in r_R(P[x; \alpha, \delta])\), which implies that \(r_R(I) + r_R(J) \subseteq r_R(P[x; \alpha, \delta])\).

To prove the reverse inclusion, let \(h = d_0 + d_1 x + \cdots + d_k x^k \in r_R(P[x; \alpha, \delta])\). Since \(R\) satisfies the SQA1 condition, we have \(P d_i = 0\), for each \(0 \leq i \leq k\). Thus there exist \(a_i \in r_R(I_0)\) and \(b_i \in r_R(J_0)\) such that \(d_i = a_i + b_i\), for each \(0 \leq i \leq k\). Assume that \(f = a_0 + a_1 x + \cdots + a_k x^k\) and \(g = b_0 + b_1 x + \cdots + b_k x^k\). Then \(h = f + g, f \in r_R(I)\) and \(g \in r_R(J)\), by Lemma 2.1. Therefore, \(r_R(P) \subseteq r_R(I) + r_R(J)\).

As a generalization of Armendariz rings, Hirano [11] introduced quasi-Armendariz rings. A ring \(R\) is called quasi-Armendariz if whenever polynomials \(f = a_0 + a_1 x + \cdots + a_n x^n\) and \(g = b_0 + b_1 x + \cdots + b_m x^m \in R[x]\) satisfy \(fR[x]g = 0\), we have \(a_i R b_j = 0\), for each \(i, j\).

Clearly, each Armendariz ring is quasi-Armendariz, but the converse is not true in general. Birkenmeier et al. [3, Theorem 3.8] proved that if \(R\) is an Armendariz ring, then \(R\) is right SA if and only if \(R[x]\) is right SA. Now we extend this result to quasi-Armendariz rings.

**Corollary 3.3.** Let \(R\) be a quasi-Armendariz ring. Then \(R\) is right SA if and only if \(R[x]\) is right SA.

**Question 2:** Let \(R\) be an \((\alpha, \delta)\)-compatible ring and \(S = R[x; \alpha, \delta]\) be a right SA-ring. Does \(R\) satisfy SQA1 condition?

We end this section with study SA property over a special subring of upper triangular matrix rings. Let \(R\) be a ring and \(n\) a positive integer. An \((n + 1) \times (n + 1)\) matrix \(A\) with entries in \(R\) is called an upper triangular Toeplitz matrix if

\[
A = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_n \\
0 & a_0 & a_1 & \cdots & \vdots \\
0 & 0 & a_0 & \cdots & a_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & a_0
\end{pmatrix},
\]

where \(a_0, a_1, \ldots, a_n\) are elements of \(R\). For simplicity we can write

\[A = (a_i) = (a_0 \ a_1 \ a_2 \ \cdots \ a_n)\,.
\]

We denote the set of all such matrices by \(S_n(R)\) that is a subring of upper triangular matrix ring. In [3, Theorem 3.5], the authors proved that \(R\) is a right SA-ring if and only if \(T_m(R)\) is a right SA-ring, for some positive integer \(m\) (where \(T_m(R)\) denotes the set of all \(m\)-by-\(m\) upper triangular matrices over \(R\)).

In the following, we will prove an analogous result for \(S_n(R)\).
**Theorem 3.4.** Let $T = S_n(R)$ be a right SA-ring for some positive integer $n$. Then $R$ is a right SA-ring.

**Proof.** Let $I$ and $J$ be right ideals of $R$. Set $I' = S_n(I)$ and $J' = S_n(J)$. It is clear that $I'$ and $J'$ are right ideals of $T$. By assumption, there is a right ideal $K$ of $T$ such that $r_T(I') + r_T(J') = r_T(K)$. Clearly the set

$$Y = \{ c \in R \mid c = c_0 \text{ for some } C = (c_i) \in K \}$$

is a right ideal of $R$. We claim that $r_R(I) + r_R(J) = r_R(Y)$. To see this, let $x \in r_R(I)$ and $y = r_R(J)$. Since $(x \ 0 \ 0 \ \ldots \ 0) \in r_T(I')$ and $(y \ 0 \ 0 \ \ldots \ 0) \in r_T(J')$, then we have $(x + y \ 0 \ 0 \ \ldots \ 0) \in r_T(I') + r_T(J') = r_T(K)$. Thus $x + y \in r_R(Y)$ and hence $r_R(I) + r_R(J) \subseteq r_R(Y)$.

Now, let $z \in r_R(Y)$. Hence $(0 \ 0 \ \ldots \ 0 \ z) \in r_T(K) = r_T(I') + r_T(J')$. Therefore, there exist $A = (a_i) \in r(I')$ and $B = (b_i) \in r(J')$ such that $A + B = (0 \ 0 \ \ldots \ 0 \ z)$. Then $z = a_n + b_n$. Since for each $x \in I$, $(x \ 0 \ 0 \ \ldots \ 0) \in S_n(I)$, then $a_n \in r_R(I)$. Also, since for each $y \in J$, $(y \ 0 \ 0 \ \ldots \ 0) \in S_n(J)$, then $b_n \in r_R(J)$. Therefore, $z \in r_R(I) + r_R(J)$ and the proof is complete. □

**Theorem 3.5.** Let $R$ be a reduced right SA-ring. Then $T = S_n(R)$ is a right SA-ring, for each positive integer $n$.

**Proof.** Let $K$ be a right ideal of $S_n(R)$. For each $0 \leq i \leq n$, let

$$K_i = \{ a \in R \mid a \text{ is the } i\text{-th entry of some elements of } K \}.$$

Clearly, each $K_i$ is a right ideal of $R$ and $K_i \subseteq K_{i+1}$, for each $0 \leq i \leq n - 1$. Let $K^{(1)} = \{ (a_i) \in S_n(R) \mid a_j \in K_j \text{, for each } 0 \leq j \leq n \}$. Clearly, $K^{(1)}$ is a right ideal of $S_n(R)$ and $K \subseteq K^{(1)}$. Let $(a_i), (b_j) \in S_n(R)$, with $a_i b_j = 0$. Let $j \in \{ 0, 1, \ldots, n \}$. Since $R$ is reduced, one can easily show that $a_i b_j = 0$, for each $0 \leq i \leq n - j$. Then $r_T(K) = r_T(K^{(1)})$.

Let $I$ and $J$ be right ideals of $T$. As mentioned in the previous paragraph, $r_T(I) = r_T(I^{(1)})$ and $r_T(J) = r_T(J^{(1)})$. Since $R$ is right SA, hence for each $0 \leq i \leq n$, $r_R(I_i) + r_R(J_i) = r_R(K_i)$, for some right ideal $K_i$ of $R$. Since $r_R(I_{i+1}) \subseteq r_R(I_i)$ and $r_R(J_{i+1}) \subseteq r_R(J_i)$, for each $i$, hence $r_R(K_{i+1}) \subseteq r_R(K_i)$, and so we can assume that $K_i \subseteq K_{i+1}$, for each $i$. Now, by a simple calculation, one can show that $r_T(I^{(1)}) + r_T(J^{(1)}) = r_T(K^{(1)})$, and the proof is complete. □

For each positive integer $n$, it is a well known result that $S_n(R) \cong R[x]/(x^{n+1})$, where $(x^{n+1})$ denotes the ideal of $R[x]$ generated by $x^{n+1}$. Then, by using Theorems 3.4 and 3.5, we have the following result.

**Corollary 3.6.** Let $R$ be a reduced ring and $n$ be a positive integer. Then $R$ is right SA if and only if $R[x]/(x^{n+1})$ is right SA.

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**References**


