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Research Article

On sum annihilator ideals in Ore extensions

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Abstract

A ring R is called a left Ikeda-Nakayama ring (left IN-ring) if the right annihilator of the intersection of any two left ideals is the sum of the two right annihilators. As a generalization of left IN-rings, a ring R is called a right SA-ring if the sum of right annihilators of two ideals is a right annihilator of an ideal of R. It would be interesting to find conditions under which an Ore extension $R[x; \alpha, \delta]$ is IN and SA. In this paper, we will present some necessary and sufficient conditions for the Ore extension $R[x; \alpha, \delta]$ to be left IN or right SA. In addition, for an (α, δ)-compatible ring R, it is shown that: (i) If $S = R[x; \alpha, \delta]$ is a left IN-ring with $\mathrm{Idm}(R) = \mathrm{Idm}(R[x; \alpha, \delta])$, then R is left McCoy. (ii) Every reduced left IN-ring with finitely many minimal prime ideals is a semiprime left Goldie ring. (iii) If R is a commutative principal ideal ring, then R and $R[x]$ are IN. (iv) If R is a reduced ring and n is a positive integer, then R is right SA if and only if $R[x]/(x^{n+1})$ is right SA.

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1. Introduction and preliminary definitions

According to [5], a ring R is called a *left Ikeda-Nakayama ring* (left IN-ring) if $r_R(I \cap J) = r_R(I) + r_R(J)$ for all left ideals I, J of R. For example, all left self-injective rings, all left uniserial rings and all left uniform domains are left IN-ring. Kaplansky [13] introduced *Dual rings* as rings which every right or left ideal of them is an annihilator. Hajarnavis and Norton [7] proved that every dual ring is a right (and left) IN-ring. Wisbauer et al. [19] extended the notion of an Ikeda-Nakayama ring to bimodules and derived various characterizations and properties for modules with this property.

As a generalization of IN-rings, Birkenmeier et al. [3, 4] introduced SA-rings. A ring R is called a *right SA-ring*, if for any ideals I and J of R, there is an ideal K of R such that $r_R(I) + r_R(J) = r_R(K)$. They showed that this class of rings is exactly the class of rings for which the lattice of right annihilator ideals is a sub-lattice of the lattice of ideals. The class of right SA-rings includes all quasi-Baer (hence all Baer) rings and all right IN-rings (hence all right self-injective rings). Also they showed that this class is closed under direct products, full and upper triangular matrix rings and certain classes of polynomial rings.

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Throughout this paper, R denotes an associative ring with unity, $\alpha : R \longrightarrow R$ is an endomorphism, and δ is an α-derivation of R (i.e., δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$). We denote by $S = R[x; \alpha, \delta]$ the Ore extension whose elements are the polynomials over R, where addition is defined as usual and multiplication by $xb = \alpha(b)x + \delta(b)$ for any $b \in R$. For a subset $A \subseteq R$, we denote the right annihilator and left annihilator of A in R by $r_R(A)$ and $\ell_R(A)$, respectively. The set of all right zero divisors of R is denoted by $Z_r(R)$.

It is natural to ask if these properties (IN and SA) can be extended from R to $R[x; \alpha, \delta]$. The purpose of the present paper is to study Ore extensions over IN-rings and SA-rings. In this note we show that some portions of the results in [18] can be generalized to the Ore extension $R[x; \alpha, \delta]$, where the base coefficient ring R is an (α, δ)-compatible ring. In addition, in Section 2, we show that if $R[x; \alpha, \delta]$ is a left IN-ring with $\mathrm{Idm}(R[x; \alpha, \delta]) = \mathrm{Idm}(R)$, then $\ell_{R[x;\alpha,\delta]}(g) \cap R \neq \{0\}$, for each $g \in Z_r(R[x; \alpha, \delta])$. Furthermore, it is proved that every reduced left IN-ring R with finitely many minimal prime ideals is a semiprime left Goldie ring and $R[x; \alpha, \delta]$ is a left IN-ring. Finally, for a commutative principal ideal ring, it is shown that the IN property is inherited by polynomial extensions. In the third section, we investigate Ore extensions over SA-rings. For example, it is proved that if $R[x; \alpha, \delta]$ is a right SA-ring, then so is R, and the reverse is true when R satisfy SQA1 condition. In addition, it is shown that for a reduced ring R and a positive integer n, R is right SA if and only if $R[x]/(x^{n+1})$ is right SA. Moreover, each section contains some examples to show that the "(α, δ)-compatible" assumption on R is not superfluous. Also, examples of non-reduced IN-ring R such that $R[x]$ is left IN-ring are provided.

2. Skew polynomials over IN-rings

In this section, we will present some necessary and sufficient conditions for the Ore extension $R[x; \alpha, \delta]$ to be an IN ring. To fulfill this plan, we shall need to find a McCoy-like property of an IN Ore extension. The aim of our first result in this section is to state and prove it.

According to [8], an ideal I is called an *α-compatible ideal* if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. In addition, I is said to be a *δ-compatible ideal* if for each $a, b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If I is both α-compatible and δ-compatible, we say that I is an *(α, δ)-compatible ideal*. If $I = 0$ is α-compatible (resp., δ-compatible), then the ring R is called α-compatible (resp., δ-compatible). Also, if R is both α-compatible and δ-compatible, then R is said to be (α, δ)-compatible. The concept of α-compatible rings were defined in [9], as a common generalization of α-rigid rings. It was proved [9, Lemma 2.2] that R is α-rigid if and only if R is α-compatible and reduced. Clearly, each compatible endomorphism is a monomorphism.

We begin this section with the following essential lemmas.

Lemma 2.1. [10, Lemma 2.1] *Let R be an (α, δ)-compatible ring and $a, b \in R$. Then we have the following:*

1. *If $ab = 0$, then $a\alpha^n(b) = 0 = \alpha^n(a)b$ for each non-negative integer n.*
2. *If $\alpha^k(a)b = 0$ for some non-negative integer k, then $ab = 0$.*
3. *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for any non-negative integers m, n.*
4. *If $ab = 0$, then $\alpha(a)\alpha(b) = 0 = \delta(a)\delta(b)$.*
5. *If $ab = 0$, then $ax^m b = 0$ in $R[x; \alpha, \delta]$, for each $m \geq 0$.*
6. *If $ax^m b = 0$ in $R[x; \alpha, \delta]$, for some $m \geq 0$, then $ab = 0$.*

Lemma 2.2. [9, Lemma 2.3] *Let R be an (α, δ)-compatible ring. If $f = a_0 + a_1x + \cdots + a_nx^n \in R[x; \alpha, \delta]$, $r \in R$ and $fr = 0$, then $a_ir = 0$ for each i.*

We denote the set of all idempotent elements of R by $\mathrm{Idm}(R)$.

Proposition 2.3. *Let R be an (α,δ)-compatible ring. Also, let $f=a_0+a_1x+\cdots+a_nx^n$ and $g=b_0+b_1x+\cdots+b_mx^m$ be non-zero elements of $R[x;\alpha,\delta]$ such that $fg=0$. If $S=R[x;\alpha,\delta]$ is a left IN-ring with Idm(R) =Idm($R[x;\alpha,\delta]$), then $f=a_0$ or there exists $r\in R$ such that $0\neq ra_n$ and $ra_ng=0$.*

Proof. Since $fg=0$, then by Lemma 2.1, $a_nb_m=0$. Also, since $S=R[x;\alpha,\delta]$ is left IN, we have $r_S(f)+r_S(a_n)=r_S(Sf\cap Sa_n)$. Now, we consider the following two cases:

Case 1: Assume that $Sf\cap Sa_n=\{0\}$. Then there exists an idempotent $e\in R$, such that $Sf\subseteq Se$ and $Sa_n\subseteq S(1-e)$, by [5, Corollary 4]. Then $f=fe$ and $a_n=a_n(1-e)$. Hence $a_n=a_n\alpha^n(e)$, and since R is α-compatible, we have $a_n=a_ne$. Therefore, $a_n=0$, which implies that $f=a_0$.

Case 2: Assume that $Sf\cap Sa_n\neq\{0\}$. Let $\gamma^{(1)},\beta^{(1)}\in S$ such that $0\neq\gamma^{(1)}f=\beta^{(1)}a_n$. Assume that $\beta^{(1)}a_n=\beta_{10}+\beta_{11}x+\cdots+\beta_{1t_1}x^{t_1}$, with $\beta_{1t_1}\neq 0$. Clearly, $\beta_{1t_1}=r_1\alpha^{t_1}(a_n)$, for some $r_1\in R$. Since $a_nb_m=0$, hence by Lemma 2.1, $\beta_{1i}b_m=0$, for each $0\leq i\leq t_1$. Then $(\gamma^{(1)}f)g_1=(\beta^{(1)}a_n)g_1=0$, where $g_1=b_0+b_1x+\cdots+b_{m-1}x^{m-1}$. Hence $\beta_{1t_1}b_{m-1}=0$, since R is α-compatible. Since S is left IN, we have $r_S(\beta^{(1)}a_n)+r_S(\beta_{1t_1})=r_S((S\beta^{(1)}a_n)\cap(S\beta_{1t_1}))$. If $(S\beta^{(1)}a_n)\cap(S\beta_{1t_1})=\{0\}$, then by Case 1, $\beta^{(1)}a_n=\beta_{10}$. Since $\beta_{10}b_m=0=\beta_{10}g_1$, hence $\beta_{10}g=0$, and the result follows.

If $(S\beta^{(1)}a_n)\cap(S\beta_{1t_1})\neq\{0\}$, then there exist $\gamma^{(2)},\beta^{(2)}\in S$ such that $0\neq\gamma^{(2)}(\beta^{(1)}a_n)=\beta^{(2)}\beta_{1t_1}$. Assume that $\beta^{(2)}\beta_{1t_1}=\beta_{20}+\beta_{21}x+\cdots+\beta_{2t_2}x^{t_2}$, with $\beta_{2t_2}\neq 0$. Clearly, $\beta_{2t_2}=r_2\alpha^{t_2}(\beta_{1t_1})$, for some $r_2\in R$. Hence $\beta_{2t_2}=r_2\alpha^{t_2}(\beta_{1t_1})=r_2\alpha^{t_2}(r_1\alpha^{t_1}(a_n))=r_2\alpha^{t_2}(r_1)\alpha^{t_1+t_2}(a_n)$. Since $\beta_{1t_1}b_{m-1}=0$, hence by Lemma 2.1, $\beta_{2i}b_{m-1}=0$, for each $0\leq i\leq t_2$. Then $(\gamma^{(2)}\gamma^{(1)}f)g_2=(\gamma^{(2)}\beta^{(1)}a_n)g_2=(\beta^{(2)}\beta_{1t_1})g_2=0$, where $g_2=b_0+b_1x+\cdots+b_{m-2}x^{m-2}$.

By continuing this process we can find a non-zero element $\beta_{(m-1)t_{(m-1)}}\in R$ such that $\beta_{(m-1)t_{(m-1)}}g=0$ and $\beta_{(m-1)t_{(m-1)}}=r_{(m-1)}\alpha^{t_{(m-1)}}(r_{(m-2)})\alpha^{(t_{(m-1)}+t_{(m-2)})}r_{(m-3)})\ldots$ $\alpha^{(t_{(m-1)}+\cdots+t_2)}(r_1)\alpha^{(t_{(m-1)}+\cdots+t_2+t_1)}(a_n)$, for some $r_1,\ldots,r_{(m-1)}\in R$ and some non-negative integers $t_1,\ldots,t_{(m-1)}$. Then $r_{(m-1)}\ldots r_2r_1a_ng=0$, by Lemma 2.1. By considering $r=r_{(m-1)}\ldots r_2r_1$, the result follows. □

As an immediate consequence of Proposition 2.3, we get the following result.

Corollary 2.4. *Let R be an (α,δ)-compatible ring. Let $f=a_0+a_1x+\cdots+a_nx^n, g=b_0+b_1x+\cdots+b_mx^m$ be non-zero elements of $R[x;\alpha,\delta]$ satisfy $fg=0$. If $S=R[x;\alpha,\delta]$ is a left IN-ring with Idm(R) =Idm($R[x;\alpha,\delta]$), then there exists $r\in R$ such that $0\neq rf$ and $ra_ib_j=0$, for each $0\leq i\leq n$ and $0\leq j\leq m$.*

It is often taught in an elementary algebra course that if R is a commutative ring, and $f(x)$ is a zero-divisor in $R[x]$, then there is a non-zero element $r\in R$ with $f(x)r=0$. This was first proved by McCoy [16, Theorem 2]. Recall from [17] that a ring R is called *left McCoy* when the equation $f(x)g(x)=0$ over $R[x]$, where $f(x),g(x)\neq 0$, implies there exists a non-zero $r\in R$ with $rg(x)=0$.

Taking $\alpha=id_R$ and $\delta=0$, the following result is immediate from Proposition 2.3.

Corollary 2.5. *Let $S=R[x]$ be a left IN-ring with Idm(R) = Idm($R[x]$). Then R is left McCoy.*

Now, we give some classes of rings R, such that $\mathrm{Idm}(R)=\mathrm{Idm}(R[x;\alpha,\delta])$. Recall that a ring R is called *abelian* if all idempotent elements of R are central.

Example 2.6. (i) Let R be an (α,δ)-compatible ring. If $R[x;\alpha,\delta]$ is an abelian ring, then $\mathrm{Idm}(R)=\mathrm{Idm}(R[x;\alpha,\delta])$.

(ii) Let R be an abelian α-compatible ring. Then $\mathrm{Idm}(R)=\mathrm{Idm}(R[x;\alpha])$.

Proof. (i) Let $e = e_0 + e_1x + \cdots + e_nx^n$ be an idempotent element of $R[x;\alpha,\delta]$. Since $xe = ex$, we have

$$\delta(e_0) = 0; \tag{2.1}$$

$$\alpha(e_0) + \delta(e_1) = e_0;$$
$$\alpha(e_1) + \delta(e_2) = e_1;$$
$$\vdots$$
$$\alpha(e_{n-1}) + \delta(e_n) = e_{n-1};$$
$$\alpha(e_n) = e_n.$$

Since $e^2 = e$, then $e_0^2+e_1\delta(e_0)+\cdots+e_n\delta^n(e_0) = e_0$ and $e_n\alpha^n(e_n) = 0$. Then by using (2.1), we have $e_0^2 = e_0$. Now, by the abelian assumption on $R[x;\alpha,\delta]$ and by using [12, Theorem 3.13], we obtain $e \in \mathrm{Idm}(R)$.

(ii) By a similar argument as used in the proof of (i), one can show that $\mathrm{Idm}(R) = \mathrm{Idm}(R[x;\alpha])$. □

Corollary 2.7. *Let R be an (α,δ)-compatible ring and $g \in Z_r(R[x;\alpha,\delta])$. If $R[x;\alpha,\delta]$ is an abelian left IN-ring, then $\ell_{R[x;\alpha,\delta]}(g) \cap R \neq \{0\}$.*

Corollary 2.8. *Let R be an abelian α-compatible ring and $g \in Z_r(R[x;\alpha])$. If $R[x;\alpha]$ is a left IN-ring, then $\ell_{R[x;\alpha]}(g) \cap R \neq \{0\}$.*

Question 1: Let R be an (α,δ)-compatible ring and $S = R[x;\alpha,\delta]$ be a left IN-ring. Let $f = a_0 + a_1x + \cdots + a_nx^n$, $g = b_0 + b_1x + \cdots + b_mx^m$ be non-zero elements of $R[x;\alpha,\delta]$ satisfy $fg = 0$. Can we conclude $a_ib_j = 0$, for each i, j?

Let α be an endomorphism and δ an α-derivation on a ring R. Recall that an ideal I of R is called *α-ideal* if $\alpha(I) \subseteq I$; I is called a *δ-ideal* if $\delta(I) \subseteq I$; I is called an (α,δ)-ideal if it is both α- and δ-ideal. Clearly, if K is an (α,δ)-ideal of R, then $K[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$.

Proposition 2.9. *Let R be an (α,δ)-compatible ring. If $S = R[x;\alpha,\delta]$ is a left IN-ring, then for any (α,δ)-ideals I and J of R, $r_R(I) + r_R(J) = r_R(I \cap J)$.*

Proof. Let I, J be (α,δ)-ideals of R. Clearly $r_R(I) + r_R(J) \subseteq r_R(I \cap J)$. To prove the reverse inclusion, let $t \in r_R(I \cap J)$. Then $t \in r_S((I \cap J)[x;\alpha,\delta])$, by Lemma 2.2. On the other hand, $r_S(I[x;\alpha,\delta])+r_S(J[x;\alpha,\delta]) = r_S(I[x;\alpha,\delta]\cap J[x;\alpha,\delta])$, since S is a left IN-ring. Now, since $r_S((I\cap J)[x;\alpha,\delta]) = r_S(I[x;\alpha,\delta]\cap J[x;\alpha,\delta])$, it follows that $t = h(x)+k(x)$, for some $h(x) = \sum_{i=0}^n h_ix^i \in r_S(I[x;\alpha,\delta])$ and $k(x) = \sum_{i=0}^n k_ix^i \in r_S(J[x;\alpha,\delta])$. Then, since $Ih_0 = 0 = Jk_0$ and $t = h_0+k_0$, hence $t \in r_R(I)+r_R(J)$ and thus $r_R(I)+r_R(J) = r_R(I\cap J)$ as claimed. □

Lemma 2.10. *Let R be a reduced ring and $\{P_i\}_{i\in I}$ be the set of all distinct minimal prime ideals of R. If X is a non-zero left ideal of R contained in $\cap_{j\neq i}P_j$, for some $i \in I$, then $r_R(X) = P_i$.*

Proof. This follows from [6, Proposition 7.1]. □

Proposition 2.11. *Let R be a reduced left IN-ring. If R has finitely many minimal prime ideals, then ${}_RR$ has a finite left uniform dimension.*

Proof. Assume that $P_1, P_2, \ldots, P_n$ are all of the distinct minimal prime ideals of R. It is easy to see that $r_R(P_i) = \cap_{j\neq i}P_j$ for each $1 \leq i \leq n$. Now since $\cap_{i=1}^n P_i = 0$ and R is a left IN-ring, we have $r_R(P_1) + \cdots + r_R(P_n) = r_R(P_1 \cap \cdots \cap P_n) = R$. Therefore, $(\cap_{i\neq 1}P_i) \oplus \cdots \oplus (\cap_{i\neq n}P_i) = R$ and it is sufficient to prove that $\cap_{j\neq i}P_j$ is a uniform left ideal of R, for each $1 \leq i \leq n$. To see this, suppose that X, Y are non-zero left ideals of R contained in $\cap_{j\neq i} P_j$ with $X \cap Y = 0$. By using the left IN property of R and Lemma 2.10, we have $Pj = Pj + Pj = r_R(X) + r_R(Y) = r_R(X \cap Y) = R$, which is a contradiction. Therefore $\cap_{j\neq i} P_j$ is a uniform left ideal of R, for each $1 \leq i \leq n$. □

Corollary 2.12. *Let R be a reduced left IN-ring. If R has finitely many minimal prime ideals, then R is a semiprime left Goldie ring.*

Proof. It follows from Proposition 2.11 and [15, Theorem 2.15]. □

Recall that an ideal P of R is called *completely prime* whenever R/P is a domain.

Theorem 2.13. *Let R be a reduced (α, δ)-compatible left IN-ring. If R has finitely many minimal prime ideals, then $R[x; \alpha, \delta]$ is a left IN-ring.*

Proof. Let $P_1, \ldots, P_n$ be all of the distinct minimal prime ideals of R. By using Lemma 2.10 and the left IN property of R, we have $P_r + P_s = r_R(\cap_{j\neq r} P_j) + r_R(\cap_{j\neq s} P_j) = r_R(0) = R$, for each $r \neq s$. Now, by the Chinese Remainder Theorem, we have $R = R/P_1 \times \cdots \times R/P_n$. Since R is a reduced ring, hence P_i is completely prime and by Corollary 2.12 and [15, Theorem 2.5], R/P_i is a prime left Goldie ring, for each i. Also, since P_i is an annihilator ideal of R, hence P_i is an (α, δ)-compatible ideal of R, and so R/P_i is an $(\bar{\alpha}, \bar{\delta})$-compatible ring, by [8, Proposition 2.1], where $\bar{\alpha} : R/P_i \to R/P_i$ is defined by $\bar{\alpha}(a + P_i) = \alpha(a) + P_i$ and $\bar{\delta} : R/P_i \to R/P_i$ is defined by $\bar{\delta}(a + P_i) = \delta(a) + P_i$, for each $a \in R$. Then, by [14, Corollary 3.5], $R/P_i[x; \bar{\alpha}, \bar{\delta}]$ is a left Ore domain, for each i.

Finally, suppose that X, Y are left ideals of $R[x; \alpha, \delta]$. Since $R[x; \alpha, \delta] \cong R/P_1[x; \bar{\alpha}, \bar{\delta}] \times \cdots \times R/P_n[x; \bar{\alpha}, \bar{\delta}]$, hence for each i, there exist left ideals I_i, J_i of $R/P_i[x; \bar{\alpha}, \bar{\delta}]$, such that $X = I_1 \times \cdots \times I_n$ and $Y = J_1 \times \cdots \times J_n$. Then it is clear that $r_{R[x;\alpha,\delta]}(X) = r_{R/P_1[x;\bar{\alpha},\bar{\delta}]}(I_1) \times \cdots \times r_{R/P_n[x;\bar{\alpha},\bar{\delta}]}(I_n)$ and by using the fact that $R/P_i[x; \bar{\alpha}, \bar{\delta}]$ is a left Ore domain for each i, it follows that $r_{R[x;\alpha,\delta]}(X) + r_{R[x;\alpha,\delta]}(Y) = r_{R[x;\alpha,\delta]}(X \cap Y)$, which implies that $R[x; \alpha, \delta]$ is a left IN-ring. □

Now, we give an example to show that the "α-compatible" assumption on R, in Theorem 2.13 is not superfluous.

Example 2.14. Let $\mathbb{Z}_2$ be the field of integers modulo 2 and $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Clearly R is a reduced commutative IN-ring. Let $\alpha : R \to R$ be the endomorphism defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism of R, and since $(1, 0)(0, 1) = 0$ but $(1, 0)\alpha((0, 1)) \neq 0$, hence R is not α-compatible. Now let $p(x) = (1, 0) + (1, 0)x$ and $q(x) = (0, 1) + (0, 1)x \in R[x; \alpha]$. Let I and J be the left ideals of $R[x; \alpha]$ generated by $p(x)$ and $q(x)$, respectively. By a simple computation one can show that

$$I = \{(r_0, 0) + (r_0, s_1)x + \cdots + (r_t, s_{t-1})x^t + (r_t, 0)x^{t+1} | r_i, s_j \in \mathbb{Z}_2,\ t = 2i\} \cup$$

$$\{(r_0, 0) + (r_0, s_1)x + \cdots + (r_{t-1}, s_t)x^t + (0, s_t)x^{t+1}\ | r_i, s_j \in \mathbb{Z}_2,\ t = 2i + 1\}$$

and

$$J = \{(0, w_0) + (v_1, w_0)x + \cdots + (v_{k-1}, w_k)x^k + (0, w_k)x^{k+1} | v_i, w_j \in \mathbb{Z}_2, k = 2i\} \cup$$

$$\{(0, w_0) + (v_1, w_0)x + \cdots + (v_k, w_{k-1})x^k + (v_k, 0)x^{k+1} | v_i, w_j \in \mathbb{Z}_2, k = 2i + 1\}.$$

Then $I \cap J = 0$ and hence $r_{R[x;\alpha]}(I \cap J) = R[x; \alpha]$. On the other hand, for each $g = (r_0, s_0) + (r_1, s_1)x + \cdots + (r_n, s_n)x^n \in r_{R[x;\alpha]}(I)$, we have $r_0 = s_n = 0$ and $r_i + s_{i-1} = 0$, for each $1 \leq i \leq n$. Also, for each $h(x) = (v_0, w_0) + (v_1, w_1)x + \cdots + (v_m, w_m)x^m \in r_{R[x;\alpha]}(J)$, we have $w_0 = v_m = 0$ and $w_i + v_{i-1} = 0$, for each $1 \leq i \leq m$. Now, one can easily show that $(1, 1) \notin r_{R[x;\alpha]}(I) + r_{R[x;\alpha]}(J)$. Therefore, $r_{R[x;\alpha]}(I) + r_{R[x;\alpha]}(J) \neq R[x; \alpha]$, which implies that $R[x; \alpha]$ is not a left IN-ring. Thus, the "α-compatible" assumption on R in Theorem 2.13 is not superfluous.

The following example shows that we cannot eliminate the "reduced δ-compatible" assumption in Theorem 2.13.

Example 2.15. Let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t}) = 1$ where $\bar{t} = t + (t^2)$ is in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. It is clear that R is a non-reduced commutative IN-ring. Consider the differential polynomial ring $R[x;\delta]$. By [2, Example 11], $R[x;\delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. Since $\mathbb{Z}_2[y]$ is not a left self-injective ring, hence by [5, Theorem 7], $M_2(\mathbb{Z}_2)[y]$ is not a left IN-ring.

In the following, we construct some classes of commutative non-reduced IN-rings R with the property that $R[x]$ is also IN. However, the reduced condition in Theorem 2.13 plays an important role in the proof, the following examples show that it is not a necessary condition.

For the remainder of this section, R will denote a commutative ring with identity. Following Zariski and Samuel [20, page 22], we say the elements $a, b \in R$ are *relatively prime*, if $(a, b) = 1$. A *principal ideal ring* (PIR) is a ring with identity in which every ideal is principal. Any PIR is obviously Noetherian, and the PIR's may be considered the simplest type of Noetherian rings. By Zariski and Samuel [20, page 245], a PIR is called *special* if it has only one prime ideal $P \neq R$ and P is nilpotent, that is, $P^n = (0)$ for some positive integer n. If we place $P = pR$, and if we denote by m the smallest integer such that $p^m = 0$, then every non-zero element x in R may obviously be written in the form $x = ep^k$, where $0 \leq k \leq m-1$, and where $e \notin Rp$ (i.e, e and p are relatively prime). Special principal ideal rings are examples of uniserial rings.

A ring R is called *Armendariz* whenever polynomials $f = a_0 + a_1x + \cdots + a_nx^n$ and $g = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $fg = 0$, then $a_ib_j = 0$, for each i, j. The name "Armendariz ring" was chosen, because Armendariz had noted that a reduced ring satisfies this condition.

Proposition 2.16. *Let R be a special principal ideal ring. Then $S = R[x]$ is an IN-ring.*

Proof. Let R be a special principal ideal ring with maximal ideal $M = mR$ and n be the smallest integer such that $m^n = 0$. For an ideal K of S, we denote

$$K_0 = \{a \in R \mid \ a \in C_f \ \ \textit{for some} \ \ f \in K\}.$$

Now let I, J be non-zero ideals of S. It is clear that I_0, J_0 are ideals of R. Assume that $I_0 = m^kR, J_0 = m^sR$ such that $0 \leq k \leq s \leq n-1$. Since $r_R(I_0) = m^{n-k}R$, $r_R(J_0) = m^{n-s}R$ and R is an Armendariz ring, then we have $r_S(I) = r_S(I_0[x]) = m^{n-k}R[x]$ and $r_S(J) = r_S(J_0[x]) = m^{n-s}R[x]$. Hence $r_S(I) + r_S(J) = r_S(J) = m^{n-s}R[x]$.

Now we claim that $r_S(I \cap J) = r_S((I \cap J)_0)[x] = m^{n-s}R[x]$. Since $m^k \in I_0$, there exists a non-zero element $f \in I$ such that $m^k \in C_f$. Assume that $f = r_0m^{k+i_0} + r_1m^{k+i_1}x + \cdots + r_nm^{k+i_n}x^n$ such that $(r_i, m) = 1$ and $i_j = 0$ for some $0 \leq j \leq n$. Then we have $f = m^kf_1(x)$, where $f_1(x) = r_0m^{i_0} + r_1m^{i_1}x + \cdots + r_nm^{i_n}x^n$ and $i_j = 0$ for some $0 \leq j \leq n$. By a similar argument, we can show that there exists a non-zero element $g \in J$ such that $g = m^sg_1(x)$, where $g_1(x) = r'_0m^{i'_0} + r'_1m^{i'_1}x + \cdots + r'_{n'}m^{i'_{n'}}x^{n'}$, $(r'_i, m) = 1$ for all $0 \leq i' \leq n'$ and $i'_j = 0$ for some $0 \leq j \leq n'$. Thus, $(m, d) = 1$, for some $d \in C_{f_1g_1}$. Therefore $m^sf_1(x)g_1(x) \in I \cap J$ and $m^sd \in (I \cap J)_0$ where m and d are relatively prime. Hence $r_R((I \cap J)_0) \subseteq r_R(m^sR) = m^{n-s}R$. Therefore, $r_R(I \cap J) = r_R((I \cap J)_0)[x] \subseteq r_S(m^sR[x]) = m^{n-s}R[x]$. The reverse inclusion is trivial and the proof is completed. □

Theorem 2.17. [20, Theorem 33] *Every principal ideal ring R is the direct sum of principal ideal domains (PID) and special principal ideal rings.*

Theorem 2.18. *Let R be a principal ideal ring (PIR). Then $R[x]$ is an IN-ring.*

Proof. By Theorem 2.17, R can be written in the form $R_1 \times \cdots \times R_n$, where R_i is either a principal ideal domain or a special principal ideal ring for each $1 \leq i \leq n$. Then we have $R[x] = R_1[x] \times \cdots \times R_n[x]$. Now let I, J be ideals of $R[x]$. Hence, $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$, for some ideals I_i, J_i of $R_i[x]$. Clearly, $r_{R[x]}(I) = r_{R_1[x]}(I_1) \times \cdots \times r_{R_n[x]}(I_n)$. Now, since integral domains are IN-ring, hence by Proposition 2.16, one can easily prove that $r_{R[x]}(I \cap J) = r_{R[x]}(I) + r_{R[x]}(J)$. □

Corollary 2.19. *Every principal ideal ring is an Armendariz IN-ring.*

Example 2.20. Let $R = F[x]/(x^n)$, where $n \geq 2$, F is a field and (x^n) denotes the ideal of $F[x]$ generated by x^n. Then it is clear that R is a principal ideal ring. Thus, R is a non-reduced IN-ring and by Theorem 2.18, $R[y]$ is an IN-ring.

Let R be a commutative ring and M an R-module. Recall that $R \oplus M$ with coordinate-wise addition and multiplication given by $(r, m)(r', m') = (rr', rm' + mr')$ is a commutative ring with unity called the *idealization* of M or the *trivial extension* of R by M. By Anderson and Camillo [1], a right R-module M is called *Armendariz* if $m(x)f = 0$ with $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f = \sum_{i=0}^{k} f_i x^i \in R[x]$, implies $m_i f_j = 0$ for each i, j.

Example 2.21. (i) Let R be an integral domain and M a torsion-free R-module. Then $T = R \oplus M$ is a commutative non-reduced ring. We show that T is an IN-ring. To see this, it suffices to know that for a non-zero ideal I of T, either I contains an element (r, m), where $0 \neq r \in R$ and $0 \neq m \in M$, which implies $r_T(I) = 0$, or all elements of I has the form $(0, m)$, where $m \in M$, which implies $r_T(I) = 0 \oplus M$. Then it is not hard to check that T is an IN-ring.

(ii) Let R be an integral domain and M an Armendariz torsion-free R-module. Now, since M is an Armendariz torsion-free module, $M[x]$ is a torsion-free as an $R[x]$-module. Therefore, by (i), $T[x] = R[x] \oplus M[x]$ is an IN-ring.

3. Skew polynomials over SA-rings

According to [3, Definition 2.1], a ring R is called a right SA-ring, if for any ideals I and J of R there is an ideal K of R such that $r_R(I) + r_R(J) = r_R(K)$. Since $r_R(X) = r_R(RX)$ for all right ideal X of R, R is a right SA-ring, if for any right ideals X and Y of R there is a right ideal V of R such that $r_R(X) + r_R(Y) = r_R(V)$. In this section, we will present some necessary and sufficient conditions for the Ore extension $R[x; \alpha, \delta]$ to be an SA ring.

For a left (right) ideal I of R, we use $I[x; \alpha, \delta]$ to denote the set of all polynomials of $R[x; \alpha, \delta]$ with coefficients in I.

Proposition 3.1. *Let R be an (α, δ)-compatible ring. If $S = R[x; \alpha, \delta]$ is a right SA-ring, then R is a right SA-ring.*

Proof. Let I, J be right ideals of R. It is easy to show that $I[x; \alpha, \delta]$ and $J[x; \alpha, \delta]$ are right ideals of S. Since S is a right SA-ring, there exists a right ideal K of S such that $r_S(I[x; \alpha, \delta]) + r_S(J[x; \alpha, \delta]) = r_S(K)$. Now let K_0 be the right ideal of R generated by the set $\bigcup_{f \in K} C_f$. We show that $r_R(I) + r_R(J) = r_R(K_0)$. Let $b \in r_R(I)$ and $c \in r_R(J)$. Then $b \in r_S(I[x; \alpha, \delta])$ and $c \in r_S(J[x; \alpha, \delta])$, by Lemma 2.1. Thus $b + c \in r_S(K)$. Hence $b + c \in r_R(K_0)$, by Lemma 2.2. Therefore, $r_R(I) + r_R(J) \subseteq r_R(K_0)$.

Now let $d \in r_R(K_0)$. Then $d \in r_S(K)$, by Lemma 2.1. Hence there exist $h = \sum_{i=0}^{n} h_i x^i \in r_S(I[x; \alpha, \delta])$ and $g = \sum_{i=0}^{m} g_i x^i \in r_S(J[x; \alpha, \delta])$ such that $d = h + g$ and so $d = h_0 + g_0$. Since $h_0 \in r_R(I)$ and $g_0 \in r_R(J)$, we have $d \in r_R(I) + r_R(J)$. This shows that $r_R(K_0) \subseteq r_R(I) + r_R(J)$ as claimed. □

Authors in [8] introduced the SQA1 condition, which is a skew polynomial version of the quasi-Armendariz rings. Let α be a monomorphism of R and δ an α-derivation. We say R satisfies the *SQA1* condition, if whenever $f = a_0 + a_1 x + \cdots + a_n x^n$ and

$g = b_0 + b_1x + \cdots + b_mx^m \in R[x; \alpha, \delta]$ satisfy $fR[x; \alpha, \delta]g = 0$, then $a_irb_j = 0$, for each i, j and $r \in R$. They showed that if R is an (α, δ)-compatible quasi-Baer ring, then R satisfies SQA1 condition [8, Corollary 2.8].

Proposition 3.2. *Let R be an (α, δ)-compatible right SA-ring. If R satisfies the SQA1 condition, then $S = R[x; \alpha, \delta]$ is a right SA-ring.*

Proof. For an ideal K of S, let K_0 be the right ideal of R generated by the set $\bigcup_{f \in K} C_f$.

Assume that I, J are right ideals of $R[x; \alpha, \delta]$. By assumption, there is a right ideal P of R such that $r_R(I_0) + r_R(J_0) = r_R(P)$. We claim that $r_S(I) + r_S(J) = r_S(P[x; \alpha, \delta])$. To see this, let $f = a_0 + a_1x + \cdots + a_nx^n \in r_S(I)$ and $g = b_0 + b_1x + \cdots + b_mx^m \in r_S(J)$. For each $a \in I_0$, there is $r_i \in R$ and $c_i \in C_{h_i}$, for some $h_i \in I$, such that $a = \sum_{i=1}^{k} c_ir_i$. Since R satisfies the SQA1 condition and $h_iSf = 0$, for each $1 \leq i \leq k$, hence we have $c_ira_j = 0$, for each $c_i \in C_{h_i}, r \in R$, $1 \leq i \leq k$ and $0 \leq j \leq n$. Thus $aa_j = 0$, for each $0 \leq j \leq n$. It follows that $a_j \in r_R(I_0)$, for each $0 \leq j \leq m$. By a similar argument, one can show that $b_i \in r(J_0)$ for each $0 \leq i \leq m$ and hence $a_i + b_i \in r_R(P)$. Then by Lemma 2.1, we have $f + g \in r_S(P[x; \alpha, \delta])$, which implies that $r_S(I) + r_S(J) \subseteq r_S(P[x; \alpha, \delta])$.

To prove the reverse inclusion, let $h = d_0 + d_1x + \cdots + d_kx^k \in r_S(P[x; \alpha, \delta])$. Since R satisfies the SQA1 condition, we have $Pd_i = 0$, for each $0 \leq i \leq k$. Thus there exist $a_i \in r_R(I_0)$ and $b_i \in r_R(J_0)$ such that $d_i = a_i + b_i$, for each $0 \leq i \leq k$. Assume that $f = a_0 + a_1x + \cdots + a_kx^k$ and $g = b_0 + b_1x + \cdots + b_kx^k$. Then $h = f + g$, $f \in r_S(I)$ and $g \in r_S(J)$, by Lemma 2.1. Therefore, $r_S(P) \subseteq r_S(I) + r_S(J)$. □

As a generalization of Armendariz rings, Hirano [11] introduced quasi-Armendariz rings. A ring R is called *quasi-Armendariz* if whenever polynomials $f = a_0 + a_1x + \cdots + a_nx^n$ and $g = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $fR[x]g = 0$, we have $a_iRb_j = 0$, for each i, j. Clearly, each Armendariz ring is quasi-Armendariz, but the converse is not true in general. Birkenmeier et al. [3, Theorem 3.8] proved that if R is an Armendariz ring, then R is right SA if and only if $R[x]$ is right SA. Now we extend this result to quasi-Armendariz rings.

Corollary 3.3. *Let R be a quasi-Armendariz ring. Then R is right SA if and only if $R[x]$ is right SA.*

Question 2: Let R be an (α, δ)-compatible ring and $S = R[x; \alpha, \delta]$ be a right SA-ring. Does R satisfy SQA1 condition?

We end this section with study SA property over a special subring of upper triangular matrix rings. Let R be a ring and n a positive integer. An $(n + 1) \times (n + 1)$ matrix A with entries in R is called an *upper triangular Toeplitz matrix* if

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ 0 & a_0 & a_1 & \ddots & \vdots \\ 0 & 0 & a_0 & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ 0 & \dots & \dots & \dots & a_0 \end{pmatrix},$$

where $a_0, a_1, \dots, a_n$ are elements of R. For simplicity we can write

$$A = (a_i) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

We denote the set of all such matrices by $S_n(R)$ that is a subring of upper triangular matrix ring. In [3, Theorem 3.5], the authors proved that R is a right SA-ring if and only if $T_m(R)$ is a right SA-ring, for some positive integer m (where $T_m(R)$ denotes the set of all m-by-m upper triangular matrices over R).

In the following, we will prove an analogous result for $S_n(R)$.

Theorem 3.4. *Let $T = S_n(R)$ be a right SA-ring for some positive integer n. Then R is a right SA-ring.*

Proof. Let I and J be right ideals of R. Set $I^{'} = \mathrm{S}_n(I)$ and $J^{'} = \mathrm{S}_n(J)$. It is clear that $I^{'}$ and $J^{'}$ are right ideals of T. By assumption, there is a right ideal K of T such that $r_T(I^{'}) + r_T(J^{'}) = r_T(K)$. Clearly the set

$$Y = \{c \in R \mid c = c_0 \text{ for some } C = (c_i) \in K\}$$

is a right ideal of R. We claim that $r_R(I) + r_R(J) = r_R(Y)$. To see this, let $x \in r_R(I)$ and $y = r_R(J)$. Since $(x \quad 0 \quad 0 \quad \dots \quad 0) \in r_T(I^{'})$ and $(y \quad 0 \quad 0 \quad \dots \quad 0) \in r_T(J^{'})$, then we have $(x + y \quad 0 \quad 0 \quad \dots \quad 0) \in r_T(I^{'}) + r_T(J^{'}) = r_T(K)$. Thus $x + y \in r_R(Y)$ and hence $r_R(I) + r_R(J) \subseteq r_R(Y)$.

Now, let $z \in r_R(Y)$. Hence $(0 \quad 0 \quad \dots \quad 0 \quad z) \in r_T(K) = r_T(I^{'}) + r_T(J^{'})$. Therefore, there exist $A = (a_i) \in r(I^{'})$ and $B = (b_i) \in r_T(J^{'})$ such that $A + B = (0 \quad 0 \quad \dots \quad 0 \quad z)$. Then $z = a_n + b_n$. Since for each $x \in I$, $(x \quad 0 \quad 0 \quad \dots \quad 0) \in S_n(I) = I^{'}$, then $a_n \in r_R(I)$. Also, since for each $y \in J$, $(y \quad 0 \quad 0 \quad \dots \quad 0) \in S_n(I) = J^{'}$, then $b_n \in r_R(J)$. Therefore, $z \in r_R(I) + r_R(J)$ and the proof is complete. □

Theorem 3.5. *Let R be a reduced right SA-ring. Then $T = S_n(R)$ is a right SA-ring, for each positive integer n.*

Proof. Let K be a right ideal of $S_n(R)$. For each $0 \leq i \leq n$, let

$$K_i = \{a \in R \mid a \text{ is the } i\text{-th entry of some elements of } K\}.$$

Clearly, each K_i is a right ideal of R and $K_i \subseteq K_{i+1}$, for each $0 \leq i \leq n-1$. Let $K^{(1)} = \{(a_i) \in S_n(R) \mid a_j \in K_j, \text{ for each } 0 \leq j \leq n\}$. Clearly, $K^{(1)}$ is a right ideal of $S_n(R)$ and $K \subseteq K^{(1)}$. Let $(a_i), (b_j) \in S_n(R)$, with $(a_i)(b_j) = 0$. Let $j \in \{0, 1, \dots, n\}$. Since R is reduced, one can easily show that $a_i b_j = 0$, for each $0 \leq i \leq n - j$. Then $r_T(K) = r_T(K^{(1)})$.

Let I and J be right ideals of T. As mentioned in the previous paragraph, $r_T(I) = r_T(I^{(1)})$ and $r_T(J) = r_T(J^{(1)})$. Since R is right SA, hence for each $0 \leq i \leq n$, $r_R(I_i) + r_R(J_i) = r_R(K_i)$, for some right ideal K_i of R. Since $r_R(I_{i+1}) \subseteq r_R(I_i)$ and $r_R(J_{i+1}) \subseteq r_R(J_i)$, for each i, hence $r_R(K_{i+1}) \subseteq r_R(K_i)$, and so we can assume that $K_i \subseteq K_{i+1}$, for each i. Now, by a simple calculation, one can show that $r_T(I^{(1)}) + r_T(J^{(1)}) = r_T(K^{(1)})$, and the proof is complete. □

For each positive integer n, it is a well known result that $S_n(R) \cong R[x]/(x^{n+1})$, where (x^{n+1}) denotes the ideal of $R[x]$ generated by x^{n+1}. Then, by using Theorems 3.4 and 3.5, we have the following result.

Corollary 3.6. *Let R be a reduced ring and n be a positive integer. Then R is right SA if and only if $R[x]/(x^{n+1})$ is right SA.*

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