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IFHP Transformations on the Tangent Bundle with the Deformed Complete Lift Metric

Mosayeb Zohrehvand*

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ABSTRACT

Let (M_n, g) be a Riemannian manifold and TM_n the total space of its tangent bundle. In this paper, we determine the infinitesimal fiber-preserving holomorphically projective (IFHP) transformations on TM_n with respect to the Levi-Civita connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n and g^C and g^V are the complete lift and the vertical lift of g on TM_n, respectively. Morevore, we prove that every IFHP transformation on $(TM_n, \tilde{G}_f)$ is reduced to an affine and induces an infinitesimal affine transformation on (M_n, g).

Keywords: *Complete lift metric, infinitesimal fiber-preserving transformation, infinitesimal holomorphically projective transformations, adapted almost complex structure.*

AMS Subject Classification (2020): *Primary: 53B20 ; Secondary: 53C07; 53C15.*

1. Introduction

Let M_n be a connected n-dimensional manifold and TM_n the total space of its tangent bundle. It should be noted that, all the geometric objects, which will be considered in this paper, are assumed to be differentiable of the class C^∞. Also, the set of all tensor fields of type (r, s) on M_n and TM_n are denoted by $\Im^r_s(M_n)$ and $\Im^r_s(TM_n)$, respectively.

Let ∇ be an affine connection on M_n. If a transformation on M_n preserves the geodesics as point sets, then it is called a projective tansformation. Also, a transformation on M_n which preserves the connection is called affine transformation. Therefore, an affine transformation is a projective transformation which preserves the geodesics with the affine parameter.

A vector field V on M_n with the local one-parameter group $\{\phi_t\}$ is called an infinitesimal projective (resp. affine) transformation, if every ϕ_t is a projective (respectively affine) transformation on M_n.

It is well known that, a vector field V is an infinitesimal projective transformation if and and only if, for every $X, Y \in \Im^1_0(M_n)$, we have

$$(L_V\nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,$$

where Ω is a 1-form on M_n and L_V is the Lie derivation with respect to V. The 1-form Ω is called the associated 1-form of V. One can see that, V is an infinitesimal affine transformation if and only if $\Omega = 0$. For more details see [15].

Let J be an almost complex structure on (M_n, ∇), i.e. $J \in \Im^1_1(M)$, and $J^2 = -Id$. Note that, in this case the dimension of M_n is necessarily even, i.e. $n = 2m$, where $m \in \mathbb{N}$. An infinitesimal holomorphically projective transformation on M_n is a vector field V on M_n such that for every $X, Y \in \Im^1_0(M_n)$ we have

$$(L_V\nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X - \Omega(JX)JY - \Omega(JY)JX,$$

where Ω is a 1-form on M_n, and is called the associated 1-form of V. For $\Omega = 0$, it is obvious that V is an infinitesimal affine transformation. The notion of infinitesimal holomorphically projective transformations is

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*** Corresponding author***

introduced by S. Ishihara in [10] and after that many authors studied them on manifolds, e.g.[5, 7, 8, 9, 12, 13, 18].

Now let $\tilde{\phi}$ be a transformation on TM_n. If $\tilde{\phi}$ preserves the fibers, then it is called a fiber-preserving transformation. Let $\tilde{V}$ be a vector field on TM and $\{\tilde{\phi}_t\}$ the local one-parameter group generated by $\tilde{V}$. If for every t, $\tilde{\phi}_t$ is a fiber-preserving transformation, then $\tilde{V}$ is called an infinitesimal fiber-preserving transformation. Infinitesimal fiber-preserving transformations form a rich class of infinitesimal transformations on TM_n which include infinitesimal complete lift, horizontal lift and vertical lift transformations as special subclasses, (see [14]).

From a Riemannian metric g on M_n, several metric can be defined on TM_n such as 1) the Sasaki metric g^S which was introduced by Sasaki in [11], 2) the complete lift metric g^C, 3) the vertical lift metric g^V, and etc. (see [16]). It would be mentioned that g^S is a Riemannian metric, g^C is a pseudo-Riemannian metric and g^V is a degenerate form on TM_n.

Recently, a class of pseudo-Riemannian metrics on TM_n, of the form $\tilde{G}_f = g^C + (fg)^V$ is considered, where f is a nonzero differentiable function on M_n[6]. This is called the deformed complete lift metric. This new class of metrics is very interesting because for $f = 0$, the metric $\tilde{G}$ is the complete lift metric g^C and if $f = 1$ then $\tilde{G} = g^C + g^V$, thus this is a generalization of the complete lift metric g^C and of the metric $g^C + g^V$. It would be mentioned that the metric $g^C + g^V$ is called the metric I+II (see [16]). Also, the deformed complete lift metric is not a subcalss of g-natural metrics, in fact $\tilde{G}_f$ is a g-natural metric if and only if f is constant. For g-natural metrics, we quote [1, 2, 3] . On the other hand $\tilde{G}_f$ is a subclass of the synectic lift metric of g, which is defined in [4] and is of the form $\tilde{G} = g^C + a^V$, where $a \in \Im^0_2(M_n)$ is a symmetric tensor field.

Infinitesimal holomorphically projective transformations on tangent bundle of a Riemannian manifold (M, g) with respect to the complete lift metric g^C, the Sasaki metric g^S and the metric $g^C + g^V$ are considered in [5, 8] and [13].

The aim of this paper is to study of the infinitesimal fiber-preserving holomorphically projective(IFHP) transformations on TM_n with respect to the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n. Firstly, we obtained the necessary and sufficient conditions under which an infinitesimal fiber-preserving transformation on $(TM_n, \tilde{G}_f)$ is holomorphically projective. Then it is shown that every infinitesimal fiber-preserving holomorphically projective(IFHP) transformation on $(TM_n, \tilde{G}_f)$ is reduced to affine one. Finally, as special cases, the infinitesimal complete lift, horizontal lift and vertical lift holomorphically projective transformations on $(TM_n, \tilde{G}_f)$ are studied.

2. Preliminaries

Here, we give some of the basic and necessary definitions and theorems on M_n and TM_n, which are needed later. For more details see [16, 17]. Throughout this paper, indices $a, b, c, i, j, k, \ldots$ have range in $\{1, \ldots, n\}$.

Let M_n be a manifold covered by coordinate systems (U, x^i), where x^i are the coordinate functions on the coordinate neighborhood U. The tangent bundle of M_n is defined by $TM_n := \bigcup_{x\in M} T_x(M_n)$, where $T_x(M_n)$ is the tangent space of M_n at a point x. The elements of TM_n are denoted by (x, y) where $y \in T_x(M_n)$ and the natural projection $\pi : TM_n \to M_n$ is given by $\pi(x, y) := x$.

Let ∇ be the Levi-Civita connection of a Riemannian manifold (M_n, g) and its coefficients with respect to the frame field $\{\partial_i := \frac{\partial}{\partial x^i}\}$ are denoted by Γ^h_{ji} i.e. $\nabla_{\partial_j}\partial_i = \Gamma^h_{ji}\partial_h$.

Using the Levi-Civita Connection ∇, we can define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM_n, as follows

$$E_i := \partial_i - y^b\Gamma^h_{bi}\partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$. This frame field is called the adapted frame on TM_n. Setting $\delta y^h := dy^h + y^b\Gamma^h_{ab}dx^a$, one can see that $\{dx^h, \delta y^h\}$ is the dual frame of $\{E_i, E_{\bar{i}}\}$. The following lemma can be proved by the straightforwald calculations.

Lemma 2.1. *The Lie brackets of the adapted frame $\{E_i, E_{\bar{i}}\}$ satisfy the following identities:*

1. $[E_j, E_i] = y^b R^a_{ijb} E_{\bar{a}}$,
2. $[E_j, E_{\bar{i}}] = \Gamma^a_{ji} E_{\bar{a}}$,
3. $[E_{\bar{j}}, E_{\bar{j}}] = 0$,
   where R^a_{ijb} are the coefficients of the Riemannian curvature tensor of ∇.

Let X be a vector field on M_n and expressed by $X = X^i\partial_i$ on the local chart (U, x^i). We can define the horizontal lift X^H, vertical lift X^V and complete lift X^C of X on TM_n as follows

$$X^H := X^i E_i, \quad X^V := X^i E_{\bar{i}}, \quad X^C = X^i E_i + y^a \nabla_a X^i E_{\bar{i}},$$

where $\nabla_a := \nabla_{\partial_a}$.

A rich class of infinitesimal transformations on TM_n is represented by the infinitesimal fiber-preserving transformations, where include horizontal lift, vertical lift and complete lift of vector fields. The following lemma proved in [14] determines the infinitesimal fiber-preserving trasformations.

Lemma 2.2. *Let $\tilde{V} = \tilde{V}^i E_i + \tilde{V}^{\bar{i}} E_{\bar{i}}$ be a vector field on TM_n. Then $\tilde{V}$ is an infinitesimal fiber-preserving transformation if and only if $\tilde{V}^i$ are functions on M_n.*

Using Lemma 2.2, one can assume that $\tilde{V}^i := V^i(x)$. Therefore, every fiber-preserving vector field $\tilde{V}$ on TM_n induces a vector field $V = V^i\partial_i$ on M_n. By a simple calculation, the following lemma can be proved (see [19]).

Lemma 2.3. *Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be a fiber-preserving vector field on TM_m. Then*

1. $[\tilde{V}, E_i] = -(\partial_i V^a)E_a + (V^c y^b R^a_{icb} - \tilde{V}^{\bar{b}}\Gamma^a_{bi} - E_i\tilde{V}^{\bar{a}})E_{\bar{a}}$,
2. $[\tilde{V}, E_{\bar{i}}] = (V^b\Gamma^a_{bi} - E_{\bar{i}}\tilde{V}^{\bar{a}})E_{\bar{a}}$.

Now, we define a tensor field $\tilde{J} \in \Im^1_1(TM)$, as follow

$$JX^H = X^V, \;\; JX^V = -X^H,$$

for any vector field $X \in \Im^1_0(M_n)$. In other words

$$\tilde{J}E_i = E_{\bar{i}}, \;\; \tilde{J}E_{\bar{i}} = -E_i.$$

Thus we obtain

$$\tilde{J}^2 = -I,$$

which means that $\tilde{J}$ is an almost complex structure on TM_n. This is called the adapted almost complex structure. It is well known that $\tilde{J}$ is integrable if and only if M_n is locally flat (see [16]).

For a Riemannian metric g on a manifold M_n, the Sasaki metric g^S, the complete lift g^C and the vertical lift g^V of g are defined as follows, respectively:

$$\begin{aligned} g^S(X^H, Y^H) &= g(X, Y), \\ g^S(X^H, Y^V) &= 0, \\ g^S(X^V, Y^V) &= g(X, Y), \end{aligned} \tag{2.1}$$

$$\begin{aligned} g^C(X^H, Y^H) &= 0, \\ g^C(X^H, Y^V) &= g(X, Y), \\ g^C(X^V, Y^V) &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} g^V(X^H, Y^H) &= g(X, Y), \\ g^V(X^H, Y^V) &= 0, \\ g^V(X^V, Y^V) &= 0, \end{aligned} \tag{2.3}$$

for every $X, Y \in \Im^1_0(M_n)$. It would be noted that g^S is a Riemannian metric, g^C is a pseudo-Riemannian metric and g^V is a degenerate quadratic form. For more details see [16].

In [6], a new class of metrics on TM_n was introduced. It is a generalization of the complete lift metric g^C and is of the form $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n. It is called the deformed complete lift metric. It is easy to see that the deformed complete lift metric is a pseudo-Riemannian metric and it is defined by

$$\begin{aligned} \tilde{G}_f(X^H, Y^H) &= fg(X, Y), \\ \tilde{G}_f(X^H, Y^V) &= g(X, Y), \\ \tilde{G}_f(X^V, Y^V) &= 0, \end{aligned} \tag{2.4}$$

for any $X, Y \in \Im_0^1(M_n)$.

The coefficients of the Levi-Civita connection $\tilde{\nabla}$, of the pseudo Riemannian metric $\tilde{G}_f$, with respect to the adapted frame field $\{E_i, E_{\bar{i}}\}$ are computed in [6]. In fact, the following lemma is proved.

Lemma 2.4. *Let $\tilde{\nabla}$ be the Levi-Civita connection of the deformed complete lift metric $\tilde{G}_f = g^C + (fg)^V$, where f is a nonzero differentiable function on M_n, then we have*

$$\begin{aligned}
&\tilde{\nabla}_{E_j} E_i = \Gamma_{ji}^h E_h + y^k\{R_{kji}^h + \tfrac{1}{2}(f_i\delta_j^h + f_j\delta_i^h - g_{ji}f_{\cdot}^h)\}E_{\bar{h}},\\
&\tilde{\nabla}_{E_j} E_{\bar{i}} = \Gamma_{ji}^h E_{\bar{h}},\\
&\tilde{\nabla}_{E_{\bar{j}}} E_i = 0,\\
&\tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0.
\end{aligned}$$

where Γ_{ji}^h and R_{kji}^h are the coefficients of the Levi-Civita connection ∇ and the Riemannian curvature of $g := (g_{ji})$, respectively and $f_i := \partial_i f$, $f_{\cdot}^h := g^{hi} f_i$

3. Main results

Now, we study the infinitesimal fiber-preserving holomorphically projective(IFHP) transformations on $(TM_n, \tilde{G}_f)$ with the adapted almost complex structure $\tilde{J}$.

Theorem 3.1. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n the total space of its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted almost complex structure $\tilde{J}$. Then $\tilde{V}$ is an IFHP transformation with the associated one form $\tilde{\Omega}$ on TM_n if and only if there exist $\psi \in \Im_0^0(M_n)$, $V = (V^h), D = (D^h) \in \Im_0^1(M_n)$ and $C = (C_i^h) \in \Im_1^1(M_n)$, satisfying*

1. $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (V^h, D^h + y^a C_a^h + 2\psi y^h)$,
2. $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (0, 0)$,
3. $\partial_i \psi = 0$,
4. $V^a \nabla_a R_{jbi}^h + R_{abi}^h \nabla_j V^a + R_{jba}^h \nabla_i V^a + R_{jai}^h C_b^a - R_{jbi}^a C_a^h = 0$,
5. $\nabla_i C_j^h = V^a R_{iaj}^h$,
6. $L_V \Gamma_{ji}^h = \nabla_j \nabla_i V^h + V^a R_{aji}^h = 0$,
7. $L_D \Gamma_{ji}^h = \nabla_j \nabla_i D^h + D^a R_{aji}^h = -\{V^a \nabla_a M_{ji}^h + \nabla_i V^a M_{ja}^h + \nabla_j V^a M_{ia}^h - C_a^h M_{ji}^a - 2\psi M_{ji}^h\}$,

where $\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$, $\tilde{\Omega} = (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_i dx^i + \tilde{\Omega}_{\bar{i}} \delta y^i$, $M_{ij}^h := \frac{1}{2}(f_i\delta_j^h + f_j\delta_i^h - g_{ji}f_{\cdot}^h)$, $f_i := \partial_i f$, and $f_{\cdot}^h := g^{hi} f_i$.

Proof. Firstly, we prove the neccesary conditions. Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be an IFHP transformation on TM_n with respect to the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{G}_f$ and $\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}} \delta y^h$ its the associated one form, thus for any $\tilde{X}, \tilde{Y} \in \Im_0^1(TM_n)$, we have

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X} - \tilde{\Omega}(J\tilde{X})J\tilde{Y} - \tilde{\Omega}(J\tilde{Y})J\tilde{X}. \tag{3.1}$$

From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \tilde{\Omega}_{\bar{j}} E_{\bar{i}} + \tilde{\Omega}_{\bar{i}} E_{\bar{j}} - \tilde{\Omega}_j E_i - \tilde{\Omega}_i E_j,$$

we have

$$\tilde{\Omega}_j \delta_i^h + \tilde{\Omega}_i \delta_j^h = 0, \tag{3.2}$$

and

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}} \delta_i^h + \tilde{\Omega}_{\bar{i}} \delta_j^h. \tag{3.3}$$

From (3.2) one can see that

$$\tilde{\Omega}_i = 0. \tag{3.4}$$

Form (3.3) we obtain that, there exist $\psi \in \Im_0^0(M)$, $\Phi = (\Phi_i) \in \Im_1^0(M)$, $D = (D^h) \in \Im_0^1(M)$ and $C = (C_i^h) \in \Im_1^1(M)$ which satisfy

$$\tilde{\varphi} = \psi + y^a \Phi_a, \tag{3.5}$$

$$\tilde{\Omega}_{\bar{i}} = \partial_{\bar{i}} \tilde{\varphi} = \Phi_i, \tag{3.6}$$

and

$$\tilde{V}^{\bar{h}} = D^h + y^a C_a^h + 2\psi y^h + y^h y^a \Phi_a. \tag{3.7}$$

where $\psi := -\frac{1}{n-1} C_a^a$.
From

$$(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_i) = \Phi_j E_i + \Phi_i E_j,$$

and (3.4) and (3.7) we have

$$\big(\Phi_j \delta_i^h + \Phi_i \delta_j^h\big) E_h = \Big\{(\nabla_i C_j^h + 2\partial_i \psi \delta_j^h + V^a R_{aij}^h) + y^b \big(\nabla_i \Phi_j \delta_b^h + \nabla_i \Phi_b \delta_j^h\big)\Big\} E_{\bar{h}} \tag{3.8}$$

Comparing the both sides of the equation (3.8), we obtain

$$\Phi_i = 0, \ \partial_i \psi = 0, \tag{3.9}$$

$$\nabla_i C_j^h = V^a R_{iaj}^h. \tag{3.10}$$

Lastly from

$$(L_{\tilde{V}} \tilde{\nabla})(E_j, E_i) = 0,$$

and (3.9) and (3.10) we obtain that

$$\begin{aligned} 0 = &\big\{\nabla_j \nabla_i V^h + V^a R_{aji}^h\big\} E_h + \Big\{\nabla_j \nabla_i D^h + D^a R_{aji}^h + \frac{1}{2}\Big(V^a \nabla_a (f_j \delta_i^h + f_i \delta_j^h - g_{ji} f_{.}^h) \\ &+ \nabla_i V^a (f_j \delta_a^h + f_a \delta_j^h - g_{ja} f_{.}^h) + \nabla_j V^a (f_i \delta_a^h + f_a \delta_i^h - g_{ia} f_{.}^h) - C_a^h (f_i \delta_j^a + f_j \delta_i^a - g_{ji} f_{.}^a) \\ &- 2\psi (f_j \delta_i^h + f_i \delta_j^h - g_{ji} f_{.}^h)\Big) + y^b \big(V^a \nabla_a R_{jbi}^h + R_{abi}^h \nabla_j V^a + R_{jba}^h \nabla_i V^a + R_{jai}^h C_b^a - R_{jbi}^a C_a^h\big)\Big\} E_{\bar{h}}. \end{aligned} \tag{3.11}$$

From which we have

$$L_V \Gamma_{ji}^h = \nabla_j \nabla_i V^h + V^a R_{aji}^h = 0, \tag{3.12}$$

that is, $V = V^h \partial_h$ is an infinitesimal affine transformation on M_n,

$$L_D \Gamma_{ji}^h = \nabla_j \nabla_i D^h + D^a R_{aji}^h = -\left\{V^a \nabla_a M_{ji}^h + \nabla_i V^a M_{ja}^h + \nabla_j V^a M_{ia}^h - C_a^h M_{ij}^a - 2\psi M_{ji}^h\right\}, \tag{3.13}$$

where $M_{ij}^h := \frac{1}{2}(f_i \delta_j^h + f_j \delta_i^h - g_{ji} f_{.}^h)$, and

$$V^a \nabla_a R_{jbi}^h + R_{abi}^h \nabla_j V^a + R_{jba}^h \nabla_i V^a + R_{jai}^h C_b^a - R_{jbi}^a C_a^h = 0. \tag{3.14}$$

This completes the necessary conditions. The proof of the sufficient conditions is immediate. □

Theorem 3.2. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n the total space of its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted almost complex structure $\tilde{J}$. Then every infinitesimal fiber-preserving holomorphically projective transformation on TM_n is an affine one and induces an infinitesimal affine transformation on M_n.*

Proof. Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be an infinitesimal fiber-preservig holomorphically projective transformation on $(TM_n, \tilde{G}_f)$. By using (2) in Theorem 3.1, it is easy to see that $\tilde{V}$ is an infinitesimal affine transformation. Also from (6) in Theorem 3.1 it follows that $V := V^h \partial_h$ is an infinitesimal affine transformation on M. □

From Theorem 3.2, the following corollary can be immediately found.

Corollary 3.1. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n total space of its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted almost complex structure $\tilde{J}$. Then, the Lie algebra of fiber-preserving holomorphically projective vector fields on $(TM_n, \tilde{G}_f)$ is reduced to the Lie algebra of affine vector fields on $(TM_n, \tilde{G}_f)$.*

Let $V = V^h\partial_h$ be an affine vector field on M_n. Here we obtain the necessary and sufficient conditions such that complete lift, horizontal lift and veritcal lift of vector field V are affine vector fields on $(TM_n, \tilde{G}_f)$.

Theorem 3.3. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n the total space of its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$. Let $V = V^h\partial_h$ be an affine vector field on M_n, then V^C is an affine vector field on TM_n if and only if the following relations hold*

1. $V^a\nabla_a R^h_{jbi} = R^a_{jbi}\nabla_a V^h - R^h_{abi}\nabla_j V^a - R^h_{jba}\nabla_i V^a - R^h_{jai}\nabla_b V^a$,
2. $V^a\nabla_a M^h_{ji} = M^a_{ji}\nabla_a V^h - M^h_{ja}\nabla_i V^a - M^h_{ia}\nabla_j V^a$,

where $M^h_{ij} := \frac{1}{2}(f_i\delta^h_j + f_j\delta^h_i - g_{ji}f^h_{\cdot})$, $f_i := \partial_i f$, *and* $f^h_{\cdot} := g^{hi}f_i$.

Proof. Let $V = V^h\partial_h$ be an affine vector field on M_n and $V^C = V^a E_a + y^b\nabla_b V^a E_{\bar{a}}$. From Theorem3.1, one can see that V^C is a holomorphically projective vector field if and only if 4 and 7 are holds. In this case V^C is an affine vector filed. Thus V^C is an affine vector field if and only if 4 and 7 hold. □

Theorem 3.4. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n the total space of its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$ and let $V = V^h\partial_h$ be an affine vector field on M_n. Then V^H is an affine vector field on TM_n if and only if the following relations hold*

1. $V^a\nabla_a R^h_{jbi} + R^h_{abi}\nabla_j V^a + R^h_{jba}\nabla_i V^a = 0$,
2. $V^a\nabla_a M^h_{ji} + M^h_{ja}\nabla_i V^a + M^h_{ia}\nabla_j V^a = 0$,

where $M^h_{ij} := \frac{1}{2}(f_i\delta^h_j + f_j\delta^h_i - g_{ji}f^h_{\cdot})$, $f_i := \partial_i f$, *and* $f^h_{\cdot} := g^{hi}f_i$.

Proof. The proof is similar to that of Theorem 3.3. □

One can easily see that if V be an affine vector field on (M_n, g), then the vertical lift of V is an affine vector field on $(TM_n, \tilde{G}_f)$. Thus, we have the following corollary.

Corollary 3.2. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with the pseudo-Riemannian metric $\tilde{G}_f = g^C + (fg)^V$, where $0 \neq f \in \Im_0^0(M_n)$, and the adapted almost complex structure $\tilde{J}$. Then, there is a one-to-one correspondence between vertical lift holomorphically projective vector fields on $(TM_n, \tilde{G})$ and affine vector fields on (M_n, g).*

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Affiliations

MOSAYEB ZOHREHVAND
ADDRESS: Malayer University, Faculty of Mathematical Sciences and Statistic, Dept. of Mathematics, Malayer-Iran.
E-MAIL: m.zohrevand61@gmail.com, m.zohrehvand@malayeru.ac.ir
ORCID ID: orcid.org/0000-0002-3876-4060