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QUANTUM INTEGRABLE SYSTEM RELATED TO SO (2,3) GROUP

Edibe ELÇİN*, Cengiz DANE**

*Trakya University, Technical Sciences Vocational Higher School, 22030, Edirne / TURKEY
**Trakya University, Faculty of Arts and Sciences Department of Mathematics, 22030, Edirne / TURKEY
Corresponding Author: Tel. : +90 284 2352823, Fax : +90 284 2354010, e-mail : cfag@trakya.edu.tr

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Abstract: Quantum integrable system related to SO(2,3) group manifold or hyperboloid $[x\,x]=x_1^2+x_2^2-x_3^2-x_4^2-x_5^2=1$ in hyperbolic and parabolic coordinates systems is considered. The explicit expressions for wave functions , spectra on S-matrices are given.
Key words: Asymptotic expression, parabolic coordinates, Schrödinger equation, S-matrix

Introduction

There are certain results on complete systems of quantum commuting wave function, spectra and so on. The review presents, from a general point of view , the results obtained in these subjects [1] . The dynamics of some of these is closely related to free motion in the symmetric space (SS).

On the other hand, there are many coordinate systems which reduce to the separation of variables in Laplace - Beltrami operators given in [2] . There is a simple transformation of Laplace-Beltrami operator on symmetrical spaces (SS) to some Hamiltonian quantum systems only for geodesics relating to one parameter subgroup of symmetry group. Hence only distortion of the symmetry of the free particle motion on (SS) by the geodesic paths reduces to the dynamics of quantum systems.

The one dimensional integrable quantum systems related to free motion in some symmetric spaces of noncompact groups are considered in [1,3,4,5,6].The dynamics of a quantum system depends on stationary subgroup of the fixed point $\overset{0}{x}$ and the chosen coordinate systems on (SS). For the case of the (SS) with the compact stationary subgroup , the quantum system has only continuous spectrum , but for the case with the noncompact quantum system , it has discrete and continuous spectrum [7] .

We consider hyperbolic and parabolic coordinate systems on hyperboloid [x x] = 1. For a related quantum system we give explicit expressions for wave functions, spectra and S-matrix.

The quantum systems related with SO(2,3) Group Manifold

Let us consider bispherical and parabolic coordinate systems on the hyperboloid $[x\ x]=x_1^2+x_2^2-x_3^2-x_4^2-x_5^2=1$ which define a SO(2,3) group manifold.

Bispherical Coordinate Systems

The bispherical coordinates are given by

$$\begin{aligned} x_1 &= r\cosh\alpha\cos\theta \\ x_2 &= r\cosh\alpha\sin\theta\cos\varphi \\ x_3 &= r\cosh\alpha\sin\theta\sin\varphi \\ x_4 &= r\sinh\alpha\sin\psi \\ x_5 &= r\sinh\alpha\cos\psi \end{aligned} \tag{1}$$

where $0<r<\infty$, $0\le\alpha<\infty$, $0\le\varphi,\psi\le 2\pi$, $0\le\theta<\pi$.

From (1) we have

$$(g_{ab}) = diag\left(1, -r^2, r^2\cosh^2\alpha, r^2\cosh^2\alpha\sin\theta, -r^2\sinh^2\alpha\right) \tag{2}$$

$$\sqrt{g} = \left(\det(g_{ab})\right)^{1/2} = r^4\cosh\alpha\sinh\alpha\sin\theta \tag{3}$$

The component of the Laplace-Beltrami operator

$$\Delta_{LB} = \frac{-1}{\sqrt{g}}\partial_a g^{ab}\sqrt{g}\,\partial_b \quad , \qquad a,b=1,2,3,... \tag{4}$$

on the hyperboloid [x x] = 1 is in the following form :

$$\Delta_{LB}=-\left[\frac{\partial^2}{\partial\alpha^2}+(2\tanh\alpha+\coth\alpha)\frac{\partial}{\partial\alpha}\right]+\frac{1}{\cosh^2\alpha}\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]-\frac{1}{\sinh^2\alpha}\left[\frac{\partial^2}{\partial\psi^2}\right] \tag{5}$$

Free motion on the hyperboloid [x x] = 1 is defined by the equation

$$\Delta_{LB}\Phi(\alpha,\theta,\varphi,\psi)=-\sigma(\sigma+3)\Phi(\alpha,\theta,\varphi,\psi)\ . \tag{6}$$

After the substitution of

$$\Phi(\alpha,\theta,\varphi,\psi)=T(\alpha)\,S(\theta)\,e^{im\varphi}e^{in\psi} \tag{7}$$

in Eq.(6) we have

$$\frac{1}{\cosh^2}\left[\frac{d^2S(\theta)}{d\theta^2}+\cot\theta\frac{dS(\theta)}{d\theta}-\frac{m^2}{\sin^2\theta}S(\theta)\right]T(\alpha)+\left\{-\left[\frac{d^2T(\alpha)}{d\alpha^2}+(2\tanh\alpha+\coth\alpha)\frac{dT(\alpha)}{d\alpha}\right]+\frac{n^2}{\sinh^2\alpha}T(\alpha)\right\}S(\theta)=-\sigma(\sigma+3)\,T(\alpha)\,S(\theta)\ . \tag{8}$$

From the Eq.(8) we have

$$\left[\frac{d^2}{d\theta^2}+\cot\theta\frac{d}{d\theta}-\frac{m^2}{\sin^2\theta}\right]S(\theta)=-\nu(\nu+1)\,S(\theta) \tag{9}$$

and

$$\left[\frac{d^2}{d\alpha^2}+(2\tanh\alpha+\coth\alpha)\frac{d}{d\alpha}+\left(\frac{\nu(\nu+1)}{\cosh^2\alpha}-\frac{n^2}{\sinh^2\alpha}\right)\right]T(\alpha)\ =\sigma(\sigma+3)T(\alpha)\ . \tag{10}$$

For $\nu=0$, $n=0$ Eq. (10) takes the form

$$\left[\frac{d^2}{d\alpha^2}+(2\tanh\alpha+\coth\alpha)\frac{d}{d\alpha}\right]T(\alpha)=\sigma(\sigma+3)T(\alpha)\ . \tag{11}$$

By the transformation

$$T(\alpha)=(\cosh\alpha)^{-1/2}(\sinh\alpha)^{-1/2}\psi(\alpha) \tag{12}$$

Eq.(11) is reduced to the one dimensional Schrödinger equation for the continuous values of $\sigma=-\frac{3}{2}+i\rho$ and $E=\rho^2>0$

$$\frac{d^2\psi(\alpha)}{d\alpha^2}+\left[\frac{-\frac{1}{4}}{\sinh^2\alpha}-\left(\sigma+\frac{3}{2}\right)^2\right]\psi(\alpha)=0 \tag{13}$$

with the potential

$$V(\alpha)=\frac{1/4}{\sinh^2\alpha} \tag{14}$$

and the energy spectrum

$$E=-\left(\sigma+\frac{3}{2}\right)^2 \tag{15}$$

By the substitution $T(\alpha)=\cosh^{\lambda}\alpha\,W(\alpha)$ and the transformation $z=\tanh^2\alpha$, Eq.(11) is reduced to the hypergeometric equation

$$z(1-z)\frac{d^2W}{dz^2}+\left[1-\left(-\sigma+\frac{1}{2}\right)z\right]\frac{dW}{dz}-\left[\left(-\frac{\sigma}{2}\right)\left(\frac{-\sigma-1}{2}\right)\right]W=0 \tag{16}$$

By the regular solution of the equation (16) at $z=0$ we have

$$T(\alpha)=c_1\cosh^{\sigma}\alpha\,F\left(-\frac{\sigma}{2},\frac{-\sigma-1}{2};1;\tanh^2\alpha\right)\ . \tag{17}$$

On the other hand, by using the relation [9]

$$P_{\nu}^{\mu}(z)=2^{\mu}\left(z^2-1\right)^{-\mu/2}z^{\nu+\mu}\frac{1}{\Gamma(1-\mu)}\times{}_2F_1\left(\frac{1}{2}-\frac{\nu+\mu}{2},-\frac{\nu+\mu}{2};1-\mu;1-\frac{1}{z^2}\right) \tag{18}$$

For $\mu=0$ we obtain

$$P_\nu(z) = z^\nu F\left(\frac{1}{2}-\frac{\nu}{2}, -\frac{\nu}{2}; 1; 1-z^2\right) . \tag{19}$$

For $\sigma = \nu - 1$ from expression (17) we have

$$T(\alpha) = c_1 \cosh^{-1}\alpha \, \cosh^\nu \alpha \, F\left(\frac{1}{2}-\frac{\nu}{2}, -\frac{\nu}{2}; 1; \tanh^2\alpha\right) \tag{20}$$

From the relations (19) and (20)

$$T(\alpha) = c_1 (\cosh\alpha)^{-1} P_\nu(\cosh\alpha) \tag{21}$$

is found. Thus for $\nu = -\frac{1}{2} + i\rho$ we obtain

$$T(\alpha) = c_1 (\cosh\alpha)^{-1} P_{-1/2+i\rho}(\cosh\alpha) \ . \tag{22}$$

For continuous spectrum $\sigma = -\frac{3}{2} + i\rho$, $E = \rho^2 > 0$ we calculate the S-matrix using the analytical continuous formula (23) for the hypergeometric function of [8] ;

$$\begin{aligned} F(a,b:c;z) &= A_1 F(a,b;a+b-c+1;1-z) \\ &+ A_2 (1-z)^{c-a-b} F(c-a,c-b;c-a-b+1;1-z) \ , \ |\arg(1-z)| < \pi \end{aligned} \tag{23}$$

$$A_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad , \quad A_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \tag{24}$$

From the Eq. (23) and Eq. (20) we have the asymptotic expression.

$$\underset{\alpha\to\infty}{T(\alpha)} = c_1 e^{-3\alpha/2}\left[A(\rho)e^{i\rho\alpha} + \overline{A}(\rho)e^{-i\rho\alpha}\right] \tag{25}$$

where

$$A(\rho) = \frac{\Gamma(i\rho)}{\Gamma\left(\frac{i\rho+1/2}{2}\right)\Gamma\left(\frac{i\rho+3/2}{2}\right)} .$$

From Eq.(25) it follows that the normalizing factor is chosen to satisfy the condition

$$\int_{-\infty}^{\infty} T(\alpha)\overline{T(\alpha)}(\cosh\alpha)^2 \sinh\alpha \, d\alpha = \delta(\rho' - \rho) \tag{26}$$

and is found to be

$$|c_1|^2 = \frac{1}{|2\pi A(\rho)|^2} \ . \tag{27}$$

The S-matrix is found to be given by

$$S = \frac{A_1}{\overline{A_1}} = \frac{\Gamma(i\rho)\Gamma\left(\frac{-i\rho+1/2}{2}\right)\Gamma\left(\frac{-i\rho+3/2}{2}\right)}{\Gamma(-i\rho)\Gamma\left(\frac{i\rho+1/2}{2}\right)\Gamma\left(\frac{i\rho+3/2}{2}\right)} \tag{28}$$

By the transformation $z = \cos\theta$ Eq.(8) becomes Associated Legendre Equation

$$\left(1-z^2\right)\frac{d^2S}{dz^2} - 2z\frac{dS}{dz} + \left[\nu(\nu+1) - \frac{m^2}{1-z^2}\right]S = 0 \ . \tag{29}$$

By the transformation

$$S(z) = \left(z^2-1\right)^{m/2} W(z) \tag{30}$$

we have

$$\left(1-z^2\right)\frac{d^2W}{dz^2} + -2(m+1)z\frac{dW}{dz} + (\nu-m)(\nu+m+1)W = 0 \ . \tag{31}$$

By substituting $z' = \frac{1-z}{2}$ Eq. (31) is transformed to the Hypergeometric Equation and the solutions of the equation are

$$W_1 = F\left(m-\nu, m+\nu+1; m+1; \frac{1-z}{2}\right) \tag{32}$$

$$W_2 = \left(\frac{1-z}{2}\right)^{-m} F\left(-\nu, \nu+1; 1-m; \frac{1-z}{2}\right) \ . \tag{33}$$

From (30), for Eq(29) , considering the relation [9]

$$P_\nu^m(z)=\frac{1}{\Gamma(1-m)}\left(\frac{z+1}{z-1}\right)^{m/2}F\left(-\nu,\nu+1,1-m;\frac{1-z}{2}\right).[9] \tag{34}$$

we find the solution

$$S(\theta)=(-2)^m\Gamma(1-z)P_\nu^m(\cos\theta)\ . \tag{35}$$

Parabolic Coordinates

The parabolic coordinates are given by

$$\begin{aligned} x_1 &= r\left(\cosh\frac{t}{2}-\frac{1}{2}e^{\frac{t}{2}}q^2\right)\\ x_2 &= rq_1e^{\frac{t}{2}}\\ x_3 &= rq_2e^{\frac{t}{2}}\\ x_4 &= rq_3e^{\frac{t}{2}}\\ x_5 &= r\left(\sinh\frac{t}{2}+\frac{1}{2}e^{\frac{t}{2}}q^2\right) \end{aligned} \tag{36}$$

where $q^2=q_1^2-q_2^2-q_3^2$, $0<r<\infty$, $-\infty<t<\infty$, $-\infty<q_1,q_2,q_3<\infty$

From (36) we have

$$(g_{ab})=diag\left(1,-r^2/4,\ r^2e^t,\ -r^2e^t,-r^2e^t\right) \tag{37}$$

and

$$\sqrt{g}=\left(\det(g_{ab})\right)^{1/2}=\frac{r^2}{2}e^{\frac{t}{2}} \tag{38}$$

The Laplace-Beltrami operator on the parabola [x x]=1 is given as

$$\Delta_{LB}=\left\{\frac{-4}{e^{\frac{3t}{2}}}\frac{\partial}{\partial t}\left(e^{\frac{3t}{2}}\frac{\partial}{\partial t}\right)+\frac{1}{e^t}\left[\frac{\partial^2}{\partial q_1^2}-\frac{\partial^2}{\partial q_2^2}-\frac{\partial^2}{\partial q_2^2}\right]\right\} \tag{39}$$

On the other hand; the equation

$$\Delta_{LB}=\Phi(t,q_1,q_2,q_3)=-\sigma(\sigma+3)\Phi(t,q_1,q_2,q_3) \tag{40}$$

where

$$\mu_1^2-\mu_2^2-\mu_3^2=\mu^2 \tag{41}$$

and

$$\Phi(t,q_1,q_2,q_3)=T(t)e^{i\mu_1q_1}\ e^{i\mu_2q_2}\ e^{i\mu_3q_3} \tag{42}$$

is equivalent to

$$\left[4\frac{d^2}{dt^2}+6\frac{d}{dt}+\frac{\mu^2}{e^t}\right]T(t)=\sigma(\sigma+3)T(t)\quad . \tag{43}$$

Applying the transformation

$$T(t)=e^{\frac{-3t}{4}}\psi(t) \tag{44}$$

Eq .(43) becomes;

$$\frac{d^2\psi(t)}{dt^2}+\left[\frac{\mu^2}{4e^t}-\frac{\left(\sigma+\frac{3}{2}\right)^2}{4}\right]\psi(t)=0 \tag{45}$$

which reduces to the Schrödinger Equation , where energy spectrum

$$E=-\frac{\left(\sigma+\frac{3}{2}\right)^2}{4} \tag{46}$$

and the potential

$$V(t)=\frac{-\mu^2}{4e^t} \tag{47}$$

Applying the change of variable

$$z = e^{-\frac{1}{t}} \tag{48}$$

in Eq.(45), we arrive at the equation

$$z^2\frac{d^2\psi(z)}{dz^2} + z\frac{d\psi(z)}{dz} + \left[\mu^2 z^2 - \left(\sigma + \frac{3}{2}\right)^2\right]\psi(z) = 0 \tag{49}$$

In the case of $\mu^2 < 0$; changing variable as

$$z' = \mu\, z \tag{50}$$

at Eq.(49) we obtain

$$z^2\frac{d^2\psi(z)}{dz^2} + z\frac{d\psi(z)}{dz} - \left[z^2 + \left(\sigma + \frac{3}{2}\right)^2\right]\psi(z) = 0 \tag{51}$$

Modified Bessel equation. The solution of the Eq. (51) are given by

$$\psi(\mu\, z) = c_1 K_{\sigma+\frac{3}{2}}(\mu\, z) \tag{52}$$

where

$$K_{\sigma+\frac{3}{2}}(\mu\, z) = \frac{\pi}{2\sin\left(\sigma + \frac{3}{2}\right)\pi}\left[I_{-\sigma-\frac{3}{2}}(\mu\, z) - I_{\sigma+\frac{3}{2}}(\mu\, z)\right]. \tag{53}$$

Thus; the solution of the Eq. (43) is

$$T(\,t\,) = 4e^{-\frac{3t}{4}} K_{i\rho}\left(\mu\, e^{-\frac{t}{2}}\right). \tag{54}$$

This is the solution corresponding to the continous values $\sigma = -\frac{3}{2} + i\rho$

The normalization factor c_1 is

$$|c_1|^2 = \frac{\rho \sinh \rho\pi}{\pi^2} \tag{55}$$

Chosen to satisfy the condition

$$\int_{-\infty}^{\infty} T(t)\overline{T(t)}\, dt = \delta(\rho - \rho') \tag{56}$$

Considering the Asymptotic expression

$$\underset{t\to\infty}{K_{i\rho}}\left(\mu\, e^{-t/2}\right) = \frac{\pi}{2i \sinh\, \rho\pi}\left[Ae^{-i\rho t/2} + \overline{A}e^{-i\rho t/2}\right] \tag{57}$$

where

$$A = \frac{-\pi(\mu/2)^{i\rho}}{2i \sinh\, \rho\pi\, \Gamma(1 + i\rho)} \tag{58}$$

is found in the form of S-matrix

$$S = \frac{A}{\overline{A}} = -(\mu/2)^{2i\rho}\frac{\Gamma(1 - i\rho)}{\Gamma(1 + i\rho)} \tag{59}$$

in the case of $\mu^2 > 0$. By the substitution of $z' = \mu z$ in Eq. (49) we obtain

$$z^2\frac{d^2\psi(z)}{dz^2} + z\frac{d\psi(z)}{dz} + \left[z^2 - \left(\sigma + \frac{3}{2}\right)^2\right]\psi(z) = 0\,. \tag{60}$$

Since $\int |\psi(z)|^2 dz < \infty$, for discrete specturum $\sigma = l$ we obtain the solution of (57) in the form of

$$\psi(\mu\, z) = c_2 J_{l+\frac{3}{2}}(\mu\, z) \tag{61}$$

Thus for Eq.(43), the solution

$$T(t) = c_2\, e^{-\frac{3t}{4}} J_{l+\frac{3}{2}}\left(\mu e^{-\frac{t}{2}}\right) \tag{62}$$

is found . The normalization factor c_2 is

$$|\, c_2\,|^2 = 2\,l + 3 \tag{63}$$

where

$$\int_0^\infty \frac{J_{l+\frac{3}{2}}(z')J_{l+\frac{3}{2}}(z')}{z'}dz' = \frac{1}{2l+3} \tag{64}$$

and invariant volume element in parabolic coordinates is

$$dx = \frac{1}{\sqrt{g}}\frac{r^4}{2}\left(e^t\right)^{\frac{3}{2}} dt\, dq_1\, dq_2 dq_3 \quad . \tag{65}$$

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