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RESEARCH ARTICLE

Optimal Portfolio Allocation with Elliptical and Mixed Copulas

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Abstract
This research aims to investigate the asset allocation performance of three different optimization methods commonly applied in the literature for a portfolio composed of univariate returns generated from Mixed and Elliptic copulas instead of historical data. As a result, returns of five equities traded at the BIST30 index of the Turkish Stock Market were obtained. Dynamics of the univariate return series are modelled with GARCH processes with Student-t distributed innovations. Following the marginal modelling, a five-dimensional dependence structure between the series is modelled with Elliptical and Mixed copulas. From the fitted Mixed and Elliptical copula functions, daily returns of the equities are simulated which are employed by the specified optimization methods in order to find out methodology specific optimal portfolio allocations. Performance of the constructed optimal portfolios are compared according to varying risk and reward to variability ratios yielding results especially in favor of the Mixed and Student t copulas. The main contribution of this research is to be able to fill the gap in the literature on the out-of-sample portfolio allocation performance of copula functions where there are still fewer papers compared to the dependency modelling or the in-sample portfolio allocation performance of copulas.

Keywords

Portfolio Optimization, Copula Functions, GARCH, Portfolio Performance

Introduction

Optimal allocation of scarce funds between the assets of a portfolio is a long-standing debate that Markowitz first grounds in 1952. Markowitz (1952) developed the Mean-Variance (MV) model suggesting to use of quantitative measures of risk and return in asset selection problems. Since then, there is growing literature on the applications, shortcomings, possible improvements and extensions of the model under the name Modern Portfolio Theory. Significantly, the advances in technology made it possible to overcome some of the shortcomings leading the way to further developments.

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One of the earliest criticisms of the MV model is made on the risk measure. Variance as a symmetric measure of deviations from the mean is criticized for equally penalizing upside and downside deviations. Moreover, research on the characteristics of return series showed that financial returns are leptokurtic, skewed with autocorrelation and changing variance leading to a higher occurrence of joint tail movements compared to normal distribution. If the returns are skewed with high kurtosis, then variance will not be able to correctly specify the magnitude of the losses.

In the meantime, the correlation coefficient as one of the most frequently applied linear dependency measures is criticized for not being able to model the tail co-movements of prevalent returns especially at the times of financial market boom or distress. On the other hand, Copula functions (Sklar, 1959) can model nonlinear dependence structures with extreme co-movements that can be observed on the tails of the return distributions without any restriction on the types of the marginal distributions.

Considering the above-mentioned series specific characteristics and modelling issues, this paper employs Elliptical and Mixed copulas based on Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) (Bollerslev, 1986; Engle, 1982) models to capture the dynamic structure of the univariate returns together with the multivariate dependence between the series. Compared to most of the papers that mainly use the traditional method of historical returns or return forecasts of conditional mean and/or variance models, this paper employs returns simulated from copula functions to determine optimal weights of Tangency, Global Minimum CVaR (GMCVaR) and Global Minimum Variance (GMV) portfolios for the out-of-sample period by also comparing their results with the traditional methods. Additionally, as an alternative to variance, minimum risk portfolios are obtained from Conditional Value at Risk (CVaR) measure that considers the expected value of the losses exceeding the threshold of VaR. The main contribution of this research is that there is still a small number of papers investigating performance of copula functions in the out-of-sample portfolio allocation tasks.

This paper consists of seven main sections. In Literature Review, following a brief introduction to Modern Portfolio Theory, an overview of the literature on copula functions employed in portfolio allocation context are given. The methodology of the paper is explained in the Theoretical Background section. Data and Marginal Models part of the paper introduces the data and parameter estimates of the univariate marginals. Copula Fits and Return Simulations section explains the steps employed to fit the copula functions, obtain return simulations and determine the methodology specific optimal portfolio allocations. Research findings are summarized, and the paper is concluded in the last two sections.

Literature Review

Before Harry Markowitz (1952), portfolios were constructed according to a simple/naive diversification approach that assumed a positive relation between the number of assets in a portfolio and its performance with a decreasing portfolio risk by simply adding more assets to the portfolio.

In 1952, Markowitz with his publication named "Portfolio Selection" showed that it was not possible to diversify all portfolio risk with the simple diversification approach since the co-movement of asset returns were too high. He defined variance of returns as a portfolio risk estimator and expected value of the returns as the desired property of a portfolio. As a result, for the first time a quantitative return/risk framework for asset selection was proposed. Since then, the Mean-Variance (MV) model of Harry Markowitz (1952) and its further developments named *Modern Portfolio Theory* are the standard tools frequently applied in Finance. Furthermore, William Sharpe (1964), Tobin (1958), Lintner (1965), Jensen (1969), Fama (1970), Merton and Samuelson (1974), Merton (1980), Elton, Gruber, and Padberg (1978) and others contributed to the development of the MV framework.

Nevertheless, some of the assumptions of the MV model were highly criticized. One of them was the portfolio risk measure. Variance that is a symmetric measure of deviations from the portfolio mean, was found insufficient in measuring portfolio risk. As a result, varying risk measures were proposed that consider only the downside deviations from the mean, such as Semi-variance (H. Markowitz, 1959) or only the lower tail of the return distributions such as Lower Partial Moment (LPM) (Bawa & Lindenberg, 1977), VaR (Jorion, 2000) and CVaR (Acerbi & Tasche, 2002; Uryasev, 2000). Moreover, Rockafellar and Uryasev (2000, 2002) developed an algorithm that allowed minimizing portfolio CVaR similar to minimizing portfolio variance in the MV framework. Furthermore, Patton (2004) examined the impact of skewness of univariate return series and the asymmetric dependence of the returns on portfolio allocation by constructing and comparing portfolios based on a bivariate normal distribution and copula based on more flexible distributions. Results of the study showed an improved portfolio allocation performance in terms of the investor's utility when copula functions were employed compared to the bivariate normal model.

Riccetti (2013) applied the copula-GARCH model to obtain optimal weights of commodity portfolios that maximized the expected utility of an investor in terms of the CRRA utility. The researcher compared the macro asset allocation performance of copula-GARCH, mean-variance and univariate GARCH processes. According to the results of the study, Riccetti (2013) argued that the univariate GARCH(1,1) was better at macro asset allocation of commodity portfolios compared to the copula-GARCH model. Kresta (2015) employed a Student t copula based AR(1)-GARCH(1,1) model to forecast future stock returns to find maximum

Sharpe portfolios. According to the results, optimal portfolios of the copula-GARCH model yielded higher final wealth and lower maximum drawdown values compared to the portfolios of the bootstrapping method.

Acık Kemaloglu and Kızılok Kara (2015) modelled the dependence structure of four series (two indices: BIST30, BIST100 and two exchange rates: USD, EUR) by employing four copula functions in a static and dynamic context. Among the applied static and dynamic copulas, researchers indicated that dynamic tDCC was the best to model the dependence structure of the variables. According to the results of applied portfolio optimization based on CVaR risk measure, researchers argued that investing 35% of wealth in BIST30, 30% in USD, 20% in EUR and 15% in BIST100 yields a portfolio with the minimum CVaR value. In another study, Kızılok Kara and Acık Kemaloglu (2016) modelled the dependence of EUR and USD currency returns with static and dynamic copulas to find optimum CVaR portfolios with a changing point approach. Han, Li, and Xia (2017) suggested applying robust portfolio optimization methods by modelling the dynamic structure of the return series of ten indexes with DCC copula and copula based GARCH. The results of the study that used only bivariate copula functions indicated an outperformance of Worst-Case Conditional Value at Risk with copulas (WCVaR) method in the out-of-sample period, in contrast to the in-sample period in which the static robust method had higher cumulative portfolio returns. Sahamkhadam, Stephan, and Östermark (2018) applied copula based ARMA-GARCH-EVT and GARCH-EVT for modelling conditional mean, variance and the dependence structure of ten stock indexes. From the fitted models, one day ahead returns of the indices were obtained and optimal weights for the Minimum CVaR, Global Minimum Variance (GMV) and Sharpe portfolios were determined. The performance of the portfolios was evaluated with an out-of-sample back-testing approach. According to the results, ARMA-GARCH-EVT-copula based Sharpe portfolio outperformed the benchmark portfolio in terms of cumulative portfolio wealth and elliptical copulas based GARCH-EVT models performed better at reducing portfolio risk compared to the benchmark models that were based on historical returns.

Jin and Lehnert (2018) proposed various Dynamic Conditional Elliptical copulas by extending the dynamic equicorrelation (DECO) model to copula functions. Researchers employed the proposed copulas to model the dependence structure of 89 US companies listed in a credit default swap index (CDX.NA.IG). Following the dependency modelling, they estimated VaR and expected shortfall (ES) measures of equally weighted and value-weighted portfolios. Researchers constructed optimal portfolios by applying the Mean-Variance (MV) model of Markowitz (1952) and the minimum ES optimization of Rockafellar and Uryasev (2000). Results of the research indicated the importance of marginal modelling especially for dynamic high dimensional models. Moreover, researchers argued that an improvement in portfolio risk management could be achieved by accurately modelling the dependence structure of variables and choosing an optimization method that also considered the tail risk.

On the other hand, Trabelsi and Tiwari (2019) employed GPD distribution for the tails of the marginal density functions and simulated returns from Normal and Student t copulas. Researchers used the simulated returns to find minimum CVaR portfolios and compared optimal portfolio VaR and CVaR values with the historical simulation. Results of the study showed an improved market risk estimation performance when the returns were simulated from the copula functions compared to historical simulation. In a more recent study, Yu and Liu (2021) proposed an investor specific mean-CVaR optimization model consisting of individual risk tolerance assessment. Researchers categorized individual risk tolerances based on demographic characteristics and employed a fuzzy comprehensive evaluation method to determine the investors' (individuals) risk tolerances. They modelled the univariate returns with the copula-GARCH and obtained the minimum CVaR portfolios for the given expected return thresholds determined by the investors' risk tolerance.

Theoretical Background

In this research, different types of copula functions were applied to model the dependency between five stocks and to simulate the univariate returns composing a portfolio. However, in dependency modelling with copula functions, the variables are assumed to be random independent and identically distributed. Since most of the time the return series is autocorrelated with a changing variance with respect to time, a pre-model that would be able to capture series specific patterns was necessary. For this purpose, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model (Bollerslev, 1986; Engle, 1982) was employed. Building up on Engle (1982), Bollerslev (1986) defined GARCH(p,q) model as:

$$\varepsilon_t = \eta_t \sqrt{h_t}\,, \quad h_t = \omega + \sum_{j=1}^{q} \beta_j h_{t-j} + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 \tag{1}$$

where η_t is a sequence of iid random variables with zero mean and variance one. ε_t is the error term sequence and h_t is the conditional variance. When the β parameter of the conditional variance equation equals to zero, then Equation 1 would define Engle (1982)'s Autoregressive Conditional Heteroskedasticity (ARCH) model. Additionally, the parameters of $\omega, \beta\ and\ \alpha$ have the positivity constraints of $\omega > 0,\ \alpha_i \geq 0\ and\ \beta_j \geq 0$.

Modelling Dependence with Copulas

It was Sklar (1959) who first introduced *copula* as a multivariate function that associates (*ties*) univariate marginals of a multivariate distribution. More formally, let $F_1(x_1), \ldots, F_n(x_n)$ be the univariate marginal distribution functions and *F* is an *n*-dimensional multivariate distribution defined on R^n. The joint multivariate distribution *F* can be decomposed into a copula function *C* and its univariate marginals as:

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \tag{2}$$

While the variables of each univariate margin are assumed to be iid, the dependence between the variables of the margins are defined by the copula C that is given by:

$$C(u_1,\ldots,u_n;\theta) = F(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)) \quad (3)$$

where $u \in [0,1]^n$, θ is the copula parameter, $F_1^{-1},\ldots,F_n^{-1}$ are the quantile functions and $F_i^{-1}(u) = inf\{x: F_i(x) \geq u\}$. The density c of the copula C is obtained by:

$$c(u_1,\ldots,u_n) = \partial^n C(u_1,\ldots,u_n)/\partial u_1,\ldots,\partial u_n \quad (4)$$

Since this paper applies, Normal, Student t and a mixed copula from two copulas of the Archimedean family, a brief introduction is given in the following paragraphs. For a more detailed discussion consult the research of Joe (1997) and Nelsen (2006).

In dependency modelling, Normal (Gaussian) and Student t copulas are frequently applied copula types from the Elliptical family. Normal copula obtained from multivariate normal distribution is radially symmetric and has zero upper and lower tail dependence parameters. As a result, the co-movement of the variables when either taking very high or very low values cannot be modelled with Normal copula. For a multivariate random vector of $\boldsymbol{X} = (X_1,\ldots,X_n)$, an n-dimensional Normal copula function C_N is defined by:

$$C_N(u) = \int_{-\infty}^{\phi^{-1}(u_1)} \ldots \int_{-\infty}^{\phi^{-1}(u_n)} \frac{1}{\sqrt{(2\pi)^n\,|\boldsymbol{R}|}}\; exp\left(-\frac{1}{2}\boldsymbol{X}^T\boldsymbol{R}^{-1}\boldsymbol{X}\right) d\boldsymbol{X} \quad (5)$$

In Equation 5, ϕ^{-1} defines the inverse cdf of standard normal distribution, $\boldsymbol{R} \in \mathbb{R}^{nxn}$ is the linear correlation matrix. On the other hand, Student t copula derived from the multivariate Student's-t distribution is also a radially symmetric copula that has positive and equal upper (UTD) and lower tail dependence (LTD) parameters. An n-dimensional Student t copula (C_t) is defined as (Demarta & McNeil, 2005):

$$C_t = \int_{-\infty}^{t_\nu^{-1}(u_1)} \ldots \int_{-\infty}^{t_\nu^{-1}(u_n)} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu\pi)^n|\boldsymbol{R}|}} \left(1+\frac{\boldsymbol{X}^T\boldsymbol{R}^{-1}\boldsymbol{X}}{\nu}\right)^{-\frac{\nu+n}{2}} d\boldsymbol{X} \quad (6)$$

where t_ν^{-1} represents the inverse cdf of Student's-t distribution with the degrees of freedom parameter ν and $\boldsymbol{R}$ is the correlation matrix.

The third copula type employed in this paper was a mixed copula which was constructed from Gumbel (Gumbel, 1960) and Clayton copulas (Clayton, 1978) with equal weights. The need for diverse asymmetric dependence structures with enhanced flexibility led the way to the development of Archimedean copulas. An *Archimedean copula* with a strict generator function φ is defined as:

$$C(u_1,u_2,\ldots,u_n) = \varphi^{-1}(\varphi(u_1)+\varphi(u_2),\ldots,\varphi(u_n)) \quad (7)$$

where φ^{-1} is the inverse of the generator function φ which is a continuous and strictly decreasing function mapped from $[0,1]$ onto $[0,+\infty)$ with the properties of $\varphi(0) = \infty$ and

$\varphi(1) = 0$. Moreover, the properties of Clayton and Gumbel copulas are summarized in Table 1 (Nelsen, 2006). Both Clayton and Gumbel copulas have tail dependence parameters only on one tail of the distribution defined either on the lower or upper tail of the distribution.

Table 1
Properties of the N-dimensional Archimedean Copulas

Copula	Definition	Generator Func./ $\varphi_\theta(t)$	LTD, UTD
Clayton	$C_\theta^{Cl}(\mathrm{u}) = [u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1]^{\frac{-1}{\theta}}$	$\theta^{-1}(t^{-\theta} - 1)$	$(2^{-\frac{1}{\theta}}, 0)$
Gumbel	$C_\theta^{G}(\mathrm{u}) = exp\left[-((-\ln u_1)^\theta + \ldots + (-\ln u_n)^\theta)^{1/\theta}\right]$	$(-\ln t)^\theta$	$(0, 2\text{-}2^{\frac{1}{\theta}})$

Mixed Copulas

Mixed Copula was proposed as a more flexible alternative to single copulas. Since a convex union of n-dimensional single copulas is also defined as a copula, a mixture of single copulas was suggested (Nelsen, 2006). Let $\omega = (\omega_1, \ldots, \omega_j)$ be a vector of weights with $j \geq 2$ and $\omega_k \geq 0$ for all $k \in \{1, \ldots, j\}$ and let $C_1, \ldots, C_j$ be n-dimensional copulas. The *mixture* of j number of n-dimensional copulas with weight vector ω is an n-dimensional Mixed copula defined by:

$$C_{mix}(C_1, \ldots, C_j)(u; \theta) = \sum_{k=1}^{j} \omega_k C_k(u; \theta_k), \quad \sum_{k=1}^{j} \omega_k = 1, \quad u \in [0,1]^n \tag{8}$$

Mixed copulas can be constructed from any of the Elliptical and Archimedean copulas. This research employed an equally weighted ($\omega_1 = \omega_2 = 0.5, j = 2$) combination of 5-dimensional Clayton and Gumbel copulas. While Clayton copula has only lower and Gumbel copula has only upper positive tail dependence parameters, constructed mixed copula has both upper and lower tail dependence with varying dependence strengths on the tails allowing for radial asymmetry.

Portfolio Optimization

This paper approached the problem of selecting the most appropriate portfolio asset combination in three different ways. First, the traditional MV model of Harry Markowitz (1952) was applied to find Global Minimum Variance Portfolios (GMVP). According to Harry Markowitz (1952), the risk of a portfolio was defined by variance and investors would prefer the smallest risk for a certain level of return. Furthermore, returns were normally distributed and there were not any transaction costs. On the other hand, *Global Minimum Variance Portfolio* (GMVP) (Merton, 1980) is also defined as one of the portfolios on the efficient frontier but among the efficient portfolios it is the one with the smallest risk level. GMVP is obtained by solving the minimization objective of:

$$\min_{w \epsilon W} \; w^T \Sigma \, w \tag{9}$$

In Equation 9, w is the vectoral representation of the weights of returns, Σ is the covariance matrix and W is the set of feasible solutions defined by: $W = \{\text{for } \forall i = 1,2,\ldots,N,\ w \in \mathbb{R}^N : \sum_{i=1}^{N} w_i = 1 \text{ and } w_i \geq 0\}$. The set of W defines total investment and long only constraints of the optimization applied in this paper.

As a second approach, instead of variance, portfolio Conditional Value at Risk (CVaR) measure was employed as the main minimization objective of portfolio optimization which was implemented by Rockafellar and Uryasev (2000). *Global Minimum CVaR Portfolio* (GMCVaR) is defined as the portfolio with a minimum CVaR value which is on the efficient frontier constructed by mean-CVaR efficient portfolios. GMCVaR has an objective of:

$$\min_{w \epsilon W} \ CVaR_{\beta}(w) \tag{10}$$

Where β is the probability level, w is the vectoral representation of the weights of returns, and W is the set of feasible solutions defined as above. According to Rockafellar and Uryasev (2000), convex GMCVaR portfolio optimization objective can be re-written as a minimization of a linear objective with respect to linear constraints such as:

$$\min_{w \in W} \ \alpha + \frac{1}{(1-\beta)N} \sum_{n=1}^{N} z_n \tag{11}$$

$$\text{w.r.t. } \ z_n \geq 0 \quad \wedge \quad w^T r_n + \alpha + z_n \geq 0 \ \ for\ n = 1,2,\ldots,N \tag{12}$$

where r is the vector of random portfolio returns ($r \in \mathbb{R}$), α is the minimum loss, w is the weight vector and $f(w, r_n)$ is the portfolio loss function. For the purpose to rewrite nonlinear optimization objective into a linear one, the auxiliary variables z_n are defined as $z_n \geq f(w, r_n) - \alpha$. Mean-CVaR portfolio optimization doesn't necessitate elliptically distributed portfolio returns as it is in mean-variance optimization.

Finally, the third approach applied to select optimal asset allocation was the maximum return per unit of risk portfolio or in other words *Tangency Portfolio*. It is a portfolio in the mean-variance framework of Harry Markowitz (1952) with an objective of maximizing risk free rate differenced (r_f) portfolio expected return / risk ratio which is the *Sharpe Ratio* (Sharpe, 1966, 1994). Tangency or Maximum Sharpe is a portfolio on the mean-variance efficient frontier where it is tangent to the Capital Market Line (CML). The optimization problem is defined as:

$$\max_{w \in W} \ \frac{\mu^T w - r_f}{\sqrt{w^T \Sigma\, w}} \tag{13}$$

In Equation 13, μ is the vectoral representation of the mean returns, Σ and w are the covariance matrix and vectoral representation of return weights, respectively. Moreover, W is the set of feasible solutions as defined previously. In this research, the risk free rate was chosen to be equal to zero, since the attitude of investors to positive portfolio returns that might be below or above risk-free rate changes significantly compared to gaining solely negative returns.

Portfolio Performance Evaluation

Outcomes of the applied optimal asset allocation methods were evaluated with risk and reward to per unit of risk-based measures. Variance, VaR and CVaR were utilized as portfolio risk measures. Portfolio specific VaR and CVaR values were estimated with the nonparametric approach. Portfolio VaR is the negative of 5th% quantile of the portfolio returns or in other words it is the 95th% quantile of the portfolio losses:

$$VaR_{5\%}(r) = -F_r^{-1}(0.05) \tag{14}$$

In Equation 14, F_r represents the distribution of portfolio returns (r). Portfolio CVaR is the mean value of portfolio losses that are equal to and greater than portfolio VaR value:

$$CVaR_{5\%}(r) = -E_r[r|r \leq VaR_{5\%}(r)] \tag{15}$$

Across applied varying portfolio optimization methodologies, the ones that minimize portfolio risk (GMV or GMCVaR portfolios) are expected to yield lower portfolio risk measures, either variance or CVaR depending on the optimization method.

Furthermore, portfolios are compared according to their reward to variability ratios or in other words risk-adjusted performance measures. From a growing number of measures, Sharpe Ratio (Sharpe, 1966, 1994) and Omega-Sharpe Ratios (Kazemi, Schneeweis, & Gupta, 2003) were employed in this paper. As defined in the previous subsection (3.3 Portfolio Optimization), the Sharpe ratio is the risk-free rate differenced portfolio expected return / portfolio standard deviation ratio. It is a well-known and commonly applied performance measure in finance literature. Nevertheless, the ratio is criticized for being inaccurate under nonnormal return distributions. On the other hand, Keating and Shadwick (2002) suggested the *Omega* measure as a probability weighted ratio of portfolio returns above and below a threshold return (τ):

$$\Omega(\tau) = \frac{\int_\tau^b [1-F(x)]dx}{\int_a^\tau F(x)dx} \tag{16}$$

where x is the random portfolio returns and *(a,b)* are the extreme tail (minimum and maximum) realizations of the portfolio return distribution, respectively. Moreover, $F(r) = P[x \leq r]$ is the cdf of the portfolio returns.

Later, as a modification of Omega, Kazemi et al. (2003) proposed Sharpe-Omega Ratio. First, Kazemi et al. (2003) showed that Omega equals to the ratio of the European call and European put option prices that were written on the portfolio:

$$\Omega(\tau) = \frac{\int_\tau^b [1-F(x)]dx}{\int_a^\tau F(x)dx} = \frac{\int_\tau^b (x-\tau)f(x)dx}{\int_a^\tau (\tau-x)f(x)dx} = \frac{E[max(x-\tau,0)]}{E[max(\tau-x,0)]} = \frac{C(\tau)}{P(\tau)} \tag{17}$$

In Equation 17, *f(x)* represents the density of portfolio returns, *C(τ)* and *P(τ)* are the undiscounted European call and Put option prices written on the portfolio. From Equation 17, Kazemi et al. (2003) derived and defined the Sharpe-Omega as the ratio of the expected excess portfolio return over the value of the put option written on the portfolio:

$$Sharpe\text{-}Omega = \frac{\bar{\mu}_p - \tau}{P(\tau)} \tag{18}$$

where $\bar{\mu}_p$ is defined as the expected portfolio return, τ is the threshold or the minimum acceptable return (MAR). In this research, similar to the assumption of zero risk-free rate for the Sharpe ratio and Tangency portfolio optimization, MAR or the threshold return τ is also assumed to be zero.

Additionally, D Ratio suggested by Bacon (2008) was applied to compare the performance of the portfolios. D Ratio is defined as the ratio of the sum of portfolio negative and positive returns considering their frequency:

$$DRatio = \frac{n_d * \sum_{i=1}^{T} max(-r_i, 0)}{n_u * \sum_{i=1}^{T} max(r_i, 0)} \tag{19}$$

where r_i is the portfolio return i, T is the number of portfolio returns, n_d and n_u are the total number of returns that are below and above zero, respectively.

Final portfolio wealth values (W_f) of the optimal portfolios are also compared with an assumption of 100 base (beginning) value ($W_0 = 100$):

$$W_f = W_0 * \prod_{k=1}^{503} (1 + r_k) \tag{20}$$

In Equation 20, r_k is defined as the return of an optimal portfolio at day k of the out of sample period. So that an investor is assumed to rebalance its portfolio on a daily basis.

Data and Marginal Models

This research employed five equities traded at BIST30 Index of the Turkish Stock Market. The data consisted of 1,390 observations for the period of 19 June 2013 – 28 December 2018. Three of the five companies; BIM Birleşik Mağazalar A.Ş. (BIMAS), Türkiye Halk Bankası (HALKB) and Türk Hava Yolları (Turkish Airlines, THYAO) operate in Retail, Finance and Airline industries, respectively. The rest are Holding companies; Sabancı Holding (SAHOL) and Koç Holding (KCHOL) that operate in various industries. Daily closing prices of the stocks were obtained from BORSA Istanbul A.S. The returns were calculated by:

$$r_t = 100 * \ln(P_t / P_{t-1}) \tag{21}$$

Table 2
Descriptive Statistics

In-Sample Period (19/06/2013 - 30/12/2016)						**Full Sample (19/06/2013 – 28/12/2018)**				
Stat / Stock	**BIMAS**	**HALKB**	**KCHOL**	**SAHOL**	**THYAO**	**BIMAS**	**HALKB**	**KCHOL**	**SAHOL**	**THYAO**
Mean	0.021	-0.073	0.042	-0.015	-0.054	0.054	-0.067	0.028	-0.025	0.049
Median	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Min	-7.776	-13.172	-7.026	-10.795	-13.449	-7.776	-15.387	-7.192	-10.795	-13.449
Max	7.605	13.638	8.289	8.417	10.344	7.605	13.638	8.289	8.417	10.344
SD	1.688	2.346	1.726	1.913	2.217	1.628	2.445	1.730	1.829	2.380
Skewness	-0.006	-0.346	0.062	-0.027	-0.266	0.181	-0.337	0.006	-0.167	-0.274
Kurtosis	4.909	6.886	4.963	4.732	6.567	4.940	7.214	4.696	4.785	5.377
Jarque-Bera	136.3***	580.2***	144.6***	112.3***	484.7***	227.1***	1059.6***	167.9***	192.3***	346.8***
ADF stat.	-10.40**	-9.857**	-10.54**	-10.38**	-9.13**	-11.53**	-11.49**	-11.02**	-11.39**	-10.31**
LB Q(10)	0.576	0.635	0.075	0.289	0.004	0.016	0.258	0.051	0.065	0.000
LM(5)	0.000	0.011	0.083	0.040	0.000	0.000	0.009	0.016	0.000	0.000
LM(10)	0.003	0.121	0.000	0.000	0.006	0.000	0.042	0.000	0.000	0.000

Notes: *** and ** indicate significance at 1% and %5 levels, respectively. p values of the Ljung-Box and Arch LM tests are reported.

Series specific descriptive statistics are given in Table 2. According to the obtained statistics, returns are skewed and have excess kurtosis. Applied Jarque-Bera test for normality also confirmed the violation of the normality assumption of financial return series. Moreover, plots of price, return and absolute return series of stocks are given in Figure 1.

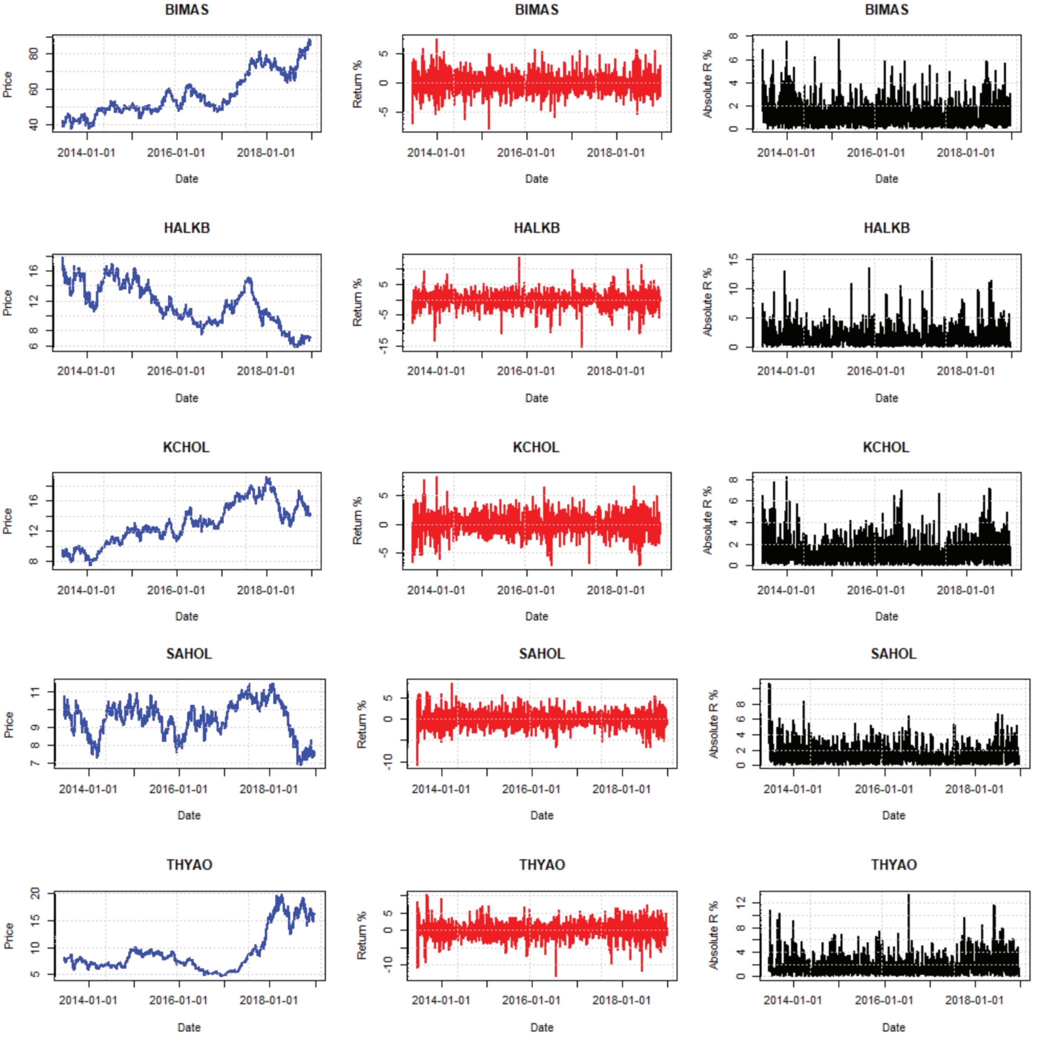

Figure 1. Plots of Price, Return and Absolute Return Series

Out of 1,390 observations spanning to five and half years, 887 of them were used for the first GARCH model parameter estimation window. GARCH(1,1) specifications with Student's t residuals efficiently captured the heteroskedastic and fat tailed return series. AR(1) was also included in a specification in case of serial autocorrelation, see THYAO in Table 3.

Table 3

GARCH(1,1) Specifications for the First Estimation Window

Return GARCH Fit	**AR(1)**	**ω**	**α_1**	**β_1**	**ν**	**LB(12)**	**LM(10)**
BIMAS GARCH(1,1)	-	0.212	0.073**	0.853***	5.871***	0.580	0.645
HALKB GARCH(1,1)	-	0.807	0.065	0.784***	5.057***	0.860	0.757
KCHOL GARCH(1,1)	-	0.081*	0.039**	0.933***	6.597***	0.204	0.752
SAHOL GARCH(1,1)	-	0.340	0.047**	0.856***	9.265***	0.669	0.613
THYAO AR(1)-GARCH(1,1)	-0.0735**	0.733**	0.089**	0.758***	5.163***	0.234	0.643

Notes: See the notes under Table 2. GARCH innovations are modelled with Student's t distribution.

In this research one day ahead, rolling windows approach was applied. As mentioned above, the first window included return observations from 1 to 887 and was used to simulate returns for the first day of the out-sample period consisting of 503 observations (the data starting from 888 to 1,390). The second fit window consisted of observations from 2 to 888 and was used to simulate returns for the second day of the out-sample period. One day ahead rolling windows approach was applied until simulations of the returns were obtained for every single day of the out-sample period. As a result, univariate GARCH processes given in Table 3 are re-estimated for a total of 503 windows keeping the window size constant.

Copula Fits and Return Simulations

Following the marginal modelling, pseudo-uniform variables were obtained from the standardized innovations of the GARCH filtered series. For the first window, when the scatterplots of the pseudo-uniform variables are examined (see Figure 2), it can be seen that the series are both lower and upper tail dependent with varying dependence strengths on the tails.

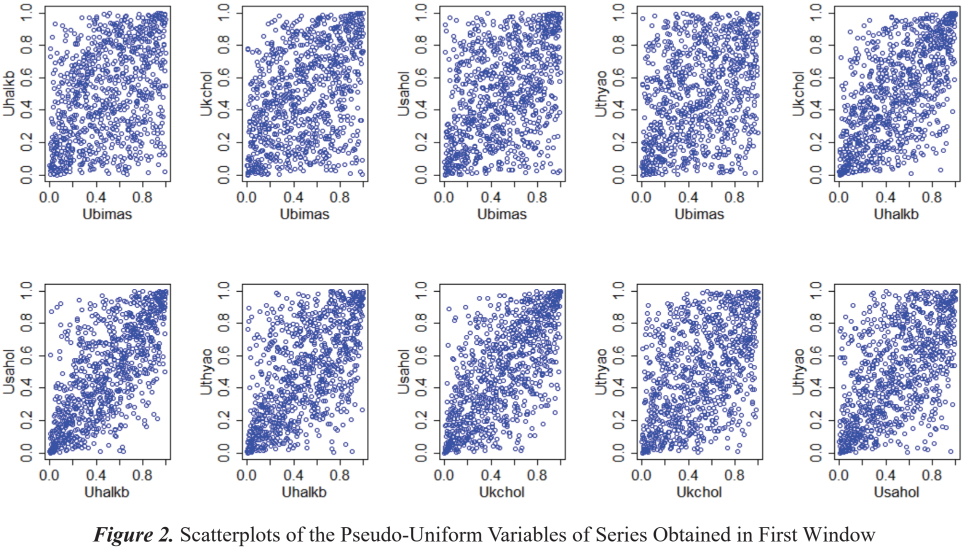

Figure 2. Scatterplots of the Pseudo-Uniform Variables of Series Obtained in First Window

Additionally, the applied multivariate Radial Symmetry Test (Genest & Nešlehová, 2014; Kojadinovic, 2017), confirmed our observation with strong evidence against radial symmetry (see also the recent work of Billio, Frattarolo, and Guégan (2022) in which a randomization based high dimensional copula radial symmetry test was proposed). Since, Student t copula is a radial symmetric copula, this finding is in favor of the mixed Clayton-Gumbel copula. Especially, if it is considered that in this research there are 503 windows in which the parameters of the copula functions were re-estimated in that case the assumption of symmetric dependence on the lower and upper parts of the multivariate distributions was not realistic.

On the next step, the parameters of Normal, Student t and the equally weighted mixture of Clayton-Gumbel copulas were estimated from the pseudo-uniform variables. For this purpose, Maximum Pseudo-Likelihood estimation method was used by employing the "Copula" package (Hofert, Kojadinovic, Maechler, & Yan, 2018) of R software (R Core Team, 2019). From the fitted copula functions, 1-day ahead returns were simulated by obtaining daily 1,000 return values for each stock. Using the daily simulated returns, optimal portfolio weights of stocks were determined by employing the three different optimization methodologies explained in the Portfolio Optimization subsection. More formally; let $u_{i,d,s}$ be pseudo-uniform variables simulated from the fitted 5-dimensional copula function. For $d=1,2,\ldots,5$, d is the return number, $i=1,2,\ldots,503$ is the data fit window and $s=1,\ldots,1{,}000$ is the number of simulation. The estimation steps, can be summarized as follows:

- $\eta_{i,d,s} = F_{i,d}^{-1}(u_{i,d,s})$, return and window specific standardized innovations are estimated.

- One day ahead 1,000 return simulations were obtained from the window and return specific GARCH equations; $r_{i,d,s} = \eta_{i,d,s}\sqrt{h_{i,d,s}}$.

- One day ahead optimal portfolio weights were estimated using the GMV, GMCVaR and Tangency portfolio optimization models from the simulated returns. The optimization conditions included the assumptions of no transaction costs, total investment of the beginning wealth and no short selling of the assets.

- The previous steps were repeated until daily optimal portfolio weights of stocks were estimated for every single day of the out-sample period.

- Once, all the copula and optimization model specific daily optimal portfolio weights were determined, they were multiplied with the realized stock returns (the corresponding out of sample period returns) and the final model specific daily optimal portfolio returns were obtained.

Additional to the copula functions, daily optimal portfolio weights were also estimated from the historical data and univariate GARCH simulations. Furthermore, equally weighted (EWP) or in other words 1/n naive portfolio was also included in the performance evaluations, since many studies reported the outperformance of EWP.

Empirical Findings

Performance of the constructed portfolios from varying methods was compared according to the estimated reward, risk and reward to variability measures summarized in Table 4. According to Table 4, portfolio with the minimum variance measure is constructed from the daily optimal weights of Global Minimum Variance portfolios of historical returns. Since variance is a symmetric estimate, it does not correctly account for the deviations below the mean when the portfolio return distribution is not symmetric.

Table 4
Performance Measures of the Optimal Portfolios

Method	Port. Mean	Variance	VaR5%	ES5%	Sharpe Ratio	Sharpe-Omega Ratio	DRatio	Portfolio Final Wealth
EWP	0.0494	1.9775	2.4721	3.2428	0.0351	0.0963	0.8390	121.96
GMVhis	0.0639	**1.3744**	1.9577	2.5573	0.0545	0.1532	0.7726	133.19
GMCVaRhis	0.0553	1.3753	1.9223	2.5591	0.0471	0.1325	0.8385	127.58
TPhis	0.0636	2.2782	2.3800	3.4356	0.0421	0.1178	0.8035	130.02
GMVgarch	0.0458	1.5623	2.0628	2.7977	0.0367	0.1001	0.8564	121.08
GMCVaRgarch	0.0423	1.5812	2.1397	2.8684	0.0336	0.0913	0.8362	118.88
TPgarch	-0.0011	2.8372	2.8477	4.1352	-0.0006	-0.0017	0.9474	92.57
GMVnc	0.0652	1.4098	1.9568	2.5597	0.0549	0.1561	0.8020	133.95
GMCVaRnc	0.0495	1.4546	1.9840	2.7408	0.0411	0.1161	0.8308	123.68
TPnc	0.0704	3.2951	2.5263	4.0771	0.0388	0.1165	0.9137	131.11
GMVst	0.0623	1.3998	1.9524	2.5213	0.0527	0.1490	0.8199	132.07
GMCVaRst	0.0655	1.4282	**1.9034**	**2.5003**	0.0548	0.1549	0.8093	134.13
TPst	0.1682	3.6940	2.7296	4.3275	0.0875	0.2854	0.6873	212.26
GMVmixed	0.0364	1.3851	1.9421	2.5290	0.0310	0.0849	0.8964	116.01
GMCVaRmixed	0.0245	1.4063	1.9396	2.5733	0.0207	0.0560	0.8850	109.20
TPmixed	**0.1860**	3.1548	2.8132	3.9176	**0.1047**	**0.3324**	**0.6706**	**235.34**

Note: Models yielding the best measures are shown in bold. EWP is the equally weighted portfolio. Following the optimization methods (GMV, GMCVaR and TP), suffixes of -his, -garch, -nc, -st and mixed are given to indicate data type used for the optimization that are historical, GARCH simulated, Normal, Student t and Mixed copula simulated returns, respectively.

On the other hand, portfolio VaR and CVaR risk estimates take into account only the lower tail of the portfolio return distribution quantifying only the losses at and below of a given probability level. Instead of variance, if portfolio VaR and/or CVaR is considered, then the Global Minimum CVaR portfolio employing returns simulated from Student t copula based GARCH(1,1) model outperformed the rest by having the smallest portfolio risk. When the optimal portfolios are compared according to the reward to variability ratios, Tangency (max Sharpe) portfolio employing the returns simulated from the Mixed copula had the biggest Sharpe and Sharpe-Omega ratios with the lowest DRatio clearly showed that the ratios were ranked with respect to the best model. Additionally, portfolio mean, and final portfolio wealth values obtained from Student t and Mixed copula-Tangency portfolios outperformed the rest with a big difference. Time development of the portfolio cumulative wealth values of GMV, GMCVaR and Tangency portfolios are given on Figure 3 and Figure 4, respectively.

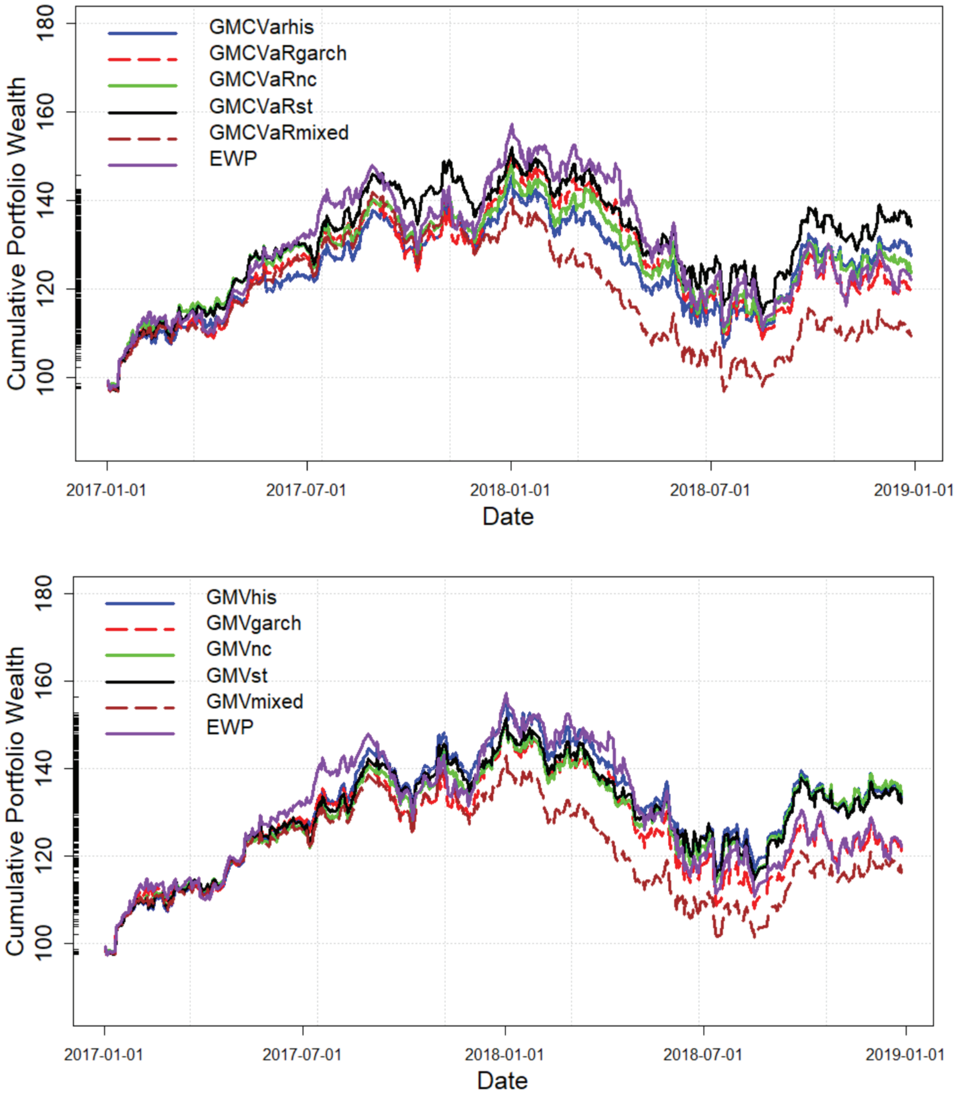

Figure 3. Time Plots of Cumulative Wealth of the GMV and GMCVaR Portfolios

From the following Figure 4, it can be seen that most of the time cumulative wealth of Student t copula-based Tangency portfolio was outperforming the rest but could not catch up the sharp rise of the Mixed copula-TP portfolio beginning in August 2018. Nevertheless, the rest of the models do not perform as well as copula-based models in terms of final portfolio wealth and reward-to-variability ratios showing the importance of tail dependence modelling. Especially, portfolios constructed with returns simulated from the univariate GARCH processes performed worst in most measures highlighting the importance of dependency modelling.

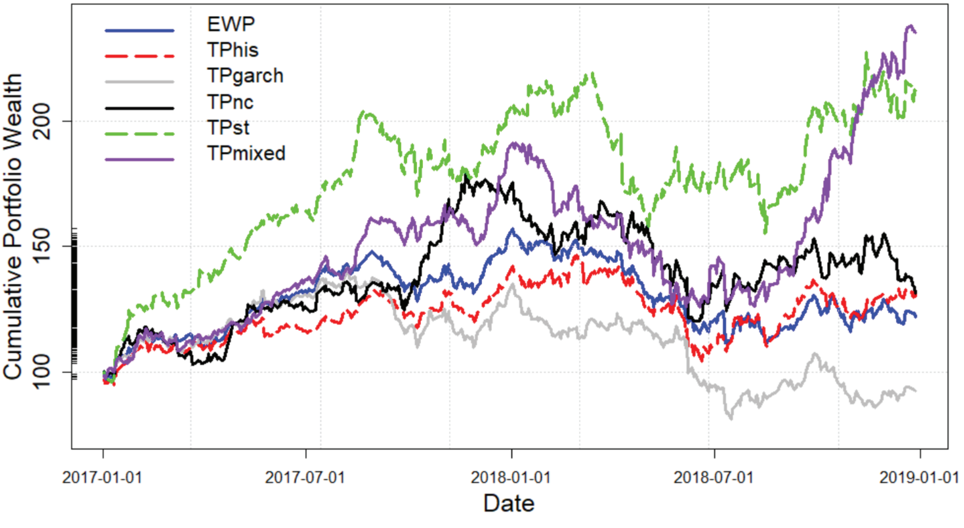

Figure 4. Time Plot of Cumulative Wealth of the Tangency Portfolios

Conclusion

This paper investigated out-of-sample performance of Normal, Student t and Clayton-Gumbel Mixed copula functions in asset allocation context with three different portfolio allocation strategies. First, optimal global minimum risk portfolios were obtained by applying Global Minimum Variance optimization method from the Mean-Variance framework. As an alternative method, optimal portfolios were constructed from Global Minimum CVaR optimization that minimizes portfolio CVaR as a risk measure instead of variance. Moreover, since the purpose of an investment is to earn the highest return relative to the per unit of risk of the investment, Tangency or maximum Sharpe portfolio optimization was employed as the third portfolio allocation strategy.

Furthermore, the performance of the copula and optimization method specific optimal portfolios were compared with the equally weighted portfolio as well as optimal portfolios constructed with GMV, GMCVaR and TP optimizations that either use historical returns or returns obtained from the univariate GARCH simulations. The results indicated that copula functions improved the out-sample asset allocation performance of the optimization models either by reducing the portfolio risk or by increasing the risk adjusted portfolio return. The usefulness of the copulas was more pronounced if extreme co-movements on the tails of the joint return distribution were more prevalent than suggested by normal distribution.

Moreover, the results of this study have the limitations of being dependent on the characteristics of the data fit and evaluation periods. The main aim of using copula functions was to model the non-linear co-movement of assets and/or the dependence that may occur on the tails of multivariate portfolio distributions. As a result, a portfolio allocation with copula functions in periods of weaker or no co-movement between the assets and without prevalence of extreme observations might yield results in favor of the other models.

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References

Acerbi, C., & Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance, 26*(7), 1487-1503. doi:10.1016/S0378-4266(02)00283-2

Bacon, C. R. (2008). *Practical Portfolio Performance Measurement and Attribution* (Vol. 546): John Wiley & Sons.

Bawa, V. S., & Lindenberg, E. B. (1977). Capital market equilibrium in a mean-lower partial moment framework. *Journal of Financial Economics, 5*(2), 189-200. doi:10.1016/0304-405X(77)90017-4

Billio, M., Frattarolo, L., & Guégan, D. (2022). High-Dimensional Radial Symmetry of Copula Functions: Multiplier Bootstrap vs. Randomization. *Symmetry*, *14*(1), 97.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics, 31*, 307-327.

Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika, 65*(1), 141-151.

Demarta, S., & McNeil, A. J. (2005). The t Copula and Related Copulas. *International Statistical Review, 73*(1), 111-129. doi:10.1111/j.1751-5823.2005.tb00254.x

Elton, E. J., Gruber, M. J., & Padberg, M. W. (1978). Simple Criteria for Optimal Portfolio Selection: Tracing Out the Efficient Frontier. *The Journal of Finance, 33*(1), 296-302. doi:10.2307/2326368

Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica, 50*(4), 987-1007. doi:10.2307/1912773

Fama, E. F. (1970). Multiperiod Consumption and Investment Decisions. *American Economic Review, 60*, 163-174.

Genest, C., & Nešlehová, J. G. (2014). On tests of radial symmetry for bivariate copulas. *Statistical Papers, 55*(4), 1107-1119.

Gumbel, E. J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association, 55*(292), 698-707.

Han, Y., Li, P., & Xia, Y. (2017). Dynamic robust portfolio selection with copulas. *Finance Research Letters, 21*, 190-200. doi:10.1016/j.frl.2016.12.008

Hofert, M., Kojadinovic, I., Maechler, M., & Yan, J. (2018). copula: Multivariate Dependence with Copulas. R package version 0.999-19.1. Retrieved from https://cran.r-project.org/package=copula

Jensen, M. C. (1969). Risk, The Pricing of Capital Assets, and The Evaluation of Investment Portfolios. *The*

Journal of Business, 42(2), 167-247. Retrieved from http://www.jstor.org/stable/2351902

Jin, X., & Lehnert, T. (2018). Large portfolio risk management and optimal portfolio allocation with dynamic elliptical copulas. *Dependence Modeling*, *6*(1), 19-46.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts* (Vol. 73). London: Chapman and Hall.

Jorion, P. (2000). *Value at Risk: The New Benchmark for Managing Financial Risk*. New York: McGraw-Hill.

Kazemi, H., Schneeweis, T., & Gupta, R. (2003). Omega as a Performance Measure. *CISDM Research Paper, June*.

Keating, C., & Shadwick, W. F. (2002). A Universal Performance Measure. *Journal of performance measurement, 6*(3), 59-84.

Kemaloglu Acik, S., & Kizilok Kara, E. (2015). Modeling dependent financial assets by dynamic copula and portfolio optimization based on CVaR. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, *64*(1), 1-13.

Kızılok Kara, E., & Acık Kemaloglu, S. (2016). Portfolio Optimization of Dynamic Copula Models for Dependent Financial Data Using Change Point Approach. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 65*, 175-188. doi:10.1501/Commua1_0000000768

Kojadinovic, I. (2017). Some copula inference procedures adapted to the presence of ties. *Computational Statistics & Data Analysis, 112*, 24-41.

Kresta, A. (2015). Application of GARCH-Copula Model in Portfolio Optimization. *Financial Assets and Investing, 6*(2), 7-20. doi:10.5817/FAI2015-2-1

Lintner, J. (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *The Review of Economics and Statistics, 47*(1), 13-37. doi:10.2307/1924119

Markowitz, H. (1952). Portfolio Selection*. *The Journal of Finance, 7*(1), 77-91. doi:10.1111/j.1540-6261.1952.tb01525.x

Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investments*. NY: Wiley.

Merton, R. C. (1980). On Estimating the Expected Return on the Market: An Exploratory Investigation. *Journal of Financial Economics, 8*(4), 323-361. doi:10.1016/0304-405X(80)90007-0

Merton, R. C., & Samuelson, P. A. (1974). Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods. *Journal of Financial Economics, 1*(1), 67-94. doi:10.1016/0304-405X(74)90009-9

Nelsen, R. B. (2006). *An Introduction to Copulas.* (2 ed.). New York: Springer-Verlag.

Patton, A. J. (2004). On the Out-of-Sample Importance of Skewness and Asymmetric Dependence for Asset Allocation. *Journal of Financial Econometrics, 2*(1), 130-168. doi:10.1093/jjfinec/nbh006

R Core Team. (2019). R: A Language and Environment for Statistical Computing: R Foundation for Statistical Computing. Retrieved from https://www.r-project.org/

Riccetti, L. (2013). A copula–GARCH model for macro asset allocation of a portfolio with commodities. *Empirical Economics, 44*(3), 1315-1336. doi:10.1007/s00181-012-0577-1

Rockafellar, R. T., & Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of risk, 2*, 21-42.

Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance, 26*(7), 1443-1471. doi:10.1016/S0378-4266(02)00271-6

Sahamkhadam, M., Stephan, A., & Östermark, R. (2018). Portfolio optimization based on GARCH-EVT-

Copula forecasting models. *International Journal of Forecasting, 34*(3), 497-506. doi:10.1016/j.ijforecast.2018.02.004

Sharpe, W. F. (1964). Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk*. *The Journal of Finance, 19*(3), 425-442. doi:10.1111/j.1540-6261.1964.tb02865.x

Sharpe, W. F. (1966). Mutual Fund Performance. *The Journal of Business, 39*(1), 119-138. Retrieved from http://www.jstor.org/stable/2351741

Sharpe, W. F. (1994). The Sharpe Ratio. *The Journal of Portfolio Management, 21*(1), 49-58. doi:10.3905/jpm.1994.409501

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges [N-dimensional distribution functions and their margins]. *Publ. Inst. Statist. Univ. Paris, 8*, 229-231.

Tobin, J. (1958). Liquidity Preference as Behavior Towards Risk. *The Review of Economic Studies, 25*(2), 65-86. doi:10.2307/2296205

Trabelsi, N., & Tiwari, A. K. (2019). Market-Risk Optimization among the Developed and Emerging Markets with CVaR Measure and Copula Simulation. *Risks, 7*(3), 78. Retrieved from https://www.mdpi.com/2227-9091/7/3/78

Uryasev, S. (2000, 28-28 March 2000). *Conditional Value-at-Risk: Optimization Algorithms and Applications.* Paper presented at the Proceedings of the IEEE/IAFE/INFORMS 2000 Conference on Computational Intelligence for Financial Engineering (CIFEr) (Cat. No.00TH8520).

Yu, C., & Liu, Y. (2021). A Personalized Mean-CVaR Portfolio Optimization Model for Individual Investment. *Mathematical Problems in Engineering*, *2021*, 8863597. doi:10.1155/2021/8863597