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FIXED-POINT THEOREMS IN EXTENDED FUZZY METRIC SPACES VIA α-ϕ-$\mathcal{M}^0$ AND β-ψ-$\mathcal{M}^0$ FUZZY CONTRACTIVE MAPPINGS

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Abstract. In this article we would like to present a new type of fuzzy contractive mappings which are called $\alpha - \phi - \mathcal{M}^0$ fuzzy contractive and $\beta - \psi - \mathcal{M}^0$ fuzzy contractive, and then we demonstrate two theorems which ensure the existence of a fixed point for these two types of mappings. And so we combine and generalize some existing notions in the literature ([5], [7]). Proved these theorems in the extended fuzzy metric spaces are in the more general version than the existing in the literature ones.

1. Introduction

The attention of fuzzy concept has been growing from the presented by Zadeh [20] in 1965. The concept of fuzzy was used a lot of fields such as mathematical analysis and general topology with many applications in economy and engineering. Recently, it is a paramount development that defining the concept of contractive mapping in fuzzy metric spaces. After the remarkable Banach [1] contraction principle, a large amount of mathematicians studied some contractive mappings to proof a fixed point exists. Afterwards, studies gained popularity with the notion of fuzzy metric space defined by Kramosil and Michalek [13], and then George and Veeramani [4] modified the concept of fuzzy metric space.

Contractivity's role in the fixed point theory is very important. There are a lot of studies in the literature regarding different versions contractive mappings in the different spaces ([2], [3], [5], [6], [8]- [17], [19]). Samet et al. [17] put forward new notions of contractive mapping and used these mappings to verify some fixed point theorems in metric spaces. Based on the same perspective, D. Gopal and C.

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Vetro [5] give some contractive mappings, which can be accepted generalizations of Samet et al. [17].

In this paper, we define new notions which are generalized versions of fuzzy contractive mappings introduced by D. Gopal and C.Vetro [5]. We study these contractions in extended fuzzy metric spaces introduced by V. Gregori et al. [7].

The new contractions are called $\alpha - \phi - \mathcal{M}^0$ fuzzy contractive mapping and $\beta - \psi - \mathcal{M}^0$ fuzzy contractive mapping. Moreover, we have proved some fixed point theorems with these mappings in this new space and so we got a generalized versions.

2. Preliminaries

Now in this section, we recall some definitions and results that will be used in the sequel.

Definition 1. *[18] A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called a continuous triangular norm (t-norm) if the following conditions hold:*

T_1 $*$ *is associative and commutative*;

T_2 $*$ *is continuous*;

T_3 $a * 1 = a$, *for all* $a \in [0,1]$;

T_4 $a * b < c * d$, *whenever* $a < c$ *and* $b < d$, *for all* $a, b, c, d \in [0,1]$.

Kramosil and Michalek [13] generalized probabilistic metric space via concept of fuzzy metric. After then George and Veeramani [4] made slight modification in this fuzzy metric concept.

Definition 2. *[4], A fuzzy metric space is a triple* $(\mathcal{X}, \mathcal{M}, *)$*, where* $\mathcal{X}$ *is a non-empty set,* $*$ *is a continuous t-norm and* $\mathcal{M}$ *is a fuzzy set on* $\mathcal{X}^2 \times (0,\infty)$*, satisfying for all* $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$ *and for all* $\mathfrak{t}, \mathfrak{s} > 0$*, the following properties:*

(ÇV₁) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) > 0$;

(ÇV₂) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 1$ *if and only if* $\mathfrak{x} = \mathfrak{y}$;

(ÇV₃) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{y}, \mathfrak{x}, \mathfrak{t})$*;*

(ÇV₄) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) * \mathcal{M}(\mathfrak{y}, \mathfrak{z}, \mathfrak{s}) \leq \mathcal{M}(\mathfrak{x}, \mathfrak{z}, \mathfrak{t} + \mathfrak{s})$;

(ÇV₅) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \cdot) : (0,\infty) \to [0,1]$ *is continuous,*

$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})$ *could be considered as the degree of closeness between* x *and* y *with regard to* t*. In the above definition, if we replace (ÇV₄) by* $(ÇV_4^*)$*,*$\forall\ \mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$ *and* $\mathfrak{t}, \mathfrak{s} > 0$;

$$(ÇV_4^*) : \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) * \mathcal{M}(\mathfrak{y}, \mathfrak{z}, \mathfrak{s}) \leq \mathcal{M}(\mathfrak{x}, \mathfrak{z}, \max\{\mathfrak{t}, \mathfrak{s}\})$$

then the triple $(\mathcal{X}, \mathcal{M}, *)$ *is said to be non-Archimedean fuzzy metric space [14].*

Definition 3. *[8] A stationary fuzzy metric space is a triple* $(\mathcal{X}, \mathcal{M}, *)$ *such that* $\mathcal{X}$ *is a non-empty set,* $*$ *is a continuous t-norm and* $\mathcal{M}$ *is a fuzzy set on* $\mathcal{X}^2$ *satisfying the following conditions, for all* $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$;

(S_1) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) > 0$;
(S_2) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) = 1$ *if and only if* $\mathfrak{x} = \mathfrak{y}$;
(S_3) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) = \mathcal{M}(\mathfrak{y},\mathfrak{x})$;
(S_4) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) * \mathcal{M}(\mathfrak{y},\mathfrak{z}) \leq \mathcal{M}(\mathfrak{x},\mathfrak{z})$.

In other words, a fuzzy metric space $(\mathcal{X},\mathcal{M},*)$ *is said to be stationary if* $\mathcal{M}$ *does not depend on* $\mathfrak{t}$.

A sequence $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ *in a stationary fuzzy metric space* $(\mathcal{X},\mathcal{M})$ *is said to be Cauchy if* $\lim\limits_{i,j\to\infty}\mathcal{M}(\mathfrak{x}_i,\mathfrak{x}_j) = 1$; *a sequence* $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ *in* $\mathcal{X}$ *converges to* $\mathfrak{x}$ *if* $\lim\limits_{i\to\infty}\mathcal{M}(\mathfrak{x}_i,\mathfrak{x}) = 1$ *[8].*

Now we recall a kind of generalized fuzzy metric space introduced by V. Gregori, J-J Minana and D. Miravet [7]. They study those fuzzy metrics $\mathcal{M}$ on $\mathcal{X}$, in the George and Veeramani's sense, such that $\wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$.

Definition 4. *[7] The term* $(\mathcal{X},\mathcal{M}^0,*)$ *is called an extended fuzzy metric space if* $\mathcal{X}$ *is a (non-empty) set,* $*$ *is a continuous t-norm and* $\mathcal{M}^0$ *is a fuzzy set on* $\mathcal{X}^2\times[0,\infty)$ *satisfying the following conditions, for each* $\mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mathcal{X}$ *and* $\mathfrak{t},\mathfrak{s}\geq 0$;
(EFM$_1$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$;
(EFM$_2$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = 1$ *if and only if* $\mathfrak{x} = \mathfrak{y}$;
(EFM$_3$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = \mathcal{M}^0(\mathfrak{y},\mathfrak{x},\mathfrak{t})$;
(EFM$_4$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) * \mathcal{M}^0(\mathfrak{y},\mathfrak{z},s) \leq \mathcal{M}^0(\mathfrak{x},\mathfrak{z},\mathfrak{t}+\mathfrak{s})$;
(EFM$_5$) $\mathcal{M}^0_{\mathfrak{x},\mathfrak{y}} : [0,\infty)\to(0,1]$ *is continuous, where* $\mathcal{M}^0_{\mathfrak{x},\mathfrak{y}}(\mathfrak{t}) = \mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t})$.

Theorem 1. *[7] Let* M *be a fuzzy set on* $\mathcal{X}^2\times(0,\infty)$, *and denote by* $\mathcal{M}^0$ *its extension to ,* $\mathcal{X}^2\times[0,\infty)$ *given by*

$$\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = \mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) \text{ for all } \mathfrak{x},\mathfrak{y},\in\mathcal{X},\ \mathfrak{t}>0 \quad \text{and}$$

$$\mathcal{M}^0(\mathfrak{x},\mathfrak{y},0) = \wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}).$$

Then, $(\mathcal{X},\mathcal{M}^0,*)$ *is an extended fuzzy metric space if and only if* $(\mathcal{X}.\mathcal{M},*)$ *is a fuzzy metric space satisfying for each* $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$ *the condition* $\wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$.

Proposition 1. *[7] Let* $(\mathcal{X},\mathcal{M},*)$ *be a fuzzy metric space. Define*

$$N_{\mathcal{M}}(\mathfrak{x},\mathfrak{y}) = \wedge_{t>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}).$$

Then, $(N_{\mathcal{M}},*)$ *is a stationary fuzzy metric on* $\mathcal{X}$ *if and only if* $\wedge_{t>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$ *for all* $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$.

It is clear that

$$\mathcal{M}^0(\mathfrak{x},\mathfrak{y},0) = \wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = N_{\mathcal{M}}(\mathfrak{x},\mathfrak{y}). \tag{1}$$

Definition 5. *[7] Let* $(\mathcal{X},\mathcal{M},*)$ *be a fuzzy metric space.* $\mathcal{M}$ *is called extendable if for each* $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$ *the condition* $\wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$ *is satisfied. In such a case, we will say that* $\mathcal{M}^0$ *is the (fuzzy metric) extension of* $\mathcal{M}$, *and that* $\mathcal{M}$ *is the restriction of* $\mathcal{M}^0$.

Proposition 2. *[7] Let* $(\mathcal{X},\mathcal{M}^0,*)$ *is complete if and only if* $(\mathcal{X},N_{\mathcal{M}},*)$ *is complete.*

Samet et al. [17] introduced a new concept of $\alpha-\psi-$ *contractive* and $\alpha-$ *admissible* mappings in metric spaces. D. Gopal and C. Vetro [5] inspired from them [17] and introduced the notions of $\alpha-\phi-$ *fuzzy contractive* mapping and $\beta-\psi-$ *fuzzy contractive* mapping. We recall the notions as follows.

Remark 1. *[5] Denote by Φ the family of all right continuous functions $\phi : [0,\infty) \longrightarrow [0,\infty)$,with $\phi(r) < r$ for all $r > 0$.Note that for every function $\phi \in \Phi$, $\lim_{n\to\infty} \phi^n(r) = 0$ for each $r > 0$,where $\phi^n(r)$ denotes the $n-th$ iterate of ϕ.*

Definition 6. *[5] Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. It is said that $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is an $\alpha-\phi-$ fuzzy contractive mapping if there exist two functions $\alpha : \mathcal{X}^2 \times (0,\infty) \longrightarrow [0,\infty)$ and $\phi \in \Phi$ such that*

$$\alpha(\mathfrak{x},\mathfrak{y},\mathfrak{t})(\frac{1}{\mathcal{M}(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t})}-1) \leq \phi(\frac{1}{\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t})}-1)$$

for all $\mathfrak{x},\mathfrak{y} \in \mathcal{X}$ and $\mathfrak{t} > 0$.

Definition 7. *[5] Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. It is said that $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is $\alpha-$ admissible if there exist a function $\alpha : \mathcal{X}^2 \times (0,\infty) \longrightarrow [0,\infty)$ such that,*

$$\alpha(\mathfrak{x},\mathfrak{y},\mathfrak{t}) \geq 1 \Longrightarrow \alpha(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t}) \geq 1$$

for all $\mathfrak{x},\mathfrak{y} \in \mathcal{X}$ and $\mathfrak{t} > 0$.

Remark 2. *[5] Let Ψ be the class of all functions $\psi : [0,1] \longrightarrow [0,1]$ such that ψ is non-decreasing and left continuous and $\psi(r) > r$ for all $r \in (0,1)$. If $\psi \in \Psi$, then $\psi(1) = 1$ and $\lim_{n\to\infty} \psi^n(r) = 1$ for all $r \in (0,1]$.*

Definition 8. *[5] Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. It is said that $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is an $\beta-\psi-$ fuzzy contractive mapping if there exist two functions $\beta : \mathcal{X}^2 \times (0,\infty) \longrightarrow (0,\infty)$ and $\psi \in \Psi$ such that,*

$$\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0 \Rightarrow \beta(\mathfrak{x},\mathfrak{y},\mathfrak{t})\mathcal{M}(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t}) \geq \psi(\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}))$$

for all $\mathfrak{x},\mathfrak{y} \in \mathcal{X}$ with $\mathfrak{x} \neq \mathfrak{y}$ and for all $\mathfrak{t} > 0$.

Definition 9. *[5] Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. It is said that $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is a $\beta-$ admissible if there exist a function $\beta : \mathcal{X}^2 \times (0,\infty) \longrightarrow (0,\infty)$ such that,*

$$\beta(\mathfrak{x},\mathfrak{y},\mathfrak{t}) \leq 1 \Longrightarrow \beta(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t}) \leq 1 \text{ for all } \mathfrak{x},\mathfrak{y}, \in \mathcal{X} \text{ and } \mathfrak{t} > 0.$$

3. Main Result

3.1. **$\alpha-\phi-\mathcal{M}^0-$ fuzzy contractive mappings.** We are ready to introduce new definitions of $\alpha-\phi-\mathcal{M}^0-$ *fuzzy contractive* and $\alpha-\mathcal{M}^0-$ *admissible*. We would like to inform you that use these mappings in the new fuzzy metric space (introduced in [7]). Then, we prove the theorem (proved in [5]) but in the new fuzzy metric spaces. And so, we obtain new results that are generalizations of those in fuzzy metric spaces.

Definition 10. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\alpha - \phi - \mathcal{M}^0 - fuzzy$ contractive mapping if*

$$\alpha(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})(\frac{1}{\mathcal{M}(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t})} - 1) \leq \phi(\frac{1}{\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})} - 1) \tag{2}$$

is ensured $\forall\ \mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially,$\Im$ is called $\alpha - \phi - 0 - fuzzy$ contractive if Equation (2) is ensured for $\mathfrak{t} = 0$.

Definition 11. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\alpha - \mathcal{M}^0 -$ admissible mapping if*

$$\alpha(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t}) \geq 1 \tag{3}$$

is ensured $\forall\ \mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially,$\Im$ is called $\alpha - 0 -$admissible if Equation (3) is ensured for $\mathfrak{t} = 0$.

Theorem 2. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a complete extendable fuzzy metric space and a mapping $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ be an $\alpha - \phi - \mathcal{M}^0 - fuzzy$ contractive ensuring the provisions given below:*

(i) $\Im$ is $\alpha - \mathcal{M}^0 -$ admissible;
(ii) $\exists\ \mathfrak{x}_0 \in \mathcal{X}$ such that $\alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, \mathfrak{t}) \geq 1,\ \forall\ \mathfrak{t} \geq 0$;
(iii) $\Im$ is continuous;
Then, $\Im$ has a fixed point.

Proof. We will examine the proof in two cases.

Case 1. $\mathfrak{t} > 0$;

In this case, since $\mathcal{M}^0(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})\ \forall\ \mathfrak{x}, \mathfrak{y} \in \mathcal{X}$, it is same situation in fuzzy metric spaces and introduced in the proof of the Theorem 3.5. [5].

Case 2. $\mathfrak{t} = 0$;

Let $\mathfrak{x}_0 \in \mathcal{X}$ such that $\alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, 0) \geq 1$.

Define the squence $\{\mathfrak{x}_n\}$ in $\mathcal{X}$ with $\mathfrak{x}_{n+1} = \Im\mathfrak{x}_n,\ \forall\ n \in \mathbb{N}$.

Provided that $\mathfrak{x}_{n+1} = \mathfrak{x}_n$ for some $n \in \mathbb{N}$, then $\mathfrak{x}^* = \mathfrak{x}_n$ is a fixed point of $\Im$.

Presume that $\mathfrak{x}_n \neq \mathfrak{x}_{n+1},\ \forall\ n \in \mathbb{N}$.

From (ii),

$$\alpha(\mathfrak{x}_0, \mathfrak{x}_1, 0) = \alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, 0) \geq 1$$

and using (i), we have

$$\alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, 0) \geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_0, \Im\mathfrak{x}_1, 0) \geq 1$$

By induction,

$$\begin{aligned} \alpha(\Im\mathfrak{x}_0, \Im\mathfrak{x}_1, 0) &\geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_1, \Im\mathfrak{x}_2, 0) \geq 1 \\ \alpha(\Im\mathfrak{x}_1, \Im\mathfrak{x}_2, 0) &\geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_2, \Im\mathfrak{x}_3, 0) \geq 1 \\ &\ldots \\ \alpha(\Im\mathfrak{x}_{n-3}, \Im\mathfrak{x}_{n-2}, 0) &\geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_{n-2}, \Im\mathfrak{x}_{n-1}, 0) \geq 1 \end{aligned}$$

and so we get,

$$\alpha(\Im\mathfrak{x}_{n-2},\Im\mathfrak{x}_{n-1},0)=\alpha(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)\geq 1,\,\forall\; n\in\mathbb{N}. \tag{4}$$

Using (1), implementing (2) with $\mathfrak{x}=\mathfrak{x}_{n-1}$, $\mathfrak{y}=\mathfrak{x}_n$, $\mathfrak{t}=0$ and using (4) respectively we obtain;

$$\begin{aligned}\frac{1}{\mathcal{M}^0(\mathfrak{x}_n,\mathfrak{x}_{n+1},0)}-1 &= \frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)}-1\\ &\leq \alpha(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)(\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)}-1)\\ &\leq \phi(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_{n-1},\mathfrak{x}_n)}-1)\\ &= \phi(\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-2},\Im\mathfrak{x}_{n-1})}-1)\end{aligned}$$

This implies that,

$$\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)}-1\leq\phi^n(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_0,\mathfrak{x}_1)}-1)$$

as $n\to\infty$

$$\lim_{n\to\infty}(\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)}-1)\leq\lim_{n\to\infty}\phi^n(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_0,\mathfrak{x}_1)}-1)$$

Since, as $n\to\infty$ and $\phi^n(r)\to 0$,

$$\lim_{n\to\infty}(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_{n+1})}-1)=0$$

and so, we obtain that

$$\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_{n+1})=1.$$

which implies that for $n<m$ and using (1) with $\mathfrak{x}=\mathfrak{x}_n$, $\mathfrak{y}=\mathfrak{x}_m$, $\mathfrak{t}=0$;

$$\mathcal{M}^0(\mathfrak{x}_n,\mathfrak{x}_m,0)=\wedge_{t>0}\mathcal{M}(\mathfrak{x}_n,\mathfrak{x}_m,t)=N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_m)$$

Using Definition 3,

$$N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_m)\geq N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_{n+1})*N_{\mathcal{M}}(\mathfrak{x}_{n+1},\mathfrak{x}_{n+2})*...*N_{\mathcal{M}}(\mathfrak{x}_{m-1},\mathfrak{x}_m)$$

and as $n\to\infty$,

$$\begin{aligned}\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_m) &\geq \lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_{n+1})*\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{n+1},\mathfrak{x}_{n+2})*...*\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{m-1},\mathfrak{x}_m)\\ &\geq 1*1*...*1\\ &\geq 1\end{aligned}$$

We obtain,

$$\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_m)=1$$

And so, we solve an important point of the proof that $\{\mathfrak{x}_n\}$ is a Cauchy squence. Since $\mathcal{X}$ is complete,

$$\exists\ \mathfrak{x}^* \in \mathcal{X}\ \ :\ \text{as } n \to \infty \text{ and } \mathfrak{x}_n \to \mathfrak{x}^*$$

Since $\Im$ is continuous, as $\mathfrak{x}_n \to \mathfrak{x}^*$we have $\Im\mathfrak{x}_n \to \Im\mathfrak{x}^*$ and using (1),

$$\mathcal{M}^0(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*, 0) = \wedge_{t>0}\mathcal{M}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*, t) = N_{\mathcal{M}}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*),\ \forall\ \mathfrak{x}_n \in \mathcal{X}.$$

And so we obtain,

$$\lim_{n\to\infty} N_{\mathcal{M}}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*) = 1.$$

By the uniqueness of the limit, we get $\mathfrak{x}^* = \Im\mathfrak{x}^*$, that is, $\mathfrak{x}^*$ is a fixed point of $\Im$. □

3.2. **$\boldsymbol{\beta - \psi - \mathcal{M}^0-}$ fuzzy contractive mappings.** We are ready to introduce new definitions of $\beta - \psi - \mathcal{M}^0 - fuzzy\ contractive$ and $\beta - \mathcal{M}^0-$ *admissible*. We would like to inform you that we use these mappings in the new fuzzy metric space (introduced in [7]). Then, we prove the theorem (proved in [5]) but in the new fuzzy metric spaces. And so, we obtain new results that are generalizations of those in fuzzy metric spaces.

Definition 12. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\beta - \psi - \mathcal{M}^0 - fuzzy\ contractive$ mapping if*

$$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) > 0 \Rightarrow \beta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t}) \geq \psi(\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})) \tag{5}$$

is ensured $\forall\ \mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially,$\Im$ is called $\beta-\psi-0-fuzzy\ contractive$ if Equation (5) is ensured for $\mathfrak{t} = 0$.

Definition 13. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\beta - \mathcal{M}^0 -$ admissible mapping if*

$$\beta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \leq 1 \Longrightarrow \beta(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t}) \leq 1 \tag{6}$$

is ensured $\forall\ \mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially,$\Im$ is called $\beta-0-$admissible if Equation (6) is ensured for $\mathfrak{t} = 0$

By adding an additional condition, we prove a fixed point theorem introduced in [5] in extendable fuzzy metric space using these new mappings. This is a new context that using the new mappings in the extendable fuzzy metric space.

Theorem 3. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable complete non-Archimedean fuzzy metric space and a mapping $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ be a $\beta - \psi - \mathcal{M}^0 - fuzzy\ contractive$ ensuring the provisions given below:*

(i) $\Im$ is $\beta - \mathcal{M}^0 -$ admissible;

(ii) $\exists\ \mathfrak{x}_0 \in \mathcal{X}$ such that $\beta(\mathfrak{x}_0, \Im\mathfrak{x}_0, \mathfrak{t}) \leq 1\ \forall\ \mathfrak{t} \geq 0$;

(iii) for each sequence $\{\mathfrak{x}_n\}$ in $\mathcal{X}$ such that $\beta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}) \leq 1\ \forall\ n \in \mathbb{N}$ and $\mathfrak{t} \geq 0$, $\exists\ k_0 \in N$ such that $\beta(\mathfrak{x}_{m+1}, \mathfrak{x}_{n+1}, \mathfrak{t}) \leq 1\ \forall\ m, n \in \mathbb{N}$ with $m > n \geq k_0$ and $\forall\ \mathfrak{t} \geq 0$;

(iv) if $\{\mathfrak{x}_n\}$ is a sequence in $\mathcal{X}$ such that $\beta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}) \leq 1 \ \forall \ n \in \mathbb{N}$ and $\mathfrak{t} \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(\mathfrak{x}_n, \mathfrak{x}, \mathfrak{t}) \leq 1 \ \forall \ n \in \mathbb{N}$ and $\forall \ \mathfrak{t} \geq 0$;

(v) $\forall \ \mathfrak{x}, \mathfrak{y} \in \mathcal{X}$ and $\forall \ \mathfrak{t} \geq 0, \exists \ \mathfrak{z} \in \mathcal{X}$ such that $\beta(\mathfrak{x}, \mathfrak{z}, \mathfrak{t}) \leq 1$ and $\beta(\mathfrak{y}, \mathfrak{z}, \mathfrak{t}) \leq 1$;

Then, $\Im$ has a unique fixed point.

Proof. We will examine the proof in two cases.

Case 1. $\mathfrak{t} > 0$;

In this case, since $\mathcal{M}^0(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}), \ \forall \, \mathfrak{x}, \mathfrak{y} \in \mathcal{X}$; it is same situation in fuzzy metric spaces and introduced in the proof of the Theorem 4.4 [5]. It is obtained that $\Im\mathfrak{x}^* = \mathfrak{x}^*$ in the [5].

Now we will show that uniqueness of the fixed point.

Presume that $\Im$ have two different fixed points; $\mathfrak{x}^*$and $\mathfrak{y}^*$.

Provided that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \leq 1$,then

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, \mathfrak{t}).$$

Since $\Im$ is $\beta - \psi - \mathcal{M}^0-$ fuzzy contractive, we have

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, \mathfrak{t}) \geq \psi(\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})).$$

Also, since $\psi(r) > r$, we obtain that

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, \mathfrak{t}) \geq \psi(\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})) > \mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}).$$

And so, we get

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) > \mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})$$

It is a contradiction.

That is, $\mathfrak{x}^*$and $\mathfrak{y}^*$ are not different points; $\mathfrak{x}^* = \mathfrak{y}^*$.

Presume that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) > 1$,then from (v),

$$\exists \ \mathfrak{z} \in X : \ \beta(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t}) \leq 1 \text{and } \beta(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t}) \leq 1.$$

From (i), we obtain,

$$\begin{aligned}
\beta(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}^*, \Im\mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{x}^*, \Im\mathfrak{z}, \mathfrak{t}) \leq 1 \\
\beta(\mathfrak{x}^*, \Im\mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}^*, \Im^2\mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{x}^*, \Im^2\mathfrak{z}, \mathfrak{t}) \leq 1 \\
&\quad ... \\
\beta(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) \leq 1
\end{aligned}$$

and so we get,

$$\beta(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) \leq 1, \forall \ n \in \mathbb{N} \text{ and } \forall \ \mathfrak{t} > 0. \tag{7}$$

Since $\Im$ is $\beta - \psi - \mathcal{M}^0-$ *fuzzy contractive*, using (7), we get,

$$\begin{aligned}
\mathcal{M}^0(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) &= \mathcal{M}(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) = \mathcal{M}(\Im\mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z}), \mathfrak{t}) \\
&\geq \beta(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z}, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z}), \mathfrak{t}) \\
&\geq \psi(\mathcal{M}(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z}, \mathfrak{t})) \\
&= \psi(\mathcal{M}(\Im\mathfrak{x}^*, \Im(\Im^{n-2}\mathfrak{z}), \mathfrak{t}))
\end{aligned}$$

And by induction we have,

$$\mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \psi^n(\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t})), \ \forall\, n \in \mathbb{N}.$$

as $n \to \infty$

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \lim_{n\to\infty} \psi^n(\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t}))$$

Since $\psi^n(r) \to 1$,

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, \mathfrak{t}) = 1 \Rightarrow \Im^n \mathfrak{z} \to \mathfrak{x}^* \tag{8}$$

and by similar way, we get

$$\begin{aligned}
\beta(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im \mathfrak{y}^*, \Im \mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{y}^*, \Im \mathfrak{z}, \mathfrak{t}) \leq 1 \\
\beta(\mathfrak{y}^*, \Im \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im \mathfrak{y}^*, \Im^2 \mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{y}^*, \Im^2 \mathfrak{z}, \mathfrak{t}) \leq 1 \\
&\ldots \\
\beta(\mathfrak{y}^*, \Im^{n-2} \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im \mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) \leq 1
\end{aligned}$$

$$\beta(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) \leq 1, \forall\, n \in \mathbb{N} \text{ and } \forall\, \mathfrak{t} > 0. \tag{9}$$

Since $\Im$ is $\beta - \psi - \mathcal{M}^0-$ *fuzzy contractive*, using (9), we get,

$$\begin{aligned}
\mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) &= \mathcal{M}(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) = \mathcal{M}(\Im \mathfrak{y}^*, \Im(\Im^{n-1} \mathfrak{z}), \mathfrak{t}) \\
&\geq \beta(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) \mathcal{M}(\Im \mathfrak{y}^*, \Im(\Im^{n-1} \mathfrak{z}), \mathfrak{t}) \\
&\geq \psi(\mathcal{M}(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}))
\end{aligned}$$

And so, by induction we have,

$$\mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \psi^n(\mathcal{M}^0(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t})), \forall\, n \in \mathbb{N}.$$

as $n \to \infty$

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \lim_{n\to\infty} \psi^n(\mathcal{M}^0(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t}))$$

Since $\psi^n(r) \to 1$,

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) = 1 \Rightarrow \Im^n \mathfrak{z} \to \mathfrak{y}^* \tag{10}$$

From (8), (10) and the uniqueness of the limit $\mathfrak{x}^* = \mathfrak{y}^*$.

Case 2. $\mathfrak{t} = 0$;

Let $\mathfrak{x}_0 \in \mathcal{X}$ such that $\beta(\mathfrak{x}_0, \Im \mathfrak{x}_0, 0) \leq 1$.

Define the sequence $\mathfrak{x}_{n+1} = \Im \mathfrak{x}_n, \forall\, n \in \mathbb{N}$. If $\mathfrak{x}_{n+1} = \mathfrak{x}_n$ for some $n \in \mathbb{N}$,then $\mathfrak{x}^* = \mathfrak{x}_n$ is a fixed point of $\Im$.

Suppose $\mathfrak{x}_{n+1} \neq \mathfrak{x}_n$, $\forall\, n \in \mathbb{N}$.

From (ii),

$$\beta(\mathfrak{x}_0, \mathfrak{x}_1, 0) = \beta(\mathfrak{x}_0, \Im \mathfrak{x}_0, 0) \leq 1$$

and using (i), we obtain

$$\beta(\mathfrak{x}_0, \Im \mathfrak{x}_0, 0) \leq 1 \Rightarrow \beta(\Im \mathfrak{x}_0, \Im \mathfrak{x}_1, 0) \leq 1.$$

By induction,

$$\begin{aligned}\beta(\Im\mathfrak{x}_0,\Im\mathfrak{x}_{1,}0) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}_1,\Im\mathfrak{x}_{2,}0)\leq 1\\ \beta(\Im\mathfrak{x}_1,\Im\mathfrak{x}_{2,}0) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}_2,\Im\mathfrak{x}_{3,}0)\leq 1\\ &...\\ \beta(\Im\mathfrak{x}_{n-3},\Im\mathfrak{x}_{n-2,}0) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}_{n-2},\Im\mathfrak{x}_{n-1,}0)\leq 1\end{aligned}$$

and so we get,

$$\beta(\Im\mathfrak{x}_{n-2},\Im\mathfrak{x}_{n-1,}0)=\beta(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)\leq 1,\ \forall\ n\in\mathbb{N}. \tag{11}$$

Implementing (5) with $\mathfrak{x}=\mathfrak{x}_{n-1}$, $\mathfrak{y}=\mathfrak{x}_n$, $\mathfrak{t}=0$ and using (11) respectively, we obtain;

$$\mathcal{M}^0(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n,0)\geq\beta(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)\mathcal{M}^0(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n,0)\geq\psi(\mathcal{M}^0(\mathfrak{x}_{n-1},\mathfrak{x}_n,0))$$

Using (1), we get,

$$\begin{aligned}N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n) &\geq \beta(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)(N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n))\\ &\geq \psi(N_{\mathcal{M}}(\mathfrak{x}_{n-1},\mathfrak{x}_n))\end{aligned}$$

And this implies that,

$$N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)\geq\psi^n(N_{\mathcal{M}}(\mathfrak{x}_0,\mathfrak{x}_1)),\ \forall\ n\in\mathbb{N}.$$

as $n\to\infty$,

$$\lim_{n\to\infty}N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)\geq\lim_{n\to\infty}\psi^n(N_{\mathcal{M}}(\mathfrak{x}_0,\mathfrak{x}_1))$$

Since $\psi^n(r)\to 1$,

$$\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_{n+1})=1.$$

The important point of the proof is setting that the sequence $\{\mathfrak{x}_n\}$ Cauchy in $\mathcal{X}$.

Suppose that it is false; there exists $0<\varepsilon<1$ and two subsequences $\{\mathfrak{x}_{p_n}\}$ and $\{\mathfrak{x}_{q_n}\}$ of $\{\mathfrak{x}_n\}$ such that q_n is the smallest index for which $p_n>q_n\geq n_0$, using (1)

$$\mathcal{M}^0(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n},0)=\wedge_{t>0}\mathcal{M}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n},t)=N_{\mathcal{M}}(\mathfrak{x}_{p_n},x_{\mathfrak{x}q_n})\leq 1-\varepsilon$$

$$\mathcal{M}^0(\mathfrak{x}_{p_{n-1}},\mathfrak{x}_{q_n},0)=\wedge_{t>0}\mathcal{M}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n},t)=N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n})>1-\varepsilon$$

and by (iii); $n_0\in\mathbb{N}$ such that, for each $n\in\mathbb{N}$ with $n\geq n_0$,there exist $p_n,q_n\in\mathbb{N}$ $\beta(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n},0)\leq 1$.

And we get

$$1-\varepsilon\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n})\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n})*N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{p_n})$$

as $n\to\infty$

$$\lim_{n\to\infty}(1-\varepsilon)\geq\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n})\geq\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n})*\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{p_n})$$

Since $\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{p_n})=1$,

$$(1-\varepsilon)\geq\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n})\geq(1-\varepsilon)$$

we obtain that

$$\lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n}) = (1-\varepsilon).$$

and similarly

$$\begin{aligned}(1-\varepsilon) &\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n}) \\ &\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * N_{\mathcal{M}}(\mathfrak{x}_{p_n+1}, \mathfrak{x}_{q_n+1}) * N_{\mathcal{M}}(\mathfrak{x}_{q_n+1}, \mathfrak{x}_{q_n}) \\ &\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * \beta(\Im\mathfrak{x}_{p_n}, \Im\mathfrak{x}_{q_n}, 0) N_{\mathcal{M}}(\Im\mathfrak{x}_{p_n}, \Im\mathfrak{x}_{q_n}) * N_{\mathcal{M}}(\mathfrak{x}_{q_n+1}, \mathfrak{x}_{q_n}) \\ &\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * \psi(N_M(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n})) * N_{\mathcal{M}}(\mathfrak{x}_{q_n}, \mathfrak{x}_{q_n+1}).\end{aligned}$$

as $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty}(1-\varepsilon) &\geq \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * \lim_{n\to\infty} \psi(N_M(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n})) * \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{q_n}, \mathfrak{x}_{q_n+1}) \\ (1-\varepsilon) &\geq \lim_{n\to\infty} \psi(N_M(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n})) \\ (1-\varepsilon) &\geq \psi(1-\varepsilon)\end{aligned}$$

It is a contradiction, because of $\psi(r) > r$.

So we have obtained that $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete,

$$\exists\, \mathfrak{x}^* \in \mathcal{X} \ : \ \text{as } n \to \infty \text{ and } \mathfrak{x}_n \to \mathfrak{x}^*$$

Using (11) and (iv);

$$\beta(\mathfrak{x}_n, \mathfrak{x}^*, 0) \leq 1, \forall n \in \mathbb{N}$$

from (5) with using (1) and S_4,

$$\begin{aligned}N_{\mathcal{M}}(\Im\mathfrak{x}^*, \mathfrak{x}^*) &\geq N_{\mathcal{M}}(\Im\mathfrak{x}^*, \Im\mathfrak{x}_n) * N_{\mathcal{M}}(\Im\mathfrak{x}_n, \mathfrak{x}^*) \\ &\geq \beta(\mathfrak{x}_n, \mathfrak{x}^*, 0) N_{\mathcal{M}}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*) * N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}^*) \\ &\geq \psi(N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}^*)) * N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}^*)\end{aligned}$$

as $n \to \infty$, $\psi(1) = 1$

$$\begin{aligned}\lim_{n\to\infty} N_{\mathcal{M}}(\Im\mathfrak{x}^*, \mathfrak{x}^*) &\geq \lim_{n\to\infty} \psi(N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}^*)) * \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}^*) \\ &\geq \psi(1) * 1 = 1\end{aligned}$$

and we obtain,

$$\lim_{n\to\infty} N_{\mathcal{M}}(\Im\mathfrak{x}^*, \mathfrak{x}^*) = 1.$$

And so, $\mathfrak{x}^* = \Im\mathfrak{x}^*$.That is, $\mathfrak{x}^*$ is a fixed point of $\Im$.

Now we will show that uniqueness of the fixed point.

Presume that $\Im$ have two different fixed points; $\mathfrak{x}^*$and $\mathfrak{y}^*$.

Provided that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, 0) \leq 1$, then since $\Im$ is $\beta-\psi-0-$ fuzzy contractive, using (1) and $\psi(r) > r$, we have

$$\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{y}^*, 0) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, 0)\mathcal{M}^0(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, 0) \geq \psi(\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{y}^*, 0)) > \mathcal{M}^0(\mathfrak{x}^*, \mathfrak{y}^*, 0)$$

$$N_{\mathcal{M}}(\mathfrak{x}^*, \mathfrak{y}^*) > N_{\mathcal{M}}(\mathfrak{x}^*, \mathfrak{y}^*)$$

it is a contradiction. That is, $\mathfrak{x}^* = \mathfrak{y}^*$.

Assume that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, 0) > 1$,then from (v)

$$\exists\ \mathfrak{z} \in X : \ \beta(\mathfrak{x}^*, \mathfrak{z}, 0) \leq 1 \text{ and } \beta(\mathfrak{y}^*, \mathfrak{z}, 0) \leq 1.$$

From (i), we obtain,

$$\beta(\mathfrak{x}^*, \Im^n \mathfrak{z}, 0) \leq 1 \text{ and } \beta(\mathfrak{y}^*, \Im^n \mathfrak{z}, 0) \leq 1, \forall\ n \in \mathbb{N}. \tag{12}$$

Since $\Im$ is $\beta - \psi - 0-$ fuzzy contractive and using (10), we obtain

$$\begin{aligned} \mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, 0) &= \mathcal{M}(\mathfrak{x}^*, \Im^n \mathfrak{z}, 0) = \mathcal{M}(\Im \mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z}), 0) = N_{\mathcal{M}}(\Im \mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z})) \\ &\geq \beta(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z}, 0) N_{\mathcal{M}}(\Im \mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z})) \\ &\geq \psi(N_{\mathcal{M}}(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z})) \end{aligned}$$

And by induction we obtain,

$$N_{\mathcal{M}}(\Im \mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z})) \geq \psi^n(N_{\mathcal{M}}(\mathfrak{x}^*, \mathfrak{z})), \forall n \in \mathbb{N}.$$

As $n \to \infty$, we get $\Im^n \mathfrak{z} \to \mathfrak{x}^*$.

And by the similary way we obtain $\Im^n \mathfrak{z} \to \mathfrak{y}^*$.So the uniqueness of the limit $\mathfrak{x}^* = \mathfrak{y}^*$ □

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