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Research Article

Orlicz dual of log-Aleksandrov–Fenchel inequality

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Abstract

In this paper, we establish an Orlicz dual of the log-Aleksandrov–Fenchel inequality, by introducing two new concepts of dual mixed volume measures, and using the newly established Orlicz dual Aleksandrov–Fenchel inequality. The Orlicz dual log-Aleksandrov–Fenchel inequality in special cases yields the classical dual Aleksandrov–Fenchel inequality and some dual logarithmic Minkowski type inequalities, respectively. Moreover, the dual log-Aleksandrov–Fenchel inequality is therefore also derived.

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1. Introduction

In 2016, Stancu [15] established the following logarithmic Minkowski inequality.

The logarithmic Minkowski inequality *If K and L are convex bodies in $\mathbb{R}^n$ that containing the origin in their interior, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\bar{v}_1 \geq \frac{1}{n}\ln\left(\frac{V(K)}{V(L)}\right). \tag{1.1}$$

with equality if and only if K and L are homothetic, where dv_1 is the mixed volume measure $dv_1 = \frac{1}{n}h_K dS_L$, and $d\bar{v}_1 = \frac{1}{V_1(L,K)}dv_1$ is its normalization, and $V_1(L,K)$ denotes the usual mixed volume of L and K, defined by

$$V_1(L,K) = \frac{1}{n}\int_{S^{n-1}} h_K dS_L.$$

The functions h_K and h_L are the support functions. If K is a nonempty closed convex set in $\mathbb{R}^n$, then

$$h_K = \max\{x\cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defines the support function h_K of K.

Recently, the logarithmic Minkowski inequality and its dual form have attracted extensive attention and research. The recent research can be found in the references [1–3, 5–7, 11–13, 16, 18–20, 23, 24]. In particularly, as a generalization of (1.1), the log-Aleksandrov–Fenchel inequality has been established in [21].

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The dual mixed volume of star bodies $K_1,\ldots,K_n$, $\widetilde{V}(K_1,\ldots,K_n)$ defined by Lutwak (see [8])

$$\widetilde{V}(K_1,\ldots,K_n)=\frac{1}{n}\int_{S^{n-1}}\rho(K_1,u)\cdots\rho(K_n,u)dS(u). \tag{1.2}$$

Here, $\rho(K,\cdot)$ denotes the radial function of star body K. The radial function of star body K is defined by

$$\rho(K,u)=\max\{c\geq 0: cu\in K\},$$

for $u\in S^{n-1}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. In the following, let $\mathcal{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$.

It is well known that in dual Brunn–Minkowski theory, dual Minkowski inequality and dual Aleksandrov–Fenchel inequality appear at the same time. So a natural question is raised: is there an Orlicz dual logarithmic Aleksandrov–Fenchel inequality relative to an Orlicz logarithmic dual Minkowski inequality? The main purpose of this article is to answer the above questions perfectly and obtain an Orlicz dual log-Aleksandrov–Fenchel inequality by introducing two new concepts of mixed dual volume measure and Orlicz multiple dual mixed volume measure, and using the Orlicz dual Aleksandrov–Fenchel inequality for the Orlicz multiple dual mixed volume. The dual logarithmic Aleksandrov–Fenchel inequality is also derived here. Moreover, the Orlicz dual log-Aleksandrov–Fenchel inequality in special cases yields the classical dual Aleksandrov–Fenchel inequality and some new logarithmic Minkowski type inequalities. Our main result is given in the following inequality.

Orlicz dual of log-Aleksandrov–Fenchel inequality *Let $\phi:(0,\infty)\to(0,\infty)$ be a convex and decreasing function such that $\lim_{t\to\infty}\phi(t)=0$ and $\lim_{t\to 0}\phi(t)=\infty$. If $L_1,K_1,\ldots,K_n\in\mathcal{S}^n$ and $1\leq r\leq n$, then*

$$\int_{S^{n-1}}\ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right)d\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)$$
$$\geq\ln\left(\phi\left(\frac{\prod_{i=1}^{r}\widetilde{V}(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}}{\widetilde{V}(L_1,K_2,\cdots,K_n)}\right)\right). \tag{1.3}$$

If φ is strictly convex, inequality holds if and only if $L_1,K_1,\ldots,K_r$ are all dilations of each other. Here, $d\widetilde{V}_\phi(L_1,K_1\cdots,K_n)$ which we call Orlicz multiple dual mixed volume probability measure of star bodies $L_1,K_1,\ldots,K_n$, defined by

$$d\widetilde{V}_\phi(L_1,K_1\cdots,K_n)=\frac{1}{n\widetilde{V}_\phi(L_1,K_1\cdots,K_n)}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u), \tag{1.4}$$

and $\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)$ is the Orlicz multiple dual mixed volume of star bodies $L_1,K_1,\ldots,K_n$, defined by [22]

$$\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u). \tag{1.5}$$

On the other hand, when $\phi(t)=1/t$, (1.3) becomes the following dual logarithmic Aleksandrov–Fenchel inequality established in [20].

The dual logarithmic Aleksandrov–Fenchel inequality *If $L_1,K_1,\ldots,K_n\in\mathcal{S}^n$ and $1\leq r\leq n$, then*

$$\int_{S^{n-1}}\ln\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)d\widetilde{V}_{-1}(L_1,K_1,\ldots,K_n)\leq\ln\left(\frac{\prod_{i=1}^{r}\widetilde{V}(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}}{\widetilde{V}(L_1,K_2,\cdots,K_n)}\right). \tag{1.6}$$

If φ is strictly convex, inequality holds if and only if $L_1,K_1,\ldots,K_r$ are all dilations of each other, where $d\widetilde{V}_{-1}(L_1,K_1\cdots,K_n)$ is as in (1.4).

When $L_1 = K_1$, inequality (1.6) becomes the classical dual Aleksandrov–Fenchel inequality as follows: If $K_1, \cdots, K_n \in \mathcal{S}^n$ and $1 \leq r \leq n$, then

$$\widetilde{V}(K_1, \cdots, K_n) \leq \prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{1}{r}},$$

with equality if and only if $K_1, \ldots, K_r$ are all dilations of each other (see [8]).

Moreover, as a special case of (1.3), the Orlicz dual logarithmic Minkowski inequality has been established by Zhao [23].

2. Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. For a compact set $K \subset \mathbb{R}^n$, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. The unit ball in $\mathbb{R}^n$ and its surface are denoted by B and S^{n-1}, respectively. Let $\mathcal{K}^n$ denote the class of nonempty compact convex subsets containing the origin in their interiors in $\mathbb{R}^n$. Let $\phi : (0, \infty) \to (0, \infty)$ be a convex and decreasing function such that $\lim_{t\to\infty} \phi(t) = 0$ and $\lim_{t\to 0} \phi(t) = \infty$ and let $\mathcal{C}$ denote the class of the convex and decreasing functions ϕ. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K, \cdot) : S^{n-1} \to [0, \infty)$, defined by

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$. Two star bodies K and L are dilates if $\rho(K, u)/\rho(L, u)$ is independent of $u \in S^{n-1}$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \mathcal{S}^n$, then (see e.g. [14])

$$\tilde{\delta}(K, L) = \|\rho(K, u) - \rho(L, u)\|_\infty.$$

2.1. L_p-dual mixed volume

The dual mixed volume $\widetilde{V}_{-1}(K, L)$ of star bodies K and L is defined by ([10])

$$\widetilde{V}_{-1}(K, L) = \lim_{\varepsilon\to 0^+} \frac{V(K) - V(K \widehat{+} \varepsilon \cdot L)}{\varepsilon}, \tag{2.1}$$

where $\widehat{+}$ is the harmonic addition. The following is a integral representation for the dual mixed volume $\widetilde{V}_{-1}(K, L)$:

$$\widetilde{V}_{-1}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+1} \rho(L, u)^{-1} dS(u). \tag{2.2}$$

The dual Minkowski inequality for the dual mixed volume states that

$$\widetilde{V}_{-1}(K, L)^n \geq V(K)^{n+1} V(L)^{-1},$$

with equality if and only if K and L are dilates. (see [9])

The dual Brunn–Minkowski inequality for the harmonic addition states that

$$V(K \widehat{+} L)^{-1/n} \geq V(K)^{-1/n} + V(L)^{-1/n},$$

with equality if and only if K and L are dilates (This inequality is due to Firey [9]).

The L_p dual mixed volume $\widetilde{V}_{-p}(K, L)$ of K and L is defined by [10]

$$\widetilde{V}_{-p}(K, L) = -\frac{p}{n} \lim_{\varepsilon\to 0^+} \frac{V(K \widehat{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon}, \tag{2.3}$$

where $K, L \in \mathcal{S}^n$ and $p \geq 1$.

The following is an integral representation for the L_p dual mixed volume: For $K, L \in \mathcal{S}^n$ and $p \geq 1$,

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(L,u)^{-p}dS(u). \quad (2.4)$$

L_p-dual Minkowski and Brunn–Minkowski inequalities were established by Lutwak [10]: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K,L)^n \geq V(K)^{n+p}V(L)^{-p},$$

with equality if and only if K and L are dilates, and

$$V(K \widehat{+}_p L)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n},$$

with equality if and only if K and L are dilates.

2.2. Mixed p-harmonic quermassintegral

In 1996, the L_p-harmonic radial addition for star bodies was defined by Lutwak [10]: If K, L are star bodies, for $p \geq 1$, the L_p-harmonic radial addition defined by

$$\rho(K \widehat{+}_p L, x)^{-p} = \rho(K,x)^{-p} + \rho(L,x)^{-p}, \quad (2.5)$$

for $x \in \mathbb{R}^n$. For convex bodies, L_p-harmonic addition was first investigated by Firey [4]. The operations of the L_p-radial addition, L_p-harmonic radial addition and the L_p-dual Minkowski, Brunn-Minkwski inequalities are fundamental notions and inequalities from the L_p-dual Brunn–Minkowski theory.

From (2.5), it is easy to see that if $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$-\frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K \widehat{+}_p \varepsilon \cdot L) - \widetilde{W}_i(L)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K.u)^{n-i+p}\rho(L.u)^{-p}dS(u). \quad (2.6)$$

Let $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, the mixed p-harmonic quermassintegral of star K and L, denoted by $\widetilde{W}_{-p,i}(K,L)$, defined by (see [17])

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i+p}\rho(L,u)^{-p}dS(u). \quad (2.7)$$

Obviously, when $K = L$, the p-harmonic quermassintegral $\widetilde{W}_{-p,i}(K,L)$ becomes the dual quermassintegral $\widetilde{W}_i(K)$. The Minkowski and Brunn–Minkowski inequalities for the mixed p-harmonic quermassintegral are following (see [17]): If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \geq \widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}, \quad (2.8)$$

with equality if and only if K and L are dilates. If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$\widetilde{W}_i(K \widehat{+}_p L)^{-p/(n-i)} \geq \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)}, \quad (2.9)$$

with equality if and only if K and L are dilates.

2.3. Orlicz multiple dual mixed volumes

The Orlicz multiple mixed volume was introduced as follows: For $\phi \in \mathcal{C}$, the Orlicz multiple dual mixed volume of star bodies $K_1, \cdots, K_n, L_n$, denoted by [22]

$$\widetilde{V}_\phi(L_1, K_1, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u). \quad (2.10)$$

Putting $L_1 = K_1$ in (2.10), the Orlicz multiple dual mixed volume $\widetilde{V}_\phi(L_1, K_1, \cdots, K_n)$ becomes the usual dual mixed volume $\widetilde{V}(K_1, \cdots, K_n)$. Putting $K_1 = L$ and $L_1 = K_2 = \cdots = K_n = K$ in (2.10), $\widetilde{V}_\phi(L_1, K_1, \cdots, K_n)$ becomes the Orlicz dual mixed volume $\widetilde{V}_\phi(K,L)$. Putting $K_1 = L$ and $L_1 = K_2 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = B$ in (3.1), $\widetilde{V}_\phi(L_1, K_1, \cdots, K_n)$ becomes i-th Orlicz dual mixed quermassintegral

$\widetilde{W}_{\phi,i}(K,L)$. When $\phi(t)=t^{-p}$ and $p\geq 1$, $\widetilde{W}_{\phi,i}(K,L)$ becomes the harmonic dual mixed p-quermassintegral, $\widetilde{W}_{-p,i}(K,L)$.

Orlicz dual Aleksandrov–Fenchel inequality for Orlicz multiple dual mixed volumes: *If $L_1,K_1,\cdots,K_n\in\mathcal{S}^n$, $\phi\in\mathcal{C}$ and $1\leq r\leq n$, then*

$$\widetilde{V}_\phi(L_1,K_1,K_2,\cdots,K_n)\geq\widetilde{V}(L_1,K_2,\cdots,K_n)\cdot\phi\left(\frac{\prod_{i=1}^r\widetilde{V}(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}}{\widetilde{V}(L_1,K_2\ldots,K_n)}\right). \tag{2.11}$$

If ϕ is strictly convex, equality holds if and only if $L_1,K_1,\ldots,K_r$ are all dilates of each other.

When $\phi(t)=t^{-p}$, $p=1$ and $K_1=L_1$, the dual Orlicz-Aleksandrov–Fenchel inequality becomes Lutwak's dual Aleksandrov–Fenchel inequality. If $K_1,\cdots,K_n\in\mathcal{S}^n$ and $1\leq r\leq n$, then

$$\widetilde{V}(K_1,\cdots,K_n)\leq\prod_{i=1}^r\widetilde{V}(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}, \tag{2.12}$$

with equality if and only if $K_1,\ldots,K_r$ are all dilates of each other. In fact inequality (2.11) yields also the following result. If $K,L\in\mathcal{S}^n$, $0\leq i<n$ and $\phi\in\mathcal{C}$, then

$$\widetilde{W}_{\phi,i}(K,L)\geq\widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{1/(n-i)}\right). \tag{2.13}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. Here $\widetilde{W}_i(K)$ is the usual dual quermassintegral of K, and $\widetilde{W}_{\phi,i}(K,L)$ is the Orlicz dual mixed quermassintegral of K and L, defined by

$$\widetilde{W}_{\phi,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i}dS(u). \tag{2.14}$$

3. Orlicz dual logarithmic Aleksandrov–Fenchel inequality

In the section, in order to derive the Orlicz dual log-Aleksandrov–Fenchel inequality, we need to define some new dual mixed volume measures.

If $L_1,K_2,\ldots,K_n\in\mathcal{S}^n$, the dual mixed volume of star bodies $L_1,K_2,\ldots,K_n$, $\widetilde{V}(L_1,K_2,\ldots,K_n)$ defined by

$$\widetilde{V}(L_1,K_2,\ldots,K_n)=\frac{1}{n}\int_{S^{n-1}}\rho(L_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u). \tag{3.1}$$

From (3.1), we introduce the dual mixed volume measure of star bodies $L_1,K_2,\ldots,K_n$.

Definition 3.1. (Dual mixed volume measure) For $L_1,K_2,\cdots,K_n\in\mathcal{S}^n$, the dual mixed volume measure of $L_1,K_2,\ldots,K_n$, denoted by $d\tilde{v}(L_1,K_2,\ldots,K_n)$, defined by

$$d\tilde{v}(L_1,K_2,\ldots,K_n)=\frac{1}{n}\rho(L_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u). \tag{3.2}$$

From definition 3.1, we get the following dual mixed volume probability measure.

$$d\widetilde{V}(L_1,K_2,\ldots,K_n)=\frac{1}{\widetilde{V}(L_1,K_2,\ldots,K_n)}d\tilde{v}(L_1,K_2,\ldots,K_n). \tag{3.3}$$

For $\phi\in\mathcal{C}$, Orlicz multiple dual mixed volume of $L_1,K_1\cdots,K_n$, denoted by $\widetilde{V}_\phi(L_1,K_1,\cdots,K_n)$, defined by

$$\widetilde{V}_\phi(L_1,K_1,\cdots,K_n)=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u). \tag{3.4}$$

From (3.4), we introduce Orlicz multiple dual mixed volume measure of star bodies $L_1,K_1\cdots,K_n$ as follows.

Definition 3.2. (Orlicz multiple dual mixed volume measure) For $L_1, K_1, \cdots, K_n \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, the Orlicz dual mixed volume measure of $L_1, K_1 \ldots, K_n$, denoted by $d\tilde{v}_\phi(L_1, K_1 \cdots, K_n, L_n)$, defined by

$$d\tilde{v}_\phi(L_1, K_1 \cdots, K_n) = \frac{1}{n}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u). \tag{3.5}$$

From definition 3.2, Orlicz multiple dual mixed volume probability measure is defined by

$$d\widetilde{V}_\phi(L_1, K_1 \cdots, K_n) = \frac{1}{\widetilde{V}_\phi(L_1, K_1 \cdots, K_n)} d\tilde{v}_\phi(L_1, K_1 \cdots, K_n). \tag{3.6}$$

Theorem 3.3 (Orlicz dual of log-Aleksandrov–Fenchel inequality)**.** *If $L_1, K_1, \ldots, K_n \in \mathcal{S}^n$, $1 \leq r \leq n$ and $\phi \in \mathcal{C}$, then*

$$\int_{S^{n-1}} \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\widetilde{V}_\phi(L_1, K_1, \ldots, K_n) \geq \ln\left(\frac{\widetilde{V}_\phi(L_1, K_1, \ldots, K_n)}{\widetilde{V}(L_1, K_2, \ldots, K_n)}\right)$$
$$\geq \ln\left(\phi\left(\frac{\prod_{i=1}^r \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{1}{r}}}{\widetilde{V}(L_1, K_2, \cdots, K_n)}\right)\right). \tag{3.7}$$

The left inequality of (3.7) with equality if and only if L_1 and K_1 are dilates, and the right inequality if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other, if φ is strictly convex.

Proof. From (3.2) ang (3.5), we have

$$\int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) \ln\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) d\tilde{v}(L_1, K_2, \ldots, L_n)$$
$$= \int_{S_{n-1}} \ln\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) d\tilde{v}_\phi(L_1, K_1, \ldots, K_n). \tag{3.8}$$

From (3.4) and in view of the Lebesgues dominated convergence theorem, we obtain

$$\int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} d\tilde{v}(L_1, K_2 \ldots, K_n) \to \widetilde{V}_\phi(L_1, K_1, \ldots, K_n)$$

as $q \to \infty$, and

$$\int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} \ln\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) d\tilde{v}(L_1, K_2, \ldots, K_n)$$
$$\to \int_{S_{n-1}} \ln\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) d\tilde{v}_\phi(L_1, K_1, \ldots, K_n)$$

as $q \to \infty$.

Consider the function $g_{L_1,K_1,\ldots,K_n} : [1, \infty] \to \mathbb{R}$, defined by

$$g_{L_1,K_1,\ldots,K_n}(q) = \frac{1}{\widetilde{V}_\phi(L_1, K_1, \ldots, K_n)} \int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} d\tilde{v}(L_1, K_2, \ldots, K_n). \tag{3.9}$$

By calculating the derivative and limit of this function, we have

$$\frac{dg_{L_1,K_1,\ldots,K_n}(q)}{dq} = \frac{n}{(q+n)^2} \cdot \frac{1}{\widetilde{V}_\phi(L_1, K_1, \ldots, K_n)}$$
$$\times \int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\tilde{v}(L_1, K_2, \ldots, K_n). \tag{3.10}$$

and

$$\lim_{q\to\infty} g_{L_1,K_1,\ldots,K_n}(q) = 1. \tag{3.11}$$

From (3.9), (3.10) and (3.11), and by using L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{q\to\infty} \ln\left(g_{L_1,K_1,\ldots,K_n}(q)\right)^{q+n} &= -\frac{n}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)} \\
&\quad \times \lim_{q\to\infty} \frac{\int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\tilde{v}(L_1,K_2,\ldots,K_n)}{g_{L_1,K_1,\ldots,K_n}(q)} \\
&= -\frac{n}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)} \\
&\quad \times \int_{S_{n-1}} \phi\left(\tfrac{\rho(K_1,u)}{\rho(L_1,u)}\right) \ln\left(\phi\left(\tfrac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\tilde{v}(L_1,K_2,\ldots,K_n).
\end{aligned}$$

Hence

$$\begin{aligned}
&\exp\left(-\frac{n}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)} \int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\tilde{v}(L_1,K_2,\ldots,K_n)\right) \\
&\quad = \lim_{q\to\infty} \left(g_{L_1,K_1,\ldots,K_n}\right)^{q+n} \\
&\quad = \lim_{q\to\infty} \left(\frac{1}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)} \int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{q+n}.
\end{aligned} \tag{3.12}$$

On the other hand, from Hölder's inequality

$$\begin{aligned}
&\left(\int_{S^{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{(q+n)/q} \left(\int_{S^{n-1}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{-n/q} \\
&\quad \le \int_{S^{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) d\tilde{v}(L_1,K_2,\ldots,K_n) \\
&\quad = \widetilde{V}_\phi(L_1,K_1,\ldots,K_n).
\end{aligned} \tag{3.13}$$

From the equality condition of Hölder's inequality, it follows that the equality in (3.13) holds if and only if $\rho(K_1,u)$ and $\rho(L_1,u)$ are proportional. This yields that equality in (3.13) holds if and only if K_1 and L_1 are dilates. Namely

$$\begin{aligned}
&\left(\frac{1}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)} \int_{S^{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)^{\frac{q}{q+n}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{q+n} \\
&\qquad \le \left(\frac{\widetilde{V}(L_1,K_2,\ldots,K_n)}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)}\right)^n,
\end{aligned}$$

with equality if and only if K_1 and L_1 are dilates. Hence

$$\begin{aligned}
&\exp\left(-\frac{n}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)} \int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\tilde{v}(L_1,K_2,\ldots,K_n)\right) \\
&\qquad \le \left(\frac{\widetilde{V}(L_1,K_2,\ldots,K_n)}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)}\right)^n,
\end{aligned}$$

with equality if and only if K_1 and L_1 are dilates. That is

$$\begin{aligned}
&\frac{1}{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)} \int_{S_{n-1}} \phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right) \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\tilde{v}(L_1,K_2,\ldots,K_n) \\
&\qquad \ge \ln\left(\frac{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)}{\widetilde{V}(L_1,K_2,\ldots,K_n)}\right),
\end{aligned}$$

with equality if and only if K_1 and L_1 are dilates. Therefore

$$\int_{S^{n-1}} \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\widetilde{V}_\phi(L_1,K_1,\ldots,K_n) \geq \ln\left(\frac{\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)}{\widetilde{V}(L_1,K_2,\ldots,K_n)}\right), \tag{3.14}$$

with equality if and only if K_1 and L_1 are dilates. The completes proof of the left inequality of (3.7).

Further, by using the Orlicz dual Aleksandrov–Fenchel inequality (2.11), we obtain

$$\int_{S^{n-1}} \ln\left(\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\right) d\widetilde{V}_\phi(L_1,K_1,\ldots,K_n)$$
$$\geq \ln\left(\phi\left(\frac{\prod_{i=1}^r \widetilde{V}(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{1}{r}}}{\widetilde{V}(L_1,K_2,\cdots,K_n)}\right)\right).$$

If φ is strictly convex, the equality holds if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

This completes the proof. □

Corollary 3.4. *If K and L are star bodies in $\mathbb{R}^n$, and $0 \leq i < n$ and $\phi \in \mathcal{C}$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right) d\widetilde{W}_{\phi,i}(L,K) \geq \ln\left(\frac{\widetilde{W}_{\phi,i}(L,K)}{\widetilde{W}_i(L)}\right) \geq \frac{1}{n-i}\ln\left(\phi\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{1/(n-i)}\right). \tag{3.15}$$

if φ is strictly convex, equality holds if and only if K and L are dilates, and where

$$d\tilde{w}_{\phi,i}(L,K) = d\tilde{v}_\phi(L,K,\underbrace{L,\ldots,L}_{n-1-i},\underbrace{B,\ldots,B}_{i}) = \frac{1}{n}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\rho(L,u)^{n-i}dS(u), \tag{3.16}$$

and

$$d\widetilde{W}_{\phi,i}(L,K) = \frac{1}{\widetilde{W}_{\phi,i}(L,K)} d\tilde{w}_{\phi,i}(L,K), \tag{3.17}$$

denotes its normalization.

Proof. This follows immediately from Theorem 3.3. □

Corollary 3.5. *If K and L are star bodies in $\mathbb{R}^n$, and $0 \leq i < n$ and $p \geq 1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right) d\widetilde{W}_{-p,i}(L,K) \geq \frac{1}{p}\ln\left(\frac{\widetilde{W}_{-p,i}(L,K)}{\widetilde{W}_i(L)}\right) \geq \frac{1}{n-i}\ln\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right). \tag{3.18}$$

each equality holds if and only if K and L are dilates, and where

$$d\tilde{w}_{-p,i}(L,K) = d\tilde{v}_{-p}(L,K,\underbrace{L,\ldots,L}_{n-1-i},\underbrace{B,\ldots,B}_{i}) = \frac{1}{n}\rho(K,u)^{-p}\rho(L,u)^{n+p}dS(u), \tag{3.19}$$

and

$$d\widetilde{W}_{-p,i}(L,K) = \frac{1}{\widetilde{W}_{-p,i}(L,K)} d\tilde{w}_{-p,i}(L,K), \tag{3.20}$$

denotes its normalization.

Proof. This follows immediately from Corollary 3.4 with $\phi(t) = t^{-p}$ and $p \geq 1$. □

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References

[1] K. Chou and X. Wang, *A logarithmic Gauss curvature flow and the Minkowski problem*, Ann. Inst. Henri Poincaré, Analyse non linéaire, **17** (6), 733-751, 2000.

[2] A. Colesanti and P. Cuoghi, *The Brunn–Minkowski inequality for the n-dimensional logarithmic capacity of convex bodies*, Potential Math. **22**, 289-304, 2005.

[3] M. Fathi and B. Nelson, *Free Stein kernels and an improvement of the free logarithmic Sobolev inequality* , Adv. Math. **317**, 193-223, 2017.

[4] W. J. Firey, *Polar means of convex bodies and a dual to the Brunn–Minkowski theorem,* Canad. J. Math. **13**, 444-453, 1961.

[5] M. Henk and H. Pollehn, *On the log-Minkowski inequality for simplices and parallelepipeds*, Acta Math. Hungarica, **155**, 141-157, 2018.

[6] S. Hou and J. Xiao, *A mixed volumetry for the anisotropic logarithmic potential*, J. Geom Anal. **28**, 2018-2049, 2018.

[7] C. Li and W. Wang, *Log-Minkowski inequalities for the L_p-mixed quermassintegrals*, J. Inequal. Appl., **2019** (1), 2019.

[8] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58**, 531-538, 1975.

[9] E. Lutwak, *Intersection bodies and dual mixed volumes,* Adv. Math. **71** 232-261, 1988.

[10] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. **60**, 365-391, 1990.

[11] S. Lv, *The φ-Brunn–Minkowski inequality*, Acta Math. Hungarica, **156**, 226-239, 2018.

[12] L. Ma, *A new proof of the log-Brunn–Minkowski inequality*, Geom. Dedicata, **177**, 75-82, 2015.

[13] C. Saroglou, *Remarks on the conjectured log-Brunn–Minkowski inequality* , Geom. Dedicata, **177**, 353-365, 2015.

[14] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Cambridge University Press, 1993.

[15] A. Stancu, *The logarithmic Minkowski inequality for non-symmetric convex bodies*, Adv. Appl. Math. **73**, 43-58, 2016.

[16] W. Wang and M. Feng, *The log-Minkowski inequalities for quermassintegrals*, J. Math. Inequal. **11**, 983995, 2017.

[17] W. Wang and G. Leng, *L_p-dual mixed quermassintegrals*, Indian J. Pure Appl. Math. **36**, 177-188, 2005.

[18] W. Wang and L. Liu, *The dual log-Brunn–Minkowski inequalities*, Taiwanese J. Math. **20**, 909-919, 2016.

[19] X. Wang, W. Xu and J. Zhou, *Some logarithmic Minkowski inequalities for nonsymmetric convex bodies*, Sci. China, **60**, 1857-1872, 2017.

[20] C.-J. Zhao, *The dual logarithmic Aleksandrov-Fenchel inequality*, Balkan J. Geom. Appl. **25**, 157-169, 2020.

[21] C.-J. Zhao, *The log-Aleksandrov-Fenchel inequality*, Mediterr J. Math. **17** (3), 1-14, 2020.

[22] C.-J. Zhao, *The dual Orlicz-Aleksandrov-Fenchel inequality*, Mathematics, **8** 2005, 2020.

[23] C.-J. Zhao, *Orlicz dual logarithmic Minkowski inequality*, Math Inequal. Appl. **24** (4), 1031-1040, 2021.

[24] G. Zhu, *The logarithmic Minkowski problem for polytopes*, Adv. Math. **262**, 909-931, 2014.