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On The Atoms and the Anti-Atoms of the Lattice of Topologies

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Abstract: In this paper, some properties of the atoms in the lattice of a family $\tau(X)$ of all topologies on a finite set X are proved. In addition, a geometric method is given to find the number of the atomic and the anti-atomic topologies, and it is observed that the number of elements of $\tau(X)$ satisfies the following expression;

$$n(2^n-2) < |\tau(X)| < (2^n-2)^{(n-1)}$$

where n = |X| (cardinality of X).

Key Words: The lattice of topologies, atoms, anti-atoms.

Topolojiler Latisinin Atom ve Anti-Atomları Üzerine

Özet: Bu çalışmada, bir sonlu X kümesi üzerine kurulan bütün topolojilerin $\tau(X)$ ailesinin latisindeki atomların bazı özellikleri ispat edildi. Ayrıca, topolojilerin atom ve anti-atom sayılarını bullmak için bir geometric yöntem verildi ve $\tau(X)$ ailesinin elemanlarının sayısı, $|\tau(X)|$ için

$$n(2^n-2) < |\tau(X)| < (2^n-2)^{(n-1)}$$

ifadesinin sağlandığı gözlemlendi.

Anahtar Kelimeler: Topolojilerin latisi, atomlar, anti-atomlar

1. Introduction

We recall that the definition of a topology on a set X is a collection τ of subsets of X (called 'open') such that

$\phi, X \in \tau$

$A, B \in \tau$ implies $A \cap B \in \tau$

$A_i \in \tau$, $i \in I$ (an arbitrary index set) implies $\cup A_i \in \tau$.

Thus the open subsets in a topology always include the empty ϕ and the set X itself and are closed under the formation of finite intersections and arbitrary unions. The family of all topologies would consist of all such collections τ on all possible sets X. However, this family is enormous – so big in fact that it is not a set. If it were, it could be equipped with a topology (in many different ways) and would therefore have to be member of itself, which is not permitted in set theory. We will avoid this logical problem by restricting our attention to the family $\tau(X)$ of all topologies on a fixed set X, which is a proper set.

It is appropriate to recall that topologies τ_1 and τ_2 on a set X are said to homeomorphic if there is a bijection $i : (X, \tau_1) \rightarrow (X, \tau_2)$ which is continuous and whose inverse is also continuous

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((X, τ) denotes the set X is equipped with the topology τ). This bicontinuity is equivalent to saying that the map i sets up a one-to-one equivalence between these subsets of X that are open in the topology τ_1 and those subsets of X that are open in the topology τ_2.

2. Lattice Structure

2.1. The Lattice $\tau(X)$

Let us recall that a lattice ([1],[2]) is a partially-ordered set in which each pair of elements $\{a,b\}$ has a least upper bound, denoted $a \vee b$, a greatest lower bound, denoted $a \wedge b$. These Lattice operations are idempotent, commutative and associative that is, they satisfy the relations

Idempotency	:	$a \vee a = a$	$a \wedge a = a$
Commutativety	:	$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
Associativity	:	$(a \vee b) \vee c = a \vee (b \vee c)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Each of the operations $\wedge$ and $\vee$ makes the lattice into a semi-group (there are no inverses).

A natural partial ordering is associated with any lattice via the definition

$$a \leq b \quad \text{if and only if} \quad a \wedge b = a \qquad (1)$$

and the first step in placing a lattice structure on $\tau(X)$ is to note the well known partial ordering of topologies defined by

$$\tau_1 \leq \tau_2 \quad \text{if} \quad \tau_1 \subset \tau_2 \qquad (2)$$

where the set inclusion sign $\subset$ includes the possibility of equality; the notation $\tau_1 < \tau_2$ will be employed whenever it is important to emphasize that $\tau_1 \neq \tau_2$. If τ_1 and τ_2 are a pair of topologies satisfying (2) (so that every τ_1-open set is also τ_2-open) then τ_1 is said to be weaker / coarser than τ_2, and τ_2 is stronger / finer than τ_1. The weakest / coarsest topology is $\{\phi, X\}$ and the strongest / finest topology is P(X) –the set of all subsets of X. These extreme topologies will sometimes be denoted by 0 and 1 respectively.

Lattice operations on $\tau(X)$ can be defined by

$\tau_1 \wedge \tau_2 := \tau_1 \cap \tau_2 = \{ A \subset X : A \text{ is open in both } \tau_1 \text{ and } \tau_2 \}$

$\tau_1 \vee \tau_2 :=$ Coarsest topology containing $\tau_1 \cup \tau_2 = \{ A \subset X : A \text{ is open in } \tau_1 \text{ or } \tau_2 \}$

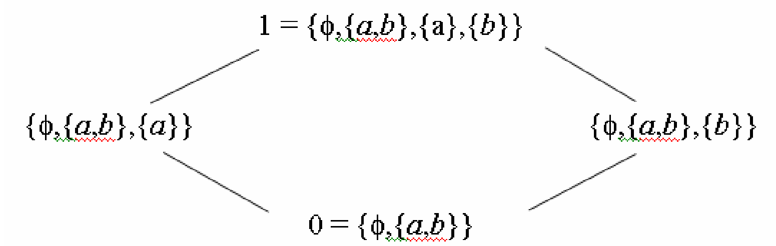
It is a standard exercise to show that these operations are compatible with the partial ordering in the sense that

(i). $\tau_1 \wedge \tau_2$ is the finest topology that is coarser than both τ_1 and τ_2,

(ii). $\tau_1 \vee \tau_2$ is the coarsest topology that is finer than both τ_1 and τ_2.

It is instructive to study a few simple examples where X is a finite set. The simplest is when the single topology $\{\phi,\{a\}\}$.

If X = $\{a,b\}$ there are four topologies arranged in the following lattice:

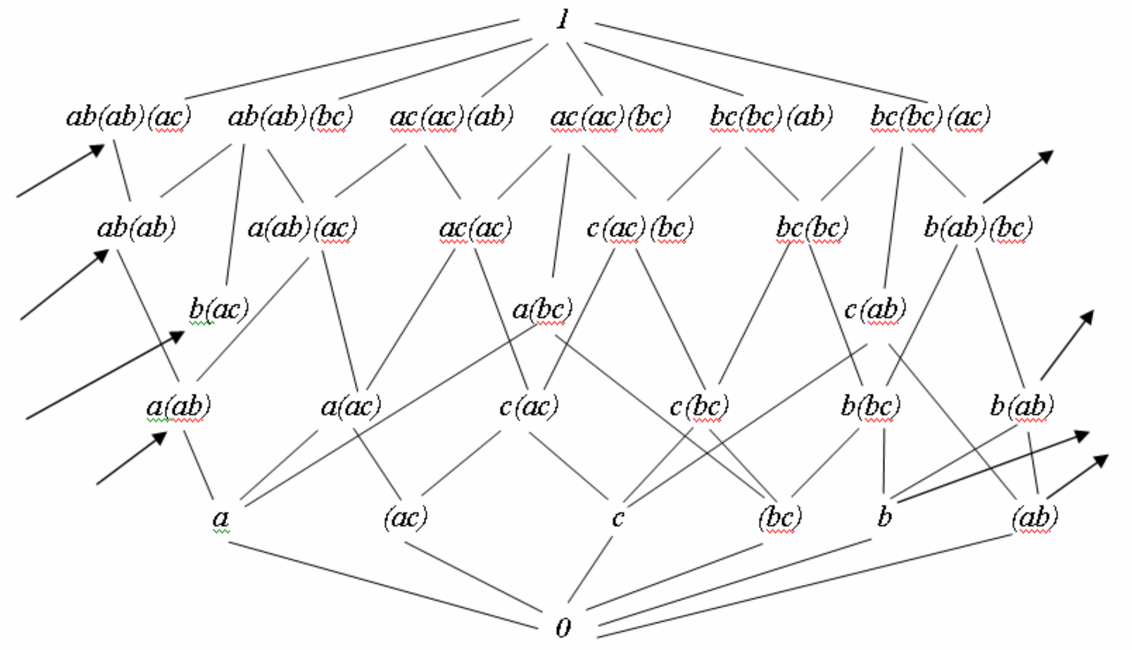


where a line drawn upwards from τ_1 to τ_2 means that;

(i). τ_1 is strictly coarser than τ_2,

(ii). there is no intermediate topology which is strictly finer than τ_1 and strictly coarser than τ_2.

The first really interesting example is when X is a set $\{a,b,c\}$ of cardinality three. The lattice diagram for this case is shown below using a notation which has been chosen for maximum topographical simplicity. For example, $ab(ab)(bc)$ means the topology whose open sets other than ϕ and X are the subsets $\{a\}$, $\{b\}$, $\{a,b\}$ and $\{b,c\}$.



The lattice of all topologies on X possesses interesting properties; a useful review article is [3]. Some examples are given below.

(i). $\tau(X)$ is complete. Hence the lattice contains the meets and joins of arbitrary sets of elements, not merely finite ones (which is all that is guaranteed by the algebraic axioms of a lattice).

(ii). $\tau(X)$ is complemented. Thus, given any topology τ, there exists some other topology τ' such that

$$\tau \wedge \tau' = 0 \qquad \text{and} \qquad \tau \vee \tau' = 1.$$

However, the complements in $\tau(X)$ are not unique. In fact, when X is infinite, each topology τ other than $\{\phi,X\}$ and P(X) has at least |X| different complements τ'.

2.2. The atoms of the lattice $\tau(X)$

2.2.1. Definition. An atom in the lattice $\tau(X)$ is a non zero topology τ_A for which

$$\tau_A \wedge \tau = \tau_A \qquad \text{or} \qquad \tau_A \wedge \tau = 0 \qquad \text{for each } \tau \in \tau(X).$$

For $A \subset X$, τ_A is an atom if $\tau \le \tau_A$ implies that $\tau = 0$ or $\tau = \tau_A$. This implies that the atoms are the elements immediately above the zero element in the lattice diagram. That is, for each $A \subset X$, $\tau_A > 0$ is no intermediate topology. Hence $\tau_A = \{\phi,X,A\}$ is an atom. In the example X = {*a,b,c*}, the atoms are the six topologies *a*, (*ac*), *c*, (*bc*), *b* and (*ab*).

Now we give some properties of the atoms.

2.2.2. Theorem. If τ is a non zero topology of a finite family $\tau(X)$, there exists an atom τ_A with $\tau_A \le \tau$.

Proof. If τ is an atom, take $\tau_A = \tau$. If not, then it follows from Definition 2.2.1 that there exist a non zero topology τ_1, different from τ with $\tau_1 \le \tau$. If τ_1 is not an atom, me continue in this way to obtain a sequence of distinct non zero topologies $\ldots \le \tau_3 \le \tau_2 \le \tau_1 \le \tau$, which, because $\tau(X)$ is finite, must terminate in an atom τ_A.

2.2.3. Theorem. If τ_A, τ_{B_1}, τ_{B_2}, τ_{B_3} ,...., τ_{B_r}, are atoms in a finite family $\tau(X)$, then $\tau_A \le (\tau_{B_1} \vee \tau_{B_2} \vee \ldots \vee \tau_{B_r})$ if and only if $\tau_A = \tau_{B_i}$, for some i with $1 \le i \le r$.

Proof. If $\tau_A \leq (\tau_{B_1} \vee \tau_{B_2} \vee ... \vee \tau_{B_r})$, then $\tau_A \wedge(\tau_{B_1} \vee \tau_{B_2} \vee ... \vee \tau_{B_r}) = \tau_A$. Thus $(\tau_A \tau_{B_1}) \vee ... \vee(\tau_A \wedge \tau_{B_r}) = \tau_A$. Since each τ_{B_i} is an atom, $(\tau_A \wedge \tau_{B_i}) = \tau_{B_i}$ or $(\tau_A \wedge \tau_{B_i}) = 0$ by Definition 2.2.1. Not all the topologies $\tau_A \wedge \tau_{B_i}$ can be 0, for this would imply $\tau_A = 0$. Hence there is some i with $1 \leq i \leq r$, for which $(\tau_A \wedge \tau_{B_i}) = \tau_{B_i}$. But τ_A is also an atom and so $\tau_A = (\tau_A \wedge \tau_{B_i}) = \tau_{B_i}$.

The implication the other way is straightforward.

2.2.4. Theorem. If $\tau_{A_1}, \tau_{A_2}, ..., \tau_{A_n}$ are all the atoms of a finite family $\tau(X)$, then $\tau = 0$ if and only if $\tau \wedge \tau_{A_i} = 0$ for all i such that $1 \leq i \leq n$.

Proof. Suppose $\tau \wedge \tau_{A_i} = 0$ for each i. If τ is non zero, it follow from Theorem 2.2.2 that there is an atom τ_{A_j} with $\tau_{A_j} \leq \tau$. Hence $\tau_{A_j} = \tau \wedge \tau_{A_j} = 0$, which is a contradiction, and so $\tau = 0$.

The converse implication is trivial.

2.2.5. Theorem. Each member τ of a finite family $\tau(X)$ can be written as a join of atoms

$$\tau = \tau_{A_\alpha} \vee \tau_{A_\beta} \vee ... \vee \tau_{A_\omega}.$$

Moreover, this expression is unique up to the order of the atoms.

Proof. Let $\tau_{A_\alpha}, \tau_{A_\beta}, ..., \tau_{A_\omega}$ be all the atoms less than or equal to τ in the partial order. It follow from the fact $\tau_1 \leq \tau_3$ and $\tau_2 \leq \tau_3$ implies $\tau_1 \vee \tau_2 \leq \tau_3$ that the join

$$\tau_A = \tau_{A_\alpha} \vee \tau_{A_\beta} \vee ... \vee \tau_{A_\omega} \leq \tau.$$

We will show that $\tau \wedge \tau'_A = 0$ which, by $\tau_1 \leq \tau'_2 = 0$, is equivalent to $\tau \leq \tau_A$. We have

$$\tau \wedge \tau'_A = \tau_{A_\alpha} \wedge \tau_{A_\beta} \wedge ... \wedge \tau_{A_\omega}$$

If τ_B is an atom in the join τ_A, say $\tau_B = \tau_{A_\alpha}$, it follows that $\tau \wedge \tau'_A \wedge \tau_B = 0$, since $\tau'_{A_\alpha} \wedge \tau_{A_\alpha} = 0$. If τ_B is an atom that is not in τ_A, then $\tau \wedge \tau'_A \wedge \tau_B = 0$ also, because $\tau \wedge \tau_B = 0$. Therefore, by Theorem 2.2.4, $\tau \wedge \tau'_A = 0$, which is equivalent to $\tau \leq \tau_A$. The antisymmetry of the partial order relation implies that $\tau = \tau_A$.

To show uniqueness, suppose that τ can be written as the join of two sets of atoms

$$\tau = \tau_{A_\alpha} \vee ... \vee \tau_{A_\omega} = \tau_{A_a} \vee ... \vee \tau_{A_z}.$$

Now, $\tau_{A_\alpha} \leq \tau$; thus, by Theorem 2.2.3, τ_{B_α} is equal to one of the atoms on the right-hand side, $\tau_{A_a}, ..., \tau_{A_z}$. Repeating this argument, we see that the two sets of atoms are the same, except possibly for their order.

2.2.6. Result. Every topology τ in the family $\tau(X)$ is uniquely determined by the atoms $\tau_A = \{\phi, X, A\}$ for each $A \subset X$ in the sense that

$$\tau = \vee\{\tau_A : \tau_A \leq \tau\}.$$

Note that $\tau_A \leq \tau$ is precisely equivalent to the statement that the subset A is open in the topology τ, that is, $A \in \tau$. Hence the lattice $\tau(X)$ is atomic.

The lattice $\tau(X)$ is also anti-atomic. That is, there exist topologies τ_A with the properties that

(i) $\tau_A < 1$ with no intermediate topologies,

(ii) every topology is uniquely determined by the anti-atoms that lie above it.

In the example X = {*a,b,c*}, the anti-atoms are the topologies *ab(ab)(bc), ab(ab)(ac), ac(ab)ac), ac(ab)(bc), bc(ab)(bc)* and *bc(ac)(bc*).

2.3. On the cardinality of τ(X) and the number of the atomic and the anti- atomic topologies

For finite X, the number of the atomic topologies is clearly only $2^{|X|}-2$, and the number of the atomic topologies is equal to the number of the anti-atomic topologies for |X| = 1-3. This is not when |X|>3. In general, the anti-atoms are topologies of the from

$$\tau_A = \{A \subset X : x \notin A \text{ or } A \in U\}$$

where U is any ultra-filter not equal to the principal ultra-filter of all subsets of X containing the point x∈X. For details see [3] or [4].

It is evidently of some interest to know how many topologies can be placed on any given X. Note first that |P(X)| = $2^{|X|}$ and |P(P(X))| = $2^{2^{|X|}}$. Now, each topology τ is member of P(X) and hence τ(X)⊂P(PX)). Thus |τ(X)| ≤ |P(P(X))| = $2^{2^{|X|}}$. For infinite X, the number of the atomic topologies is clearly only $2^{|X|}$ (the cardinality of the set P(X) of all subsets of X) but in [5], Fröhlich showed that the cardinality of the set of anti-atoms is $2^{2^{|X|}}$. Hence the cardinality of C is equal to its set- theoretic maximum $2^{2^{|X|}}$.

The size of τ(X) that have been computed for finite X are listed in Table 1 taken from [5].

|X|	|τ(X)|
1	1
2	4
3	29
4	355
5	6942
6	209527
7	9535241

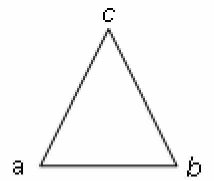
Table 1

The value of |τ(X)| for an arbitrary finite set X does not seem to have been worked out explicitly although it is known that if n = |X|

$$2^n \le |\tau(X)| \le 2^{n(n-1)} \qquad (3)$$

Now we give a method to establish the atomic and the anti-atomic topologies of the lattice τ(X) of an arbitrary finite set X.

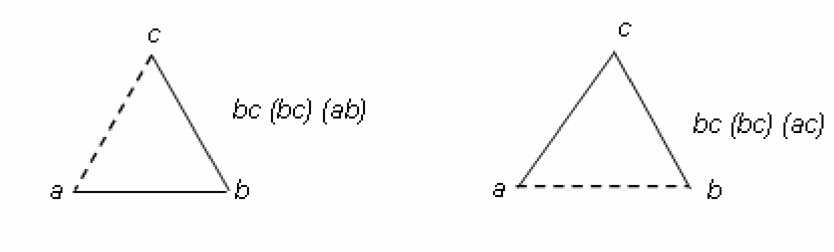
Let X be a set {*a,b,c*} of cardinality three. Each element of X can be written as a vertex of the following triangle:



We know that the atomic topologies are *a, b, c, (ab), (ac)*, and *(bc)*. Note that *a, b,* and *c* are the vertices of the triangle and *(ab), (ac)*, and *(bc)* are the edges of the triangle. Hence the atomic topologies of the lattice τ(X) of the set X = {*a,b,c*} can be represented as the vertices and the edges of a triangle.

Now, we find the complements of the atomic topologies. The complements of the atom τ_A are some topologies τ'_A which are satisfied conditions $\tau_A \wedge \tau'_A = 0$ and $\tau_A \vee \tau'_A = 1$. For the atom a, the complements are *bc(bc)(ac)* and *bc(bc)(ab)*.

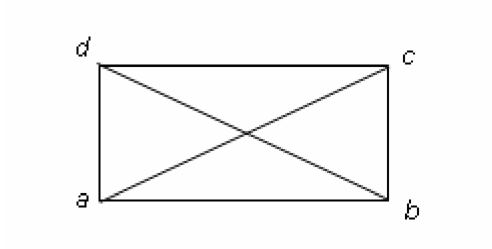
Thus the complements of the atom *a* can be written as a join of the vertices (except *a*) and the edges of a triangle in sense bellow:



Similarly, the complements of the atom *b* are *ac(ac)(ab)* and *ac(ac)(bc)*, and the complements of the atom *c* are *ab(ab)(ac)* and *ab(ab)(bc)*. Note that the complements of each atom *(ab), (ac)* and *(bc)* are two of the above six complements.

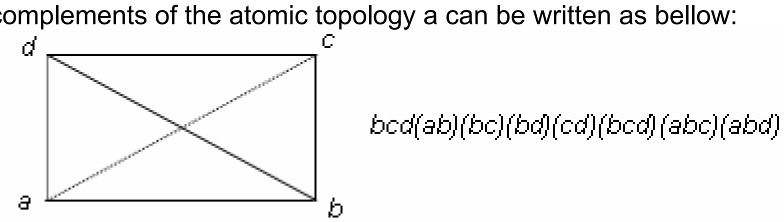
Then the complements of the atoms *a, b, c, (ab), (ac),* and *(bc)* are the topologies *ab(ab)(ac), ab(ab)(ac), ac(ac)(ab), ac(ac)(bc), bc(bc)(bc),* and *bc(bc)(ac).* Note that these topologies are the anti-atoms of the lattice τ(X). Thus, it is needed to find the complements of the atoms *a, b* and *c* (vertices of a triangle) to obtain the anti-atomic topologies of the lattice τ(X).

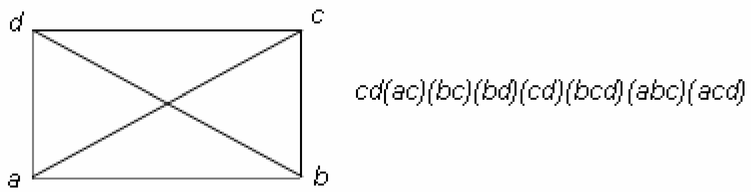
Let X be a set {*a,b,c,d*} of cardinality four. We consider that X is a rectangle with the vertices *a, b, c* and *d* as bellow:

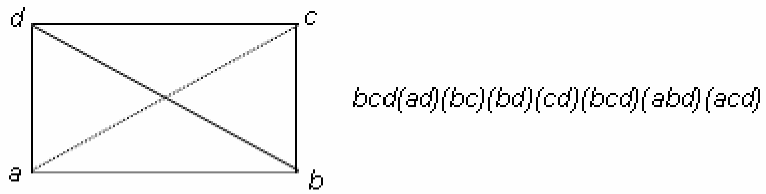


In this rectangle, there are four vertices, six edges and four triangle. That is, there are fourteen atomic topologies of the lattice τ(X). These atomic topologies are *a, b, c, d, (ab)*, *(ac), (ad), (bc), (bd), (cd), (abc), (abd), (acd)* and *(bcd).*

The complements of the atomic topology a can be written as bellow:







Then there are three complements of the atom *a*. Similarly, the complements of the atoms *b, c,* and *d* can be also found. Thus, there are twelve anti-atomic topologies of the lattice τ(X).

If we continue in this way, then we obtain the following theorem.

2.3.1. Theorem. Let X be a finite set {*a,b,...,z*} of cardinality n. We consider that X is a polygon with vertices n. Then,

(i). If A is a set of all the atomic topologies of the lattice $\tau(X)$, the number of the atomic topologies is

$|A|$ = |vertices| + |edges| + |triangles| + ... + |polygons with vertices n-1| = 2^n-2,

(ii). The number of the anti-atomic topologies is $|A'| = n(n-1)$ where $n = |X|$.

Proof. It follows by induction over n.

2.3.2. Theorem. For the family $\tau(X)$ of all topologies on a finite set X of cardinality $n \geq 3$, $n.|A| < |\tau(X)| < |A|^{(n-1)}$ that is

$$n.(2^n-2) < |\tau(X)| < (2^n-2)^{(n-1)}.$$

Proof. By a simple induction, for n = 3, we obtain that $18 < |\tau(X)| < 36$ or $18 < 29 < 36$ (see Table 1). For n = 4, we have $56 < 355 < 2744$. If we continue by induction, we obtain $n.(2^n-2) < |\tau(X)| < (2^n-2)^{(n-1)}$.

2.3.2. Remark. The reader may check that our expression in Theorem 2.3.2 is a better approach than the expression (3).

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