On the Relations Between Fuzzy Topologies and α Cut Topologies

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Abstract: In this study, some relations have been generated between fuzzy sets and their cross-sections, and further relations have been derived between fuzzy topological spaces and α-cut topologies by using these relations.

Key words and phrases: Fuzzy set, α-cut, fuzzy topological spaces, α-cut topologies

Fuzzy Topolojiler ve α-kesit Topolojiler Arasındaki Bağıntılar Üzerine

Özet: Bu çalışmada fuzzy kümeler ve bunların α-kesitleri arasında bazı bağıntılar üretilmiş ve bu bağıntıları kullanarak, fuzzy topolojik uzaylar ve α-kesit topolojileri arasında bazı ilişkiler ortaya koyulmuştur.

Anahtar sözcük ve deyimler: Fuzzy küme, α-kesit, fuzzy topolojik uzaylar, α-kesit topolojiler

Introduction

Standard symbols and deduction formulas of logistics have been used in the definitions, assumptions, and proofs that have been provided in this paper. Universal symbolic language of mathematics have been used instead of a national language, where applicable.

Let $(X,\mathcal{F})$ be a fuzzy topological space (*f*.t.s), A be a fuzzy set in space $(X,\mathcal{F})$, $\alpha\in(0,1]=I_o$, $A_\alpha=\{x|\ A(x)\geq\alpha\}$ be a α-cut of set A and $\mathcal{F}_\alpha=\{A_\alpha \mid A\in F\}$ be the family of α-cuts of the elements of $\mathcal{F}$. It has been proven that family $\mathcal{F}_\alpha$ a basis for at least one topology on X, and some fuzzy-crisp relations have been derived by determining some sufficient conditions for X to be a topological space.

Preliminaries

$(\ I := [0,1]\ )\ (\ Y^X = \{f \mid f{:}X\to Y \text{ function}\}\ )$

$(\ f\in Y^X\ )(\ A\in I^X\ )(\ B\in I^Y\ )(\ \mathcal{A}\subset I^X\ ) :\Rightarrow$

1. $(\ f^{-1}[B]\ )(x) = B(\ f(x)\ )$ [1]

2. $(\vee\mathcal{A})(x) := \sup\{A(x) \mid A\in\mathcal{A}\}$, $(\wedge\mathcal{A})(x) := \inf\{A(x) \mid A\in\mathcal{A}\}$,

3. $(X,\mathcal{F})$ *f*.t.s. $:\Leftrightarrow$

$(\mathbf{0}\in\mathcal{F})(\mathbf{1}\in\mathcal{F})(A,B\in\mathcal{F}\Rightarrow A\wedge B\in\mathcal{F})(\mathcal{A}\subset\mathcal{F}\Rightarrow\vee\mathcal{A}\in F)$

A open $:\Leftrightarrow A\in\mathcal{F}$, A closed $:\Leftrightarrow \backslash A\in\mathcal{F}$

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4. $A^o = \vee\{ B \mid (B \le A)(B \in \mathcal{F}) \}, \quad A \in \mathcal{F} \Leftrightarrow A = A^o$ [2]

5. $\mathcal{B}$ base for $\mathcal{F}$ $\;:\Leftrightarrow\; (\mathcal{B} \subset \mathcal{F})\,[\,A \in \mathcal{F} \Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(A = \vee\mathcal{B}^*)\,]$ [3]

6. $(\alpha \in \mathbf{I})\,(A \in \mathbf{I}^X) \Rightarrow \quad A_\alpha = \{x \mid A(x) \ge \alpha\}, \quad \mathbf{0}_\alpha = \begin{cases} \phi, & \alpha \in (0,1] \\ X, & \alpha = 0 \end{cases}$

7. $\mathcal{A}_\alpha := \{A_\alpha \mid A \in \mathcal{A}\}, \quad |\mathcal{A}_\alpha| \le |\mathcal{A}|$

$f : (X,\mathcal{F}) \to (Y,\mathcal{E}) \Rightarrow$

8. f fuzzy continuous (*f*-c.) $:\Leftrightarrow\; (A \in \mathcal{E} \Rightarrow f^{-1}[A] \in \mathcal{F})$

9. $(X,\mathcal{F})\ T_o \;:\Leftrightarrow$

$(\forall x,y \in X)\,(\forall \alpha \in \mathbf{I_o})\,(\exists A \in \mathcal{F})\,[\,(P_x^\alpha \in A)(P_y^\alpha \notin A) \vee (P_x^\alpha \notin A)(P_y^\alpha \in B)\,]$

10. $P \in \mathbf{I}^X,\ P(x) = \begin{cases} \alpha, & x = a \\ 0, & x \ne a \end{cases} \;:\Leftrightarrow\; P = P_a^\alpha$

$A \in \mathbf{I}^X \quad \Rightarrow (\,P_x^\alpha \in A :\Leftrightarrow \alpha \le A(x)\,)$

Some Results For α-cuts of Fuzzy Sets

$A,B \in \mathbf{I}^X,\ \alpha \in \mathbf{I},\ \mathcal{A} \subset \mathbf{I}^X \quad \Rightarrow$

1. $\mathbf{A \le B \;\Leftrightarrow\; A_\alpha \subset B_\alpha \;,\quad A = B \;\Leftrightarrow\; A_\alpha = B_\alpha}$

2. $\mathbf{(A \vee B)_\alpha = A_\alpha \cup B_\alpha \;,\quad (A \wedge B)_\alpha = A_\alpha \cap B_\alpha}$ [4]

3. $\boldsymbol{\cup\mathcal{A}_\alpha \subset (\vee\mathcal{A})_\alpha \;,\quad (\wedge\mathcal{A})_\alpha \subset \cap\mathcal{A}_\alpha}$

Proof:

$A \in \mathcal{A} \overset{2.2.}{\Rightarrow} (A \le \vee\mathcal{A})(\wedge\mathcal{A} \le A)$

$\overset{3.1}{\Rightarrow} (A_\alpha \subset (\vee\mathcal{A})_\alpha)((\wedge\mathcal{A})_\alpha \subset A_\alpha) \Rightarrow \cup\mathcal{A}_\alpha \subset (\vee\mathcal{A})_\alpha,$

$\overset{2.7}{\Rightarrow} (\cup\mathcal{A}_\alpha \subset (\vee\mathcal{A})_\alpha)((\wedge\mathcal{A})_\alpha \subset \cap\mathcal{A}_\alpha)$

4. $\boldsymbol{(\exists \mathcal{A} \subset \mathbf{I}^X)(\exists \alpha \in \mathbf{I})((\vee\mathcal{A})_\alpha \ne \cup\mathcal{A}_\alpha)}$ **,**

$\boldsymbol{\exists \mathcal{A} \subset \mathbf{I}^X)(\exists \alpha \in \mathbf{I})((\wedge\mathcal{A})_\alpha \ne \cap\mathcal{A}_\alpha)}$

Proof : (Example)

$X = R,\quad a \in \mathbf{I} :\Rightarrow A^a = \{(x,a) \mid x \in R\},\ \mathcal{A} = \{A^a \mid a \in (1/2, 2/3)\} \subset 2^X,$

for $\alpha = 2/3$ $\quad \mathcal{A}_{2/3} = \{(A^a)_{2/3} \mid a \in (1/2, 2/3)\} = \{\phi\},\quad \cup\mathcal{A}_{2/3} = \phi,$

$\vee\mathcal{A} = A^{2/3} = \{(x,2/3) \mid x \in R\},\ (\vee\mathcal{A})_{2/3} = R, \ldots$

5. $\boldsymbol{|\mathcal{A}| < \aleph_o \Rightarrow ((\vee\mathcal{A})_\alpha = \cup\mathcal{A}_\alpha)((\wedge\mathcal{A})_\alpha = \cap\mathcal{A}_\alpha)}$

Proof

$x \in (\vee\mathcal{A})_\alpha \overset{2.6}{\Leftrightarrow} (\vee\mathcal{A})(x) \ge \alpha \overset{2.2.}{\Leftrightarrow} \sup\{A(x) \mid A \in \mathcal{A}\} \ge \alpha$

$\overset{\text{hi p.}}{\Leftrightarrow} \max\{A(x) \mid A \in \mathcal{A}\} \ge \alpha \Leftrightarrow (\exists A \in \mathcal{A})(A(x) \ge \alpha)$

$\overset{2.6.}{\Leftrightarrow} (\exists A \in \mathcal{A})(x \in A_\alpha) \Leftrightarrow x \in \cup\mathcal{A}_\alpha ,$

$(\wedge\mathcal{A})_\alpha \overset{2.6}{=} \{x \mid (\wedge\mathcal{A})(x) \ge \alpha\} \overset{2.2.}{=} \{x \mid \inf\{A(x) \mid A \in \mathcal{A}\} \ge \alpha\}$

$\overset{\text{hip.}}{=} \{x \mid \min\{A(x) \mid A \in \mathcal{A}\} \ge \alpha\} = \{x \mid A \in \mathcal{A} \Rightarrow A(x) \ge \alpha\}$

2.6.

$= \{x \mid A \in \mathcal{A} \Rightarrow x \in A_\alpha\} = \cap \mathcal{A}_\alpha$.

6. $\vee\mathcal{A} \in \mathcal{A} \Rightarrow (\vee\mathcal{A})_\alpha = \cup\mathcal{A}_\alpha$, $\wedge\mathcal{A} \in \mathcal{A} \Rightarrow (\wedge\mathcal{A})_\alpha = \cap\mathcal{A}_\alpha$

Proof :

$\vee\mathcal{A} \in \mathcal{A} \overset{2.7.}{\Rightarrow} (\vee\mathcal{A})_\alpha \in \mathcal{A}_\alpha \Rightarrow (\vee\mathcal{A})_\alpha \subset \cup\mathcal{A}_\alpha \overset{3.3.}{\Rightarrow} (\vee\mathcal{A})_\alpha = \cup\mathcal{A}_\alpha$

:

$\wedge\mathcal{A} \in \mathcal{A} \overset{2.7.}{\Rightarrow} (\wedge\mathcal{A})_\alpha \in \mathcal{A}_\alpha \Rightarrow (\wedge\mathcal{A})_\alpha \supset \cap\mathcal{A}_\alpha \overset{3.3.}{\Rightarrow} (\wedge\mathcal{A})_\alpha = \cap\mathcal{A}_\alpha$

7. $(\backslash A)_\alpha = \backslash A_{1-\alpha} \cup A^{-1}(1-\alpha)$

Proof :

$x \in (\backslash A)_\alpha \Leftrightarrow (\backslash A)(x) \geq \alpha \Leftrightarrow 1 - A(x) \geq \alpha \Leftrightarrow A(x) \leq 1-\alpha$

$\Leftrightarrow A(x) < 1-\alpha \vee A(x) = 1-\alpha \Leftrightarrow x \notin A_{1-\alpha} \vee x \in A^{-1}(1-\alpha)$

$\Leftrightarrow x \in \backslash A_{1-\alpha} \vee x \in A^{-1}(1-\alpha) \Leftrightarrow x \in \backslash A_{1-\alpha} \cup A^{-1}(1-\alpha)$.

8. $(A \backslash B)_\alpha = (A_\alpha \backslash B_{1-\alpha}) \cup (A_\alpha \cap B^{-1}(1-\alpha))$ [4]

9. $(f \in Y^X)(A \in I^Y)(\alpha \in I) \Rightarrow (f^{-1}[A])_\alpha = f^{-1}[A_\alpha])$

Proof :

$x \in (f^{-1}[A])_\alpha \overset{2.6}{\Leftrightarrow} (f^{-1}[A])(x) \geq \alpha \overset{2.1.}{\Leftrightarrow} A(f(x)) \geq \alpha$

$\overset{2.6.}{\Leftrightarrow} f(x) \in A_\alpha \Leftrightarrow x \in f^{-1}[A_\alpha]$

Some Relatıons Between Fuzzy Topologies and α-Cut Topologies:

1. $(X,\mathcal{F})$ ***f*.t.s.** $\Rightarrow (\forall\alpha \in I_o)(\exists(X,\mathcal{E})$ **t.s.**$)(\mathcal{F}_\alpha$ **base for** $\mathcal{E})$

Proof :

$(X,\mathcal{F})$ *f*.t.s. $\overset{2.3}{\Rightarrow}$ $\mathbf{1} \in \mathcal{F}$

$\overset{2.6\text{-}2.7}{\Rightarrow} (\forall\alpha \in I_o)(\mathbf{1}_\alpha = X \in \mathcal{F}_\alpha)$ **... (a)**

$A, B \in \mathcal{F}_\alpha \overset{2..7.}{\Rightarrow} (\forall \alpha \in I_o)(\exists C,D \in \mathcal{F})(A = C_\alpha)(B = D_\alpha)$

$\overset{\text{hi p. - 2..3.}}{\Rightarrow} C \wedge D \in \mathcal{F} \overset{2..7.}{\Rightarrow} (C \wedge D)_\alpha \in \mathcal{F}_\alpha$

$\overset{3..2.}{\Rightarrow} C_\alpha \cap D_\alpha = A \cap B \in \mathcal{F}_\alpha$ ...(b),

(a),(b) $\Rightarrow (\forall\alpha \in I_o)(\exists(X,\mathcal{E})$ t.s.$)(\mathcal{F}_\alpha$ base for $\mathcal{E})$ **.**

2. $(\mathcal{F} \subset I^X)(|\mathcal{F}| < \aleph_o) \Rightarrow [(X,\mathcal{F})$ ***f*.t.s.** $\Leftrightarrow (\forall\alpha \in I_o)((X,\mathcal{F}_\alpha)$ **t.s.**$)]$

Proof : **i)**$\Rightarrow$:

$(X,\mathcal{F})$, *f*.t.s. $\overset{2.3}{\Rightarrow} (0 \in \mathcal{F})(1 \in \mathcal{F})$

$\overset{2.6\text{-}2.7}{\Rightarrow} (\forall\alpha \in I_o)(0_\alpha = \phi \in \mathcal{F}_\alpha)(1_\alpha = X \in \mathcal{F}_\alpha)$ ...(a),

$A, B \in \mathcal{F}_\alpha \overset{4.1}{\Rightarrow} A \cap B \in \mathcal{F}_\alpha$ ...(b),

$\left.\begin{array}{l} \mathcal{A} \subset \mathcal{F}_\alpha \overset{2.7.}{\Rightarrow} (\exists\mathcal{B} \subset \mathcal{F})(\mathcal{A} = \mathcal{B}_\alpha) \\ \mathcal{F} \text{ f.t.} \end{array}\right\} \Rightarrow \vee\mathcal{B} \in \mathcal{F}$

$\overset{2..7}{\Rightarrow} (\vee\mathcal{B})_\alpha \in \mathcal{F}_\alpha \overset{\text{hip,-3.5.}}{\Rightarrow} \cup\mathcal{B}_\alpha = \cup\mathcal{A} \in \mathcal{F}_\alpha$ ...(c), (a),(b),(c) $\Rightarrow$ **(** $\alpha\in I_o \Rightarrow (X, \mathcal{F}_\alpha)$ t.s. **),**

ii) ⇒: $(X,\mathcal{F})$ not *f*.t.s.

$\overset{2.4.}{\Rightarrow} (\mathbf{0}\notin \mathcal{F}) \vee (\mathbf{1}\notin \mathcal{F}) \vee (\exists A,B\in \mathcal{F})(A\wedge B\notin \mathcal{F}) \vee (\exists \mathcal{A}\subset \mathcal{F})(\vee\mathcal{A}\notin \mathcal{F})$ ...(a),

$\left.\begin{array}{l} \mathbf{0}\notin F \Rightarrow (\forall A\in \mathcal{F})(\exists a\in X)(0< A(a) \le 1) \\ \alpha = A(a)/2 \end{array}\right\} \Rightarrow$

$(\exists\alpha\in I_o)\ (\forall A\in \mathcal{F})\ (\exists a\in X)\ (A(a) > \alpha) \Rightarrow$

$(\exists\alpha\in I_o)\ (\forall A\in \mathcal{F})\ (\exists a\in A_\alpha) \Rightarrow (\exists\alpha\in I_o)\ (\phi\notin \mathcal{F}_\alpha)$ ...(b),

$\mathbf{1}\notin F \Rightarrow (\forall A\in \mathcal{F})\ (\exists x\in X)(0 \le A(x) <1)$

$\Rightarrow (\forall A\in \mathcal{F})\ (A_1 \ne X) \Rightarrow (\exists\alpha\in I_o)\ (X\notin \mathcal{F}_\alpha)$ ...(c),

$(\exists A,B\in \mathcal{F})(A\wedge B\notin \mathcal{F}) \overset{3.1}{\Rightarrow}$

$[\exists\, \alpha\in I_o \setminus \{\alpha \mid (\exists C\in \mathcal{F})\ ((A\wedge B)_x = C_x)\ (A\wedge B \ne C)\}]$

$\mathbf{(}\ A_\alpha, B_\alpha\in \mathcal{F}_\alpha \mathbf{)}\ \mathbf{(}\ (A\wedge B)_\alpha \notin \mathcal{F}_\alpha \mathbf{)}$

$\overset{3.2.}{\Rightarrow} (\exists\alpha\in I_o)\ \mathbf{(}\ A_\alpha, B_\alpha\in \mathcal{F}_\alpha \mathbf{)}\ \mathbf{(}\ A_\alpha\cap B_\alpha \notin \mathcal{F}_\alpha \mathbf{)}$.. (d),

$(\exists \mathcal{A}\subset \mathcal{F})\ (\vee\mathcal{A}\notin \mathcal{F}) \overset{3.1.}{\Rightarrow}$

$\left.\begin{array}{l} K= I \setminus \{x \mid (\exists C\in \mathcal{F})\ ((\vee\mathcal{A})_x = C_x)\ (\vee\mathcal{A} \ne C)\}\ne\phi \\ \alpha\in K \end{array}\right\} \Rightarrow$

$(\exists\alpha\in I_o)\ (\mathcal{A}_\alpha\subset \mathcal{F}_\alpha)\ ((\vee\mathcal{A})_\alpha \notin \mathcal{F}_\alpha) \overset{\text{Hip.-3.5.}}{\Rightarrow}$

$(\exists \mathcal{B} = \mathcal{A}_\alpha\subset \mathcal{F}_\alpha)\ (\cup\mathcal{B} = \cup\mathcal{A}_\alpha \notin \mathcal{F}_\alpha)$ ...(e),

(a),(b),(c),(d), (e) $\Rightarrow (\exists\, \alpha\in I_o)\ [\ (X,\mathcal{F}_\alpha)$ not t.s.].

Example:

$X=(0,1),\ A = \{(x, -x/6+2/3) \mid x\in X\},\ B =\{(x, x/2 +1/4) \mid x\in X\}$,

$\mathcal{F} = \{\mathbf{0}, \mathbf{1}, A, B, A\vee B, A\wedge B\} \Rightarrow |\mathcal{F}| = 6 < \aleph_o$,

$(A\vee B)(x) = \begin{cases} x/2 +1/4, & 0 < x \le 5/8 \\ -x/6+2/30, & 5/8 < x < 1 \end{cases}$, $(A\wedge B)(x) = \begin{cases} -x/6+2/30, & 0 < x \le 5/8 \\ x/2 +1/4, & 5/8 < x < 1 \end{cases}$

$A_\alpha = \begin{cases} X, & 0 < \alpha \le ½ \\ (0, 4\text{-}6\alpha], & 1/2 < \alpha < 2/3 \\ \phi, & 2/3 \le \alpha \le 1 \end{cases}$, $B_\alpha = \begin{cases} X, & 0 < \alpha \le 1/4 \\ [2\alpha-1/2, 1), & 1/4 < \alpha < 3/4 \\ \phi, & 3/4 \le \alpha \le 1 \end{cases}$

$(A\vee B)_\alpha = A_\alpha\vee B_\alpha = \begin{cases} X, & 0 < \alpha \le 9/16 \\ (0, 4\text{-}6\alpha] \cup [2\alpha-1/2, 1), & 9/16 < \alpha < 2/3 \\ [2\alpha-1/2, 1), & 2/3 \le \alpha < 3/4 \\ \phi, & 3/4 \le \alpha \le 1 \end{cases}$

$(A\wedge B)_\alpha = A_\alpha\wedge B_\alpha = \begin{cases} X, & 0 < \alpha \le 1/4 \\ [2\alpha-1/2, 1), & 1/4 < \alpha < 1/2 \\ [2\alpha-1/2, 4\text{-}6\alpha], & 1/2 \le \alpha \le 9/16 \\ \phi, & 9/16 < \alpha \le 1 \end{cases}$

$\mathcal{F}_\alpha = \{ \mathbf{0}_\alpha, \mathbf{1}_\alpha, A_\alpha, B_\alpha, (A\vee B)_\alpha, (A\wedge B)_\alpha \} = \{ \phi, X, A_\alpha, B_\alpha, A_\alpha \vee B_\alpha, A_\alpha \wedge B_\alpha \}$

$$= \begin{cases} \{ \phi, X \}, & 0 < \alpha \le 1/4 \ \vee \ 3/4 < \alpha \le 1 \\ \{\phi, X, [2\alpha - 1/2, 1) \}, & 1/4 < \alpha \le 1/2 \ \vee \ 2/3 < \alpha < 3/4 \\ \{\phi, X, [2\alpha-1/2,1), (0, 4-6\alpha], [2\alpha-1/2, 4-6\alpha] \}, & 1/2 < \alpha \le 9/16 \\ \{\phi, X, [2\alpha-1/2,1), (0, 4-6\alpha], (0, 4-6\alpha] \cup [2\alpha-1/2,1) \}, & 9/16 < \alpha < 2/3 \end{cases}$$

$(\forall \alpha \in I_o)((X, \mathcal{F}_\alpha)$ t.s.) .

3. $\{ \cup\mathcal{A} \mid \mathcal{A} \subset \mathcal{F} \} \subset \mathcal{F} \subset I^X \Rightarrow [(X,\mathcal{F})$ ***f*.t.s.** $\Leftrightarrow (\forall \alpha \in I_o)((X,\mathcal{F}_\alpha)$ **t.s.**)]

Proof : from 3.6., 4.1. and 4.2.

4. ($\exists (X,\mathcal{F})$ ***f*.t.s.**) $(\exists \alpha \in I_o)((X,\mathcal{F}_\alpha)$ **not t.s.**)

Proof : (Example)

$X = R, \quad A^a(x) = \begin{cases} 0, & x < a \\ x-a, & a \le x \le a+1 \\ 1, & a+1 < x \end{cases}$, $F = \{0,1\} \cup \{ A^a \mid a \ge 0 \} \Rightarrow$

$\exists (X,\mathcal{F})$ *f*.t.s. , but for $\alpha = 1/2$, $\mathcal{F}_\alpha = \{\phi, R\} \cup \{ [a+1/2, \infty) \mid a \ge 0 \}$and

$\mathcal{A} = \{ [a+1/2, \infty) \mid a > 0 \} \subset F_\alpha$, $\cup\mathcal{A} = (1/2, \infty) \notin \mathcal{F}_\alpha$ and $(R, \mathcal{F}_\alpha)$ not t.s.

5. $(\exists X)(\exists \mathcal{A} \subset I^X)(\forall \alpha \in I_o)((X, \mathcal{A}_\alpha)$ **t.s.**) ($(X, \mathcal{A})$ **not *f*-t.s.**)

Proof : (Example)

$X = R$, $A^b = \{(x,b) \mid x \in R\}$, $\mathcal{A} = \{0,1\} \cup \{ A^b \mid 0<b<1/2 \}$, $\alpha \in I_o \Rightarrow$

$\mathcal{A}_\alpha = \{0,1\}$, $(X, \mathcal{A}_\alpha)$ t.s., but $(X, \mathcal{A})$ not *f*-t.s. .

6. ($(X,\mathcal{F})$ ***f*.t.s.**) $(\alpha \in I_o)$ ($(X,\mathcal{F}_\alpha)$ **t.s.**) $(A \subset X) \Rightarrow (A^o)_\alpha \subset (A_\alpha)^o$

Proof :

$$\left.\begin{array}{ll} A^o \le A & \overset{3.1.}{\Rightarrow} (A^o)_\alpha \subset A_\alpha \\ A^o \in F & \overset{2.7.}{\Rightarrow} (A^o)_\alpha \in \mathcal{F}_\alpha \end{array}\right\} \overset{hip.}{\Rightarrow} (A^o)_\alpha \subset (A_\alpha)^o$$

7. ($(X,\mathcal{F})$ ***f*.t.s.**) $(\alpha \in I_o)$ ($(X,\mathcal{F}_\alpha)$ **t.s.**) $(A \in \mathcal{F}) \Rightarrow (A^o)_\alpha = (A_\alpha)^o$

Proof.:

$$\left.\begin{array}{l} A \in \mathcal{F} \Rightarrow A^o = A \overset{3\text{-}1.}{\Rightarrow} (A^o)_\alpha = A_\alpha \\ A \in \mathcal{F} \overset{2.7.}{\Rightarrow} A_\alpha \in \mathcal{F}_\alpha \overset{hip.}{\Rightarrow} (A_\alpha)^o = A_\alpha \end{array}\right\} \Rightarrow (A^o)_\alpha = (A_\alpha)^o$$

8. ($(X,\mathcal{F})$ ***f*.t.s.**) ($(Y, \mathcal{E})$ ***f*.t.s.**)

$(\alpha \in I_o)$ ($(X,\mathcal{F}_\alpha)$ **t.s.**) ($(Y, \mathcal{E}_\alpha)$ **t.s.**) ($f : (X,\mathcal{F}) \to (Y,\mathcal{E})$ ***f*-c.**)

$\Rightarrow$ **f:** $(X,\mathcal{F}_\alpha) \to (Y,\mathcal{E}_\alpha)$ **c.**

Proof.:

$B \in \mathcal{E}_\alpha \overset{2.7.}{\Rightarrow} (\exists A \in \mathcal{E})(B = A_\alpha) \overset{Hip\text{-}2.10\text{-}}{\Rightarrow} f^{-1}[A] \in \mathcal{F}$

$\overset{2.7.}{\Rightarrow} (f^{-1}[A])_\alpha \in \mathcal{F}_\alpha \overset{3.9.}{\Rightarrow} f^{-1}[A_\alpha] = f^{-1}[B] \in \mathcal{F}_\alpha$

/ $f: (X,\mathcal{F}_\alpha) \to (Y,\mathcal{E}_\alpha)$ c.

9. $f : (X,\mathcal{F}) \to (Y,\mathcal{E})$ **f-c.** $\Leftrightarrow (\forall A\in I^Y)(\ f^{-1}[A^o] \leq (f^{-1}[A])^o\ )$

Proof.: **i.** $\Rightarrow$:

$$\left.\begin{array}{l} A\in I^Y \Rightarrow A^o\in\mathcal{E} \overset{\text{Hip.}}{\Rightarrow} f^{-1}[A^o]\in\mathcal{F} \\ A^o \leq A \Rightarrow f^{-1}[A^o] \leq f^{-1}[A] \end{array}\right\} \overset{2.4}{\Rightarrow} f^{-1}[A^o] \leq (f^{-1}[A])^o$$

ii. $\Leftarrow$:

$$A\in\mathcal{E} \Rightarrow A^o \overset{2.4}{=} A \Rightarrow f^{-1}[A^o] = f^{-1}[A]$$

$$\overset{\text{hip.}}{\Rightarrow} f^{-1}[A] \leq (f^{-1}[A])^o \overset{2.4}{\Rightarrow} f^{-1}[A] = (f^{-1}[A])^o$$

$$\overset{2.4}{\Rightarrow} f^{-1}[A]\in\mathcal{F}$$

/ f f-c.

10.. $(\ (X,\mathcal{F})\ T_o\ )(\ \beta\in I_o\ )(\ (X,\mathcal{F}_\beta\ )\ t.s\ ) \Rightarrow (X,\mathcal{F}_\beta\ )\ T_o$

Proof.:

$$(X,\mathcal{F})\ T_o \overset{2.10}{\Rightarrow}$$

$$(\ \forall x,y\in X\ )(\forall\alpha\in I_o)(\ \exists A\in\mathcal{F}\ )[\ (P_x^\alpha\in A)(\ P_y^\alpha\notin A)\vee(P_x^\alpha\notin A)(\ P_y^\alpha\in B)]$$

$$\overset{2.7.}{\Rightarrow} (\ \forall x,y\in X\ )(A_\beta\in\mathcal{F}_\beta\ )[\ (x\in A_\beta)(y\notin A_\beta)\vee(x\notin A_\beta)(y\in B_\beta)]$$

$$\Rightarrow (X,\mathcal{F}\ )\ T_o$$

Example: $X = \{a, b\}$, $\mathcal{F} = \{0, 1, P_a^{1/3}\}$, $A = P_a^{1/3}$

$$A(x) = \begin{cases} 0, & x\neq a \\ 1/3, & x = a \end{cases}$$, for $\alpha = 1/4$, $A_\alpha = \{a\}$, $\mathcal{F}_\alpha = \{\phi, X, \{a\}\}\ T_o$

Concluding Remarks

In this study, various basic theorems on fuzzy sets and their a-cross-sections, and topological fuzzy-crisp relations resulting from these theorems have been introduced. Similarly, many fuzzy-topological concepts can be related with their equivalent crisp-topological concepts.

In the proofs presented here, pure symbolic and formal language of mathematics, which is a universal mean of communication, has been deliberately used instead of a national language.

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