Mathematical Modellings on Heat Distribution in Composite Elements and Convergence Analysis

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Özet: Bu çalışmada *birleşik elemanlarda sıcaklık dağılımı* bulunduğu zaman ortaya çıkan problemin matematik modeli *Laplace Yöntemi* ile seri şeklinde çözümü bulunmuş, özellikle mühendislik bilimleri açısından öneme sahip olan *yakınsaklık analizleri* yapılmış ve gözenekli malzemelere özgü porozite katsayısı incelenmiştir(Bkz. [1,2, 3 ve 4]).

Anahtar Kelimeler: Sıcaklık Dağılımı, Serilerin Yakınsaklığı, Laplace Yöntemi

Birleşik Elemanlarda Sıcaklık Dağılımı Üzerine Matematik Modellemeler ve Yakınsaklık Analizi

Abstract: In this study, the solution of *heat distribution problem* is presented *in composite elements* as series by using *Laplace method* is found. Convergent analysis that is of importance in means of engineering sciences is performed and the porosity coefficient specific to porous materials are studied(See [1, 2, 3 and 4]).

Key Words: Heat Distribution, Convergence of The Series, Laplace Method

1. Introduction

Heat distribution and *thermal diffusion* is an issue faced in many industrial problems. The problem becomes rather complex because of the porous nature of some of the units forming composite elements and dimensions of other large air spaces of regular geometrical shapes and the thermal diffusion being influenced by all mechanisms of conduction, convection and radiation. Thus the need of solving these problems comes out. In this study, the boundary and the initial conditions are considered and *Laplace method* is used in the solution of mathematical structure of given physical

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system by *modelling with partial differential equation.* As a result, functions of *one dimensional heat distribution by time* are obtained. Serial solutions have been found in different initial and boundary conditions for composite elements consisting of two different material, and with different geometry and composite structure one or two surfaces of which are isolated or exposed to convectional heat transfer. The *convergence analysis* of this *serial solutions* are made and porosity coefficient specific to materials are obtained.

2. Heat Distribution in Composite Elements

Let "*t*" denotes time, *t*>0, through distance *x*, and let the two-variable function *u*(*x,t*) represent heat distribution made dimensionless:

$$\frac{\partial u(x,t)}{\partial t} = K\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{c}R(x,t) - h\left[u(x,t) - u_0\right],\ \left(x \in [0,a) \cup (a,b], t > 0\right) \tag{1}$$

This equation expresses "*Mathematical Model of Heat Distribution*" in the body(material) *composed of two different elements* (See [1, 4]).

In the expression given in equation (1),

c represents heat capacity per unit volume,
R(*x,t*) represents heat produced through distance *x* and for time *t*,
K represents thermal diffusivity,
h represents surface heat transfer coefficient positive value,
u_0 represents environmental temperature.

Also, magnitudes other than functions *u*(*x,t*) and *R* (*x,t*) in equation (1) are generally assumed constant in practice.
Assume that two bodies (materials) of heat conduction coefficients β_1 and β_2 contact at the boundary '*a*' (Figure 1).

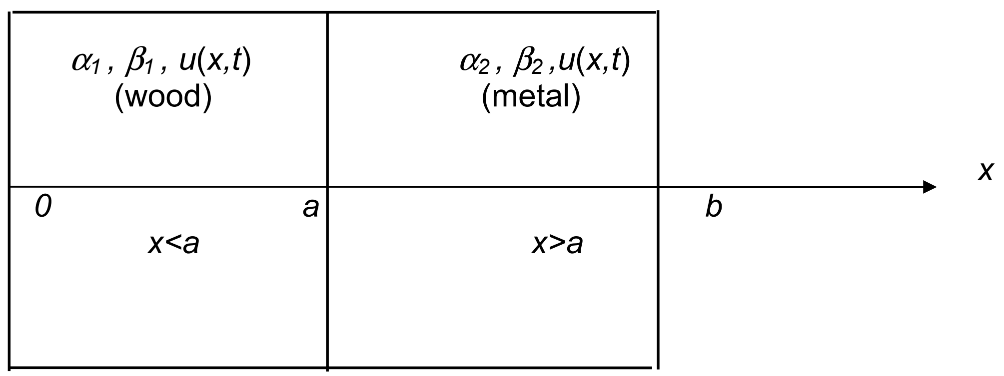

Figure 1

The condition is needed at the boundary $x=a$ where the two materials (bodies) combine. Generally there may be heat production or heat absorption at the point $x=a$ (See [1, 4]). In this case,

$\lim_{\substack{x\to 0 \\ x<a}} u(x.t) = u(a-0,t)$ and $\lim_{\substack{x\to 0 \\ x>a}} u(x,t) = u(a+0,t)$

can be defined. Thus, the equations

$$\alpha_1 u(a-0,t) = \alpha_2 u(a+0,t) \quad (t>0)$$

$$\beta_1 u_x (a-0,t) = \beta_2 u_x (a+0,t) \quad (t>0)$$

can be written(See [2, 4]).

Where α_1 , α_2 and β_1 , β_2 are positive constants.

3. Laplace Method, Convergence Analysis of Serial Solutions

Before covering mathematical models and solutions of heat distributions in composite elements, *the convergence tests* that will be used in the solution as a series can be defined as follows(See [5]):

Definition 1.

For $n\in$ N, a_n's denoting certain real numbers, an expression written as

$$a_1 + a_2 + \dots\dots\dots + a_n + \dots\dots\dots = \sum_{n=1}^{\infty} a_n$$

is called an infinite series. The expression $S_n = \sum_{k=1}^{n} a_k$, $\sum_{n=1}^{\infty} a_n$ is called n. partial sum and the sequence of S_n terms is called sequence (progression) of sums of $\sum_{n=1}^{\infty} a_n$ series.

Definition 2.

In a given $\sum_{k=1}^{\infty} a_k$, if (S_n) sequence of partial sums is convergent and $\lim_{n\to\infty} S_n = S$, then it can be said that the series given is convergent and the sum is S.

A sequence consisting of real numbers is convergent at R, if and only if this sequence is a *Cauchy sequence*. The convergence of the series in this study will be examined by using *Cauchy Section Criteria.*

Cauchy Section Criteria:

In a series with the general term a_n, if

$$\frac{a_{n+1}}{a_n} \leq a < 1 \tag{2}$$

starting from a certain point (for example for $n \geq k$) then the positive termed series $\sum a_n$ is convergent, however if always $\frac{a_{n+1}}{a_n} \geq 1$ starting from a certain point then the positive termed series $\sum a_n$ is divergent.

Theorem 1.

Absolute Convergent Series:

In a given series $\sum a_n$ of random terms, if the series $\sum |a_n|$ formed by the absolute values of the terms is convergent then the series $\sum a_n$ is said to be *absolute convergent*.

4. Heat Dispersion for Two Different Composite Materials Isolated at one Side

Let the thermal conduction and diffusivity coefficients of two different composite materials isolated at one side be defined as K_1, k_1 for $0 < x < a$ at the region I. and K_2, k_2 for $x > a$ at the region II.
One dimensional, two variable function dependent on x and time t, made dimensionless as $u = u(x,t)$ expressing the heat dispersion, the contact of two different materials at the point $x = a$ under the following conditions is given(See [1, 4]).

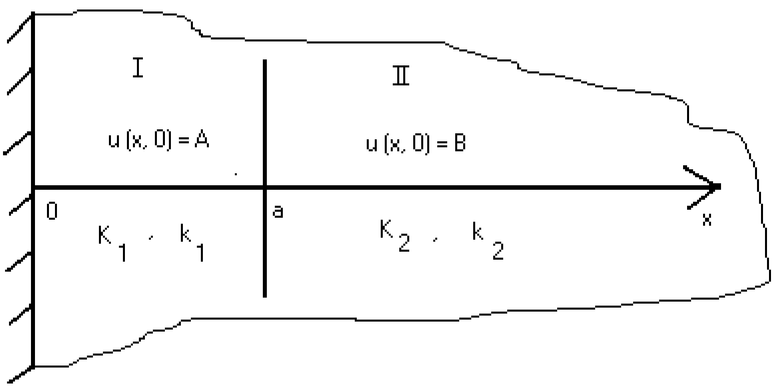

Figure 2

$$u_t(x,t) = k_1 u_{xx}(x,t), \quad (0<x<a, \ t>0), \tag{3}$$

$$u_t(x,t) = k_2 u_{xx}(x,t), \quad (x>a, \ t>0). \tag{4}$$

Initial conditions:

$$u(x,0) = A\,, \qquad (0<x<a,\ t>0), \tag{5}$$

$$u(x,0) = B\,, \quad (x>a,\ t>0). \tag{6}$$

External boundary conditions:

$$u_x(0,t) = 0\,, \qquad (0<x<a,\ t>0), \tag{7}$$

$$\lim_{x\to\infty} u(x,t) = B\,, \qquad (x>a,\ t>0). \tag{8}$$

Contacting conditions at the internal boundary(x=a):

$$u(a-0,t) = u(a+0,t), \quad K_1 u_x(a-0,t) = K_2 u_x(a+0,t)\,. \tag{9}$$

The solution of problems in forms of (3) and (4) for two different materials at given *initial and boundary conditions* by applying *Laplace Transformations*.

For 0 < *x* < *a*;

If *Laplace transformation* is applied on equation (3), *initial and boundary conditions* (5), (7); and necessary arrangements are made, then

$$k_1 \frac{d^2u}{dx^2} - su = -A\,, \tag{10}$$

$$u_x(0,s) = 0 \tag{11}$$

are obtained. If method of undetermined coefficients is used in equation (10) and *the boundary condition* (11) is considered:

$$u(x,s) = c_1 \cosh\sqrt{\frac{s}{k_1}}x + \frac{A}{s}\,, \quad (0<x<a) \tag{12}$$

is found.

For *x* >*a*;

If *Laplace transformation* is applied in equation (4), *initial and boundary conditions* (6), (8); and the same arrangements are made:

$$u(x,s) = c_2 e^{-\sqrt{\frac{s}{k_2}}x} + \frac{B}{s}\,, \qquad (x>a). \tag{13}$$

If equations of continuity (9) are applied in equations (12) and (13), the following equations are obtained:

$$c_1 \cosh \delta a - c_2 e^{-\delta\mu a} = \frac{B-A}{s} \tag{14}$$

$$c_1 \frac{K_1}{\sqrt{k_1}} \sinh \delta a + c_2 \frac{K_2}{\sqrt{k_2}} e^{-\delta a \mu} = 0 \tag{15}$$

Where,

$$\delta = \sqrt{\frac{s}{k_1}}, \qquad \mu = \sqrt{\frac{k_1}{k_2}}, \qquad \delta\mu = \sqrt{\frac{s}{k_2}}, \qquad \lambda = \frac{K_1\sqrt{k_2} - K_2\sqrt{k_1}}{K_1\sqrt{k_2} + K_2\sqrt{k_1}}.$$

Constants c_1 and c_2 can be determined by applying *Cramer method* on equations (14) and (15). Therefore

$$\Delta = \begin{vmatrix} \cosh \delta a & -e^{-\delta\mu a} \\ \frac{K_1}{\sqrt{k_1}} \sinh \delta a & \frac{K_2}{\sqrt{k_2}} e^{-\delta\mu a} \end{vmatrix} = \frac{e^{-\delta\mu a}}{2}(e^{\delta a} - \lambda e^{-\delta a}) \frac{K_1\sqrt{k_2} + K_2\sqrt{k_1}}{\sqrt{k_1}\sqrt{k_2}}$$

is found. Then,

$$c_1 = \frac{1}{\Delta} \begin{vmatrix} \frac{B-A}{s} & -e^{-\delta\mu a} \\ 0 & \frac{K_2}{\sqrt{k_2}} e^{-\delta\mu a} \end{vmatrix} = \frac{(B-A)(1-\lambda)}{s} \frac{e^{-\delta a}}{1-\lambda e^{-2\delta a}} \tag{16}$$

$$c_2 = \frac{1}{\Delta} \begin{vmatrix} \cosh \delta a & \frac{B-A}{s} \\ \frac{K_1}{\sqrt{k_1}} \sinh \delta a & 0 \end{vmatrix} = \frac{(A-B)(1+\lambda)}{2s} \frac{1-e^{-2\delta a}}{e^{-\delta\mu a}(1-\lambda e^{-2\delta a})} \tag{17}$$

are obtained. If (16) is replaced in the solution (12) then:

$$u(x,s) = \frac{(B-A)(1-\lambda)}{2s} \frac{e^{-\delta(a-x)} + e^{-\delta(a+x)}}{1-\lambda e^{-2\delta a}} + \frac{A}{s}$$

is obtained.

Where the sum of the series

$$\frac{1}{1-\lambda e^{-2\delta a}}=\sum_{n=0}^{\infty}\lambda^{n}e^{-2\delta a_{n}} \tag{18}$$

and m=2n + 1 can be written. If these expressions replaced in the equation and necessary arrangements are made:

$$u(x,s)=\frac{(B-A)(1-\lambda)}{2s}\sum_{n=0}^{\infty}\lambda^{n}(e^{-\delta(ma-x)}+e^{-\delta(ma+x)})+\frac{A}{s}$$

is found. If the inverse *Laplace transformation* is applied on the last expression, then the solution

$$u(x,t)=A+\frac{(B-A)(1-\lambda)}{2}\sum_{n=0}^{\infty}\lambda^{n}\left[erfc\frac{(2n+1)a-x}{2\sqrt{k_1 t}}+erfc\frac{(2n+1)a+x}{2\sqrt{k_1 t}}\right],\ (0<x<a) \tag{19}$$

is obtained.

Similarly, for $x>a$, if inverse *Laplace transformation* is applied on solution found after replacing the expression (17) in equation (13) and considering sum of the series (18), then the solution

$$u(x,t)=B+\frac{(A-B)(1+\lambda)}{2}\sum_{n=0}^{\infty}\lambda^{n}\left\{erfc\left[\frac{2na+\mu(x-a)}{2\sqrt{k_1 t}}\right]-erfc\left[\frac{(2n+2)a+\mu(x-a)}{2\sqrt{k_1 t}}\right]\right\},\ (x>a) \tag{20}$$

is found. When special conditions $u(x,0)=A,(0<x<a)$ and $u(x,0)=0,(x>a)$ are replaced in equations (19) and (20), the equations

$$u(x,t)=A-A\frac{1-\lambda}{2}\sum_{n=0}^{\infty}\lambda^{n}\left[erfc\frac{(2n+1)a-x}{2\sqrt{k_1 t}}+erfc\frac{(2n+1)a+x}{2\sqrt{k_1 t}}\right],\quad (0<x<a) \tag{21}$$

$$u(x,t)=A\frac{(1+\lambda)}{2}\sum_{n=0}^{\infty}\lambda^{n}\left\{erfc\left[\frac{2na+\mu(x-a)}{2\sqrt{k_1 t}}\right]-erfc\left[\frac{(2n+2)a+\mu(x-a)}{2\sqrt{k_1 t}}\right]\right\},\ (x>a) \tag{22}$$

are obtained(See [2]).

Now, let us examine the convergence of the serial solutions found as equations (21) and (22). In order to show that these solutions are convergent, we need to show that:

$$\sum_{n=0}^{\infty} \lambda^n erfc\left[\frac{(2n+1)a-x}{2\sqrt{k_1 t}}\right] \tag{23}$$

$$\sum_{n=0}^{\infty} \lambda^n erfc\left[\frac{(2n+1)a+x}{2\sqrt{k_1 t}}\right] \tag{24}$$

$$\sum_{n=0}^{\infty} \lambda^n erfc\left[\frac{2na+\mu(x-a)}{2\sqrt{k_1 t}}\right] \tag{25}$$

$$\sum_{n=0}^{\infty} \lambda^n erfc\left[\frac{(2n+2)+\mu(x-a)}{2\sqrt{k_1 t}}\right] \tag{26}$$

are convergent. Let us first show that the series formed of the absolute values of the terms of the series (23):

$$\sum_{n=0}^{\infty} \left| \lambda^n erfc\left[\frac{(2n+1)a-x}{2\sqrt{k_1 t}}\right] \right|$$

is convergent:

Since the general term is

$$a_n = \left| \lambda^n erfc\left[\frac{(2n+1)a-x}{2\sqrt{k_1 t}}\right] \right|,$$

from the definition of equation (2),

$$\frac{a_{n+1}}{a_n} = |\lambda| \left| \frac{erfc\left[\frac{(2n+3)a-x}{2\sqrt{k_1 t}}\right]}{erfc\left[\frac{(2n+1)a-x}{2\sqrt{k_1 t}}\right]} \right|$$

is obtained.

If $a_1 = \dfrac{(2n+3)a - x}{2\sqrt{k_1 t}}$, $a_2 = \dfrac{(2n+1)a - x}{2\sqrt{k_1 t}}$ are selected and the definition

$a_1 > a_2 \Rightarrow erfca_1 < erfca_2$ is taken into consideration, then $\left|\dfrac{erfca_1}{erfca_2}\right| < 1$. It is obvious that $|\lambda|<1$. Then,

$\left|\dfrac{a_{n+1}}{a_n}\right| = |\lambda|\left|\dfrac{erfca_1}{erfca_2}\right| < 1$. So, $\sum_{n=0}^{\infty}|a_n|$ is *absolute convergent*. If a series is *absolutely convergent*, then the series is convergent. By using the same method, it may be shown that series (24), (25) and (26) are convergent as well.

5. Heat Dispersion for Two Different Porous Composite Materials Isolated at Both sides

Let the thermal diffusivity of two porous materials well-welded on each other be K_1 and K_2. Let the two-variable function $u = u(x,t)$ made dimensionless, one dimensional, dependent on x, time t, show the heat dispersion(See [1, 4]).

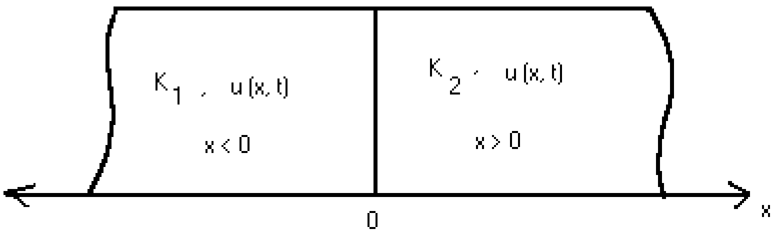

Figure 3

In the system assumed as endless bar on both sides and with no isolation at neither sides, let $x < 0$, $x > 0$, *initial and boundary conditions* be given in three conditions:

$$u_t(x,t) = K_1 u_{xx}(x,t), \quad (x<0,\ t>0), \tag{27}$$

$$u_t(x,t) = K_2 u_{xx}(x,t), \quad (x>0,\ t>0). \tag{28}$$

The initial and boundary conditions:

1. $u(x,0) = 0$, $\lim_{x\to-\infty} u(x,t) = 0$, $(x<0)$,

$u(x,0) = u_0$, $\lim_{x \to +\infty} u(x,t) = u_0$, (*x*>0).

2. $u(x,0) = u_0$, $\lim_{x \to -\infty} u(x,t) = u_0$, (*x*<0),

$u(x,0) = 0$, $\lim_{x \to +\infty} u(x,t) = 0$, (*x*>0).

3. $u(x,0) = v_0$, $\lim_{x \to -\infty} u(x,t) = v_0$, (*x*<0),

$u(x,0) = u_0$, $\lim_{x \to +\infty} u(x,t) = u_0$, (*x*>0).

The contact conditions at the internal boundary. (at the point *x* = 0)

$$u(+0,t) = \alpha u(-0,t), \quad K_2 u_x(+0,t) = K_1 u_x(-0,t) \tag{29}$$

(See [4]).

If *Laplace transformation* is applied and necessary arrangements are made on 1., 2., and 3. *initial and boundary conditions* of problem (27), (28);

1. $u(x,s) = c_1 e^{x\sqrt{\frac{s}{k_1}}}$, (*x*<0),

$u(x,s) = c_2 e^{-x\sqrt{\frac{s}{k_2}}} + \dfrac{u_0}{s}$, (*x*>0). (30)

2. $u(x,s) = c_1 e^{x\sqrt{\frac{s}{k_1}}} + \dfrac{u_0}{s}$, (*x*<0),

$u(x,s) = c_2 e^{-x\sqrt{\frac{s}{k_2}}}$, (*x*>0). (31)

3. $u(x,s) = c_1 e^{x\sqrt{\frac{s}{k_1}}} + \dfrac{v_0}{s}$, (*x*<0),

$u(x,s) = c_2 e^{-x\sqrt{\frac{s}{k_2}}} + \dfrac{u_0}{s}$, (*x*>0). (32)

solutions are obtained.

The constants in the expressions given in (30), (31) and (32) can be found with the help of internal contact(al) conditions of (29) (See [4]). The ratio given in equation (29) is specific to porous materials when $\alpha \neq 1$.

In the final part of the problem, taken

$$R = \frac{u(x,t),(x>0)}{u(x,t),(x<0)} \tag{33}$$

$\lim_{t\to\infty} R = \alpha$ for is observed for three conditions.

For condition 1;

After conditions (29) are applied and necessary operations are made on equation (30), taken $\beta = \sqrt{\frac{K_1}{K_2}}$,

$$u(x,s) = \frac{u_0}{\alpha+\beta}\frac{e^{x\sqrt{\frac{s}{k_1}}}}{s}, \qquad (x < 0),$$

$$u(x,s) = -\frac{u_0\beta}{\alpha+\beta}\frac{e^{-x\sqrt{\frac{s}{k_2}}}}{s} + \frac{u_0}{s}, \qquad (x > 0)$$

are obtained. If *inverse Laplace transformations* are applied in this equations, the equations

$$u(x,t) = \frac{u_0}{\alpha+\beta} erfc(-\frac{x}{2\sqrt{K_1 t}}), \quad (x < 0), \tag{34}$$

$$u(x,t) = \frac{u_0}{\alpha+\beta}(\alpha + \beta erfc(\frac{x}{2\sqrt{K_2 t}})), \qquad (x > 0) \tag{35}$$

are obtained. If the solution methods used for condition 1 is applied also on conditions 2 and 3 :

For condition 2;

$$u(x,t) = \frac{u_0}{\alpha+\beta}(\beta + \alpha erfc(-\frac{x}{2\sqrt{K_1 t}})), \qquad (x< 0), \tag{36}$$

$$u(x,t) = \frac{u_0\alpha\beta}{\alpha+\beta} erfc(\frac{x}{2\sqrt{K_2 t}}), \qquad (x > 0). \tag{37}$$

For condition 3;

$$u(x,t) = \frac{(u_0 - \alpha v_0)}{\alpha + \beta} erfc(-\frac{x}{2\sqrt{K_1 t}}) + v_0 \,, \qquad (x< 0), \tag{38}$$

$$u(x,t) = \beta \frac{(\alpha v_0 - u_0)}{\alpha + \beta} erfc(\frac{x}{2\sqrt{K_2 t}}) + u_0 \,, \qquad (x > 0). \tag{39}$$

are obtained.

If equations (34) and (35) given for condition 1 are transformed into (33), then the ratio

$$R_{condition\ 1} = \frac{\beta erf(\frac{x}{2\sqrt{K_2 t}}) + \alpha}{erfc(-\frac{x}{2\sqrt{K_1 t}})} \tag{40}$$

is found. Similarly, we obtain

$$R_{condition\ 2} = \frac{\alpha \beta erfc(\frac{x}{2\sqrt{K_2 t}})}{\beta + \alpha erfc(-\frac{x}{2\sqrt{K_1 t}})} \tag{41}$$

$$R_{condition\ 3} = \frac{\dfrac{(\alpha v_0 - u_0)\beta}{(\alpha+\beta) erfc(\frac{x}{2\sqrt{K_2 t}})} + u_0}{\dfrac{(u_0 - \alpha v_0)}{(\alpha+\beta) erfc(-\frac{x}{2\sqrt{K_1 t}})} + v_0} \tag{42}$$

respectively.

For t =0, $R_{condition\ 1} \to \infty$, $R_{condition\ 2} = 0$, $R_{condition\ 3} = \frac{u_0}{v_0}$

For $t \to \infty$, $R_{condition\ 1} \to \alpha$, $R_{condition\ 2} = \alpha$, $R_{condition\ 3} = \alpha$

are obtained.

If we denote $R_{condition\ 1}$ = A , $R_{condition\ 2}$ = B , $R_{condition\ 3}$ = C , figure 4 is obtained between R and t. As observed in figure 4, the ratio for $t \to \infty$ is continuously equal to the number α. Improving the problem, by using conditions (29) it's possible to increase the number of parameters such as number of elements, problem size.

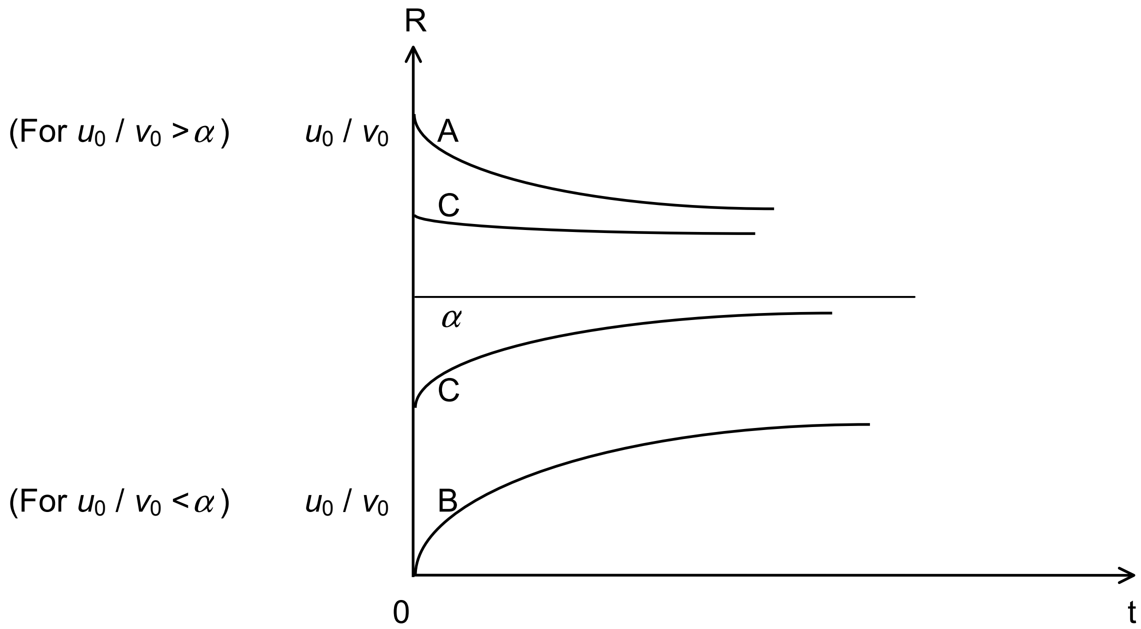

Figure 4

6. Conclusion

In this study, heat equations used in modeling composite material and basic, mathematical definitions were given. Determining heat distribution problems unique to composite materials, boundary and initial conditions were determined. These problems were elected from the literature and solved by *Laplace transformation*. Serial solutions were obtained and doing *convergency analysis*, coefficient of porosity special to porous material was examined. In the one-dimensional heat distribution problems obtained here, *Laplace transformation* was used for two composite materials, and *numerical methods* were used for composite materials more than two in number(See [6]).

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