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ALTERNATIVE RIDGE PARAMETERS IN LINEAR MODEL

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ABSTRACT

The ridge regression estimator produces efficient estimates than the Ordinary Least Square Estimator in a linear regression model that has multicollinearity problem. However, the efficiency of the ridge estimator depends on the choice of the ridge parameter, *k*. This parameter being the biasing parameter that shrinks the coefficient as it tends towards positive infinity needs to be chosen optimally to minimize the mean squared errors of the parameters. In this study, the ridge parameters are classified into different forms, various types and diverse kinds. These classifications resulted into proposing some other techniques of Ridge parameter estimation. Investigation of the existing and proposed ridge parameters were done by conducting Monte-Carlo experiments. Results from simulation study and reallife data application show that some newly proposed ridge parameters are among those that provide efficient estimates.

Keywords: Linear regression, Ridge regression, Biasing parameters, Multicollinearity

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DOĞRUSAL REGRESYONDA ALTERNATİF RİDGE PARAMETRELERİ

ÖZ

Ridge regresyon tahmin edicisi çoklu iç ilişki problemi olan doğrusal regresyon modelinde En Küçük Kareler tahmin edicisinden daha etkin sonuçlar verir. Fakat, Ridge tahmin edicisinin performansı Ridge parametresinin seçimine bağlıdır. Ridge parametreleri farklı türlerde sınıflandırılmaktadır. Bu nedenle Ridge parametrelerinin tahmini için farklı teknikler önerilmiştir. Varolan ve yeni önerilen Ridge parametrelerinin karşılaştırılması için Monte Carlo simülasyon çalışması yapılmıştır. Simülasyon çalışması ve gerçek veri seti sonuçlarına bakıdığında önerilen Ridge parametresi tahmincilerinin etkin sonuçlar verdiği gösterilmiştir.

Anahtar Kelimeler: Doğrusal regresyon, Ridge regresyon, Çoklu iç ilişki, Yanlılık parametresi

1. INTRODUCTION

The history of multicollinearity can be traced as far back as 1932 by a renowned scientist named Frisch where he identified the possible relationship between the independent variables and dependent variable (Hanan and Nurul, 2015). Multicollinearity refers to a situation in which predictor variables in a multiple regression model are highly correlated. When perfect, the regression coefficients using the ordinary least square (OLS) method are indeterminate and their standard errors are infinite. If it is high but not perfect, the regression coefficients are determinate but possess large standard errors (Gujarati, 1995). One of the ways to handle multicollinearity in linear regression is the use of ridge regression (RR). This was suggested by Hoerl and Kennard (1970) and expressed as:

$$\hat{\beta}_R = (X'X + K)^{-1}X'Y \tag{1}$$

where $\hat{\beta}_R$ is a $p \times 1$ vector of ridge estimates of the unknown coefficients, X is an $n \times p$ matrix of independent variables which is assumed to be orthogonal, the biasing parameter K is a diagonal matrix of non-negative entries for generalized ridge regression and a non-negative constant for ordinary ridge regression ($K = kI,$ where I is an identity matrix). When that when k = 0, then the ridge estimator given in (1) returns the Ordinary Least Squares (OLS) estimator given as:

$$\hat{\beta}_{OLS} = (X'X)^{-1}X'Y \tag{2}$$

Hoerl and Kennard (1970) revealed that with a positive value of ridge parameter, k, the ridge estimator provides a smaller Mean Squared Error (MSE) compared with the OLS estimator. Different techniques for estimating the ridge parameters have been developed in literature. These include those proposed by Hoerl and Kennard (1970), McDonald and Galarneau (1975), Lawless and Wang (1976), Dempster et al. (1977), Gibbons (1981), Kibria (2003), Khalaf and Shukur (2005), Alkhamisi et al. (2006), Alkhamisi and Shukur (2008), Batach et al. (2008), Muniz and Kibria (2009), Dorugade and Kashid (2010), Mansson et al. (2010), Khalaf (2013), Ghadhan and Mohamed (2014), Kibria and Shipra (2016), Bhat (2016), Lukman and Ayinde (2017), Fayose and Ayinde (2019). Algama (2018a) proposed methods of selecting biasing parameters in Generalized Linear Models (GLM) and also in Algama (2018b) some modified versions of ridge parameter estimators for gamma models were proposed.

In this study, we will review some available methods in literature to estimate the value of *k*. The main objective of this paper is to propose some biasing parameter estimators based on the work of Batach et al. (2008). The rest of the paper is organized as follows: in section 2, we proposed ridge parameters based on Batach et al. (2008). In Section 3 and 4, we present the simulation results and the reallife application respectively and in section 5, we draw conclusion.

2. METHODOLOGY

The ridge parameters have been classified into different forms and various types by Lukman and Ayinde (2017). In this study, we introduced the concept of diverse kinds (original (o), reciprocal (r), square root (SR), reciprocal of square root (RSR), pth root (PR), reciprocal of Pth root (RPR)) into the generalized ridge parameter proposed by Batach et al. (2008) and Fayose and Ayinde (2019). This involves performing the corresponding operation on the existing generalized risge parameter. Also, the existing methods of classification by Lukman and Ayinde (2017) involved performing some mathematical operation such as Arithmetic Mean (AM), Geometric Mean (GM), Harmonic Mean (HM), Mid-Range (MR), Minimum (Min) and Maximum (Max) on the generalized ridge parameter to form a constant. This results in the development of more ridge parameters.

The generalized ridge parameter introduced by Batach et al. (2008) is

$$\widehat{K}_{B_i} = \frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\left\{\left[\left(\frac{\widehat{\alpha}_i^4\lambda_i^2}{4\widehat{\sigma}^2}\right) + \left(\frac{6\widehat{\alpha}_i^4\lambda_i}{\widehat{\sigma}^2}\right)\right]^{\frac{1}{2}} - \left(\frac{\widehat{\alpha}_i^2\lambda_i}{2\widehat{\sigma}^2}\right)\right\} \quad (3)$$

while that of Fayose and Ayinde (2019) is

$$\widehat{K}_{FA_i} = \frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\left\{\left[\left[\left(\frac{\widehat{\alpha}_i^4\lambda_{min}^2}{4\widehat{\sigma}^2}\right) + \left(\frac{6\widehat{\alpha}_i^4\lambda_{min}}{\widehat{\sigma}^2}\right)\right]^{\frac{1}{2}} - \left(\frac{\widehat{\alpha}_i^2\lambda_{min}}{2\widehat{\sigma}^2}\right)\right]\right\} \quad (4)$$

2.1. Classification of Batach et al. (2008) Generalized Ridge Parameter

The classification of this ridge parameter into different forms, various types and diverse kinds is summarized in Table 1 to Table 6.

Table 1. Summary of different forms and various types of Batach et al. (2008) for original kind (O)

	Various Types of K					
FORMS	O	R	SR	RSR	PR	RPR
FM1	$\widehat{K}1_B^{FM1O}$* Proposed	$\widehat{K}1_B^{FM1R} = \left(\widehat{K}_B^{FM1O}\right)^{-1}$ Proposed	$\widehat{K}1_B^{FM1SR} = \left(\widehat{K}_B^{FM1O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{FM1RSR} = \left(\widehat{K}_B^{FM1O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{FM1PR} = \left(\widehat{K}1_B^{FM1O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{FM1RPR} = \left(\widehat{K}1_B^{FM1O}\right)^{-\frac{1}{P}}$ Proposed
FM2	$\widehat{K}1_B^{FM2O}$* Proposed	$\widehat{K}1_B^{FM2R} = \left(\widehat{K}_B^{FM2O}\right)^{-1}$ Proposed	$\widehat{K}1_B^{FM2SR} = \left(\widehat{K}_B^{FM2O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{FM2RSR} = \left(\widehat{K}_B^{FM2O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{FM2PR} = \left(\widehat{K}1_B^{FM2O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{FM2RPR} = \left(\widehat{K}1_B^{FM2O}\right)^{-\frac{1}{P}}$ Proposed
FM3	$\widehat{K}1_B^{FM3O}$ * Proposed	$\widehat{K}1_B^{FM3R} = \left(\widehat{K}_B^{FM3O}\right)^{-1}$ Proposed	$\widehat{K}1_B^{FM3SR} = \left(\widehat{K}_B^{FM3O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{FM3RSR} = \left(\widehat{K}_B^{FM3O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{FM3PR} = \left(\widehat{K}1_B^{FM3O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{FM3RPR} = \left(\widehat{K}1_B^{FM3O}\right)^{-\frac{1}{P}}$ Proposed
VM	$\widehat{K}1_B^{VMO} = \text{Max}\left(\widehat{K}_{Bi}\right)$ Proposed	$\widehat{K}1_B^{VMR} = \left(\widehat{K}_B^{VMO}\right)^{-1}$ Proposed	$\widehat{K}1_B^{VMSR} = \left(\widehat{K}_B^{VM0}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{VMRSR} = \left(\widehat{K}_B^{VM0}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{VMPR} = \left(\widehat{K}1_B^{VM0}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{VMRPR} = \left(\widehat{K}1_B^{VMO}\right)^{-\frac{1}{P}}$ Proposed

Table 1 (Continue). Summary of different forms and various types of Batach et al. (2008) for original kind (O)

AM	$\widehat{\mathbf{K}}\mathbf{1}_B^{AMO} = \frac{1}{p}\sum_{i=1}^{p}\left(\widehat{K}_{Bi}\right)$ **Proposed**	$\widehat{\mathbf{K}}\mathbf{1}_B^{AMR} = \left(\widehat{\mathbf{K}}_B^{AMO}\right)^{-1}$ **Proposed**	$\widehat{\mathbf{K}}\mathbf{1}_B^{AMSR} = \left(\widehat{\mathbf{K}}_B^{AM0}\right)^{\frac{1}{2}}$ **Proposed**	$\widehat{\mathbf{K}}\mathbf{1}_B^{AMRSR} = \left(\widehat{\mathbf{K}}_B^{AM0}\right)^{-\frac{1}{2}}$ **Proposed**	$\widehat{\mathbf{K}}\mathbf{1}_B^{AMPR} = \left(\widehat{\mathbf{K}}\mathbf{1}_B^{AM0}\right)^{\frac{1}{P}}$ **Proposed**	$\widehat{\mathbf{K}}\mathbf{1}_B^{AMRPR} = \left(\widehat{\mathbf{K}}\mathbf{1}_B^{AMO}\right)^{-\frac{1}{P}}$ **Proposed**

HM	$\widehat{K}1_B^{HMO}$ * Proposed	$\widehat{K}1_B^{HMR} = \left(\widehat{K}_B^{HMO}\right)^{-1}$ Proposed	$\widehat{K}1_B^{HMSR} = \left(\widehat{K}_B^{HMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{HMRSR} = \left(\widehat{K}_B^{HMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{HMPR} = \left(\widehat{K}1_B^{HMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{HMRPR} = \left(\widehat{K}1_B^{HMO}\right)^{-\frac{1}{P}}$ Proposed
GM	$\widehat{K}1_B^{GMO} = \left[\prod_{i=1}^{p}\left(\widehat{K}_{Bi}\right)\right]^{\frac{1}{p}}$ Proposed	$\widehat{K}1_B^{GMR} = \left(\widehat{K}_B^{GMO}\right)^{-1}$ Proposed	$\widehat{K}1_B^{GMSR} = \left(\widehat{K}_B^{GMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{GMRSR} = \left(\widehat{K}_B^{GMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{GMPR} = \left(\widehat{K}1_B^{GMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{GMRPR} = \left(\widehat{K}1_B^{GMO}\right)^{-\frac{1}{P}}$ Proposed
M	$\widehat{K}1_B^{MO} = \text{Median}\left(\widehat{K}_{Bi}\right)$ Proposed	$\widehat{K}1_B^{MR} = \left(\widehat{K}_B^{MO}\right)^{-1}$ Proposed	$\widehat{K}1_B^{MSR} = \left(\widehat{K}_B^{MO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{MRSR} = \left(\widehat{K}_B^{MO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{MPR} = \left(\widehat{K}1_B^{MO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{MRPR} = \left(\widehat{K}1_B^{MO}\right)^{-\frac{1}{P}}$ Proposed
MR	$\widehat{K}1_B^{MRO} = \frac{1}{2}\left(\max\left(\widehat{K}_{Bi}\right) + \min\left(\widehat{K}_{Bi}\right)\right)$ Proposed	$\widehat{K}1_B^{MRR} = \left(\widehat{K}_B^{MRO}\right)^{-1}$ Proposed	$\widehat{K}1_B^{MRSR} = \left(\widehat{K}_B^{MRO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{MRRSR} = \left(\widehat{K}_B^{MRO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}1_B^{MRPR} = \left(\widehat{K}1_B^{MRO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}1_B^{MRRPR} = \left(\widehat{K}1_B^{MRO}\right)^{-\frac{1}{P}}$ Proposed

Note: Asterisked estimators are given below

$$\hat{K}_B^{FM1O} = \frac{\hat{\sigma}^2}{\max(\hat{\alpha}_i^2)}\left\{\left[\left(\frac{\max(\hat{\alpha}_i^4)\max(\lambda_i)}{4\hat{\sigma}^2}\right)+\left(\frac{6\max(\hat{\alpha}_i^4)\max(\lambda_i)}{\hat{\sigma}^2}\right)\right]^{\frac{1}{2}}-\left(\frac{\max(\hat{\alpha}_i^2)\max(\lambda_i)}{2\hat{\sigma}^2}\right)\right\} \quad (5)$$

$$\hat{K}_B^{FM2O} = \frac{\hat{\sigma}^2}{\max(\hat{\alpha}_i)^2}\left\{\left[\left(\frac{\max(\hat{\alpha}_i)^4\max(\lambda_i)}{4\hat{\sigma}^2}\right)+\left(\frac{6\max(\hat{\alpha}_i)^4\max(\lambda_i)}{\hat{\sigma}^2}\right)\right]^{\frac{1}{2}}-\left(\frac{\max(\hat{\alpha}_i)^2\max(\lambda_i)}{2\hat{\sigma}^2}\right)\right\}$$

(6)

$$\hat{K}_B^{FM3O} = \frac{\hat{\sigma}^2}{\max(\hat{\alpha}_i^2)}\left\{\left[\left(\frac{\max(\hat{\alpha}_i^4\lambda_i)}{4\hat{\sigma}^2}\right)+\left(\frac{6\max(\hat{\alpha}_i^4\lambda_i)}{\hat{\sigma}^2}\right)\right]^{\frac{1}{2}}-\left(\frac{\max(\hat{\alpha}_i^2\lambda_i)}{2\hat{\sigma}^2}\right)\right\} \quad (7)$$

Table 2. Summary of different forms and various types of Batach et al. (2008) for reciprocal kind (R)

	Various Types of K					
FORMS	O	R	SR	RSR	PR	RPR
FM1	$\widehat{K}2_B^{FM1O} = \frac{1}{\widehat{K}1_B^{FM1O}}$ Proposed	$\widehat{K}2_B^{FM1R} = \left(\widehat{K}2_B^{FM1O}\right)^{-1}$ Proposed	$\widehat{K}2_B^{FM1SR} = \left(\widehat{K}2_B^{FM1O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{FM1RSR} = \left(\widehat{K}2_B^{FM1O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{FM1PR} = \left(\widehat{K}2_B^{FM1O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{FM1RPR} = \left(\widehat{K}2_B^{FM1O}\right)^{-\frac{1}{P}}$ Proposed
FM2	$\widehat{K}2_B^{FM2O} = \frac{1}{\widehat{K}1_B^{FM2O}}$ Proposed	$\widehat{K}2_B^{FM2R} = \left(\widehat{K}2_B^{FM2O}\right)^{-1}$ Proposed	$\widehat{K}2_B^{FM2SR} = \left(\widehat{K}2_B^{FM2O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{FM2RSR} = \left(\widehat{K}2_B^{FM2O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{FM2PR} = \left(\widehat{K}2_B^{FM2O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{FM2RPR} = \left(\widehat{K}2_B^{FM2O}\right)^{-\frac{1}{P}}$ Proposed
FM3	$\widehat{K}2_B^{FM3O} = \frac{1}{\widehat{K}1_B^{FM3O}}$ Proposed	$\widehat{K}2_B^{FM3R} = \left(\widehat{K}2_B^{FM3O}\right)^{-1}$ Proposed	$\widehat{K}2_B^{FM3SR} = \left(\widehat{K}2_B^{FM3O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{FM3RSR} = \left(\widehat{K}2_B^{FM3O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{FM3PR} = \left(\widehat{K}2_B^{FM3O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{FM3RPR} = \left(\widehat{K}2_B^{FM3O}\right)^{-\frac{1}{P}}$ Proposed
VM	$\widehat{K}2_B^{VMO} = \frac{1}{\widehat{K}1_B^{VMO}}$ Proposed	$\widehat{K}2_B^{VMR} = \left(\widehat{K}2_B^{VMO}\right)^{-1}$ Proposed	$\widehat{K}2_B^{VMSR} = \left(\widehat{K}2_B^{VMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{VMRSR} = \left(\widehat{K}2_B^{VMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{VMPR} = \left(\widehat{K}2_B^{VMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{VMRPR} = \left(\widehat{K}2_B^{VMO}\right)^{-\frac{1}{P}}$ Proposed
AM	$\widehat{K}2_B^{AMO} = \frac{1}{\widehat{K}1_B^{AMO}}$ Proposed	$\widehat{K}2_B^{AMR} = \left(\widehat{K}2_B^{AMO}\right)^{-1}$ Proposed	$\widehat{K}2_B^{AMSR} = \left(\widehat{K}2_B^{AMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{AMRSR} = \left(\widehat{K}2_B^{AMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{AMPR} = \left(\widehat{K}2_B^{AMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{AMRPR} = \left(\widehat{K}2_B^{AMO}\right)^{-\frac{1}{P}}$ Proposed
HM	$\widehat{K}2_B^{HMO} = \frac{1}{\widehat{K}1_B^{HMO}}$ Proposed	$\widehat{K}2_B^{HMR} = \left(\widehat{K}2_B^{HMO}\right)^{-1}$ Proposed	$\widehat{K}2_B^{HMSR} = \left(\widehat{K}2_B^{HMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{HMRSR} = \left(\widehat{K}2_B^{HMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{HMPR} = \left(\widehat{K}2_B^{HMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{HMRPR} = \left(\widehat{K}2_B^{HMO}\right)^{-\frac{1}{P}}$ Proposed
GM	$\widehat{K}2_B^{GMO} = \frac{1}{\widehat{K}1_B^{GMO}}$ Proposed	$\widehat{K}2_B^{GMR} = \left(\widehat{K}2_B^{GMO}\right)^{-1}$ Proposed	$\widehat{K}2_B^{GMSR} = \left(\widehat{K}2_B^{GMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{GMRSR} = \left(\widehat{K}2_B^{GMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{GMPR} = \left(\widehat{K}2_B^{GMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{GMRPR} = \left(\widehat{K}2_B^{GMO}\right)^{-\frac{1}{P}}$ Proposed
M	$\widehat{K}2_B^{MO} = \frac{1}{\widehat{K}1_B^{MO}}$ Proposed	$\widehat{K}2_B^{MR} = \left(\widehat{K}2_B^{MO}\right)^{-1}$ Proposed	$\widehat{K}2_B^{MSR} = \left(\widehat{K}2_B^{MO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{MRSR} = \left(\widehat{K}2_B^{MO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{MPR} = \left(\widehat{K}2_B^{MO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{MRPR} = \left(\widehat{K}2_B^{MO}\right)^{-\frac{1}{P}}$ Proposed
MR	$\widehat{K}2_B^{MRO} = \frac{1}{\widehat{K}1_B^{MRO}}$ Proposed	$\widehat{K}2_B^{MRR} = \left(\widehat{K}2_B^{MRO}\right)^{-1}$ Proposed	$\widehat{K}2_B^{MRSR} = \left(\widehat{K}2_B^{MRO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{MRRSR} = \left(\widehat{K}2_B^{MRO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}2_B^{MRPR} = \left(\widehat{K}2_B^{MRO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}2_B^{MRRPR} = \left(\widehat{K}2_B^{MRO}\right)^{-\frac{1}{P}}$ Proposed

Table 3. Summary of different forms and various types of Batach et al. (2008) for square root kind (SR)

	Various Types of K					
FORMS	O	R	SR	RSR	PR	RPR
FM1	$\widehat{K}3_B^{FM1O} = \sqrt{\widehat{K}1_B^{FM1O}}$ Proposed	$\widehat{K}3_B^{FM1R} = \left(\widehat{K}3_B^{FM1O}\right)^{-1}$ Proposed	$\widehat{K}3_B^{FM1SR} = \left(\widehat{K}3_B^{FM1O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{FM1RSR} = \left(\widehat{K}3_B^{FM1O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{FM1PR} = \left(\widehat{K}3_B^{FM1O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{FM1RPR} = \left(\widehat{K}3_B^{FM1O}\right)^{-\frac{1}{P}}$ Proposed
FM2	$\widehat{K}3_B^{FM2O} = \sqrt{\widehat{K}1_B^{FM2O}}$ Proposed	$\widehat{K}3_B^{FM2R} = \left(\widehat{K}3_B^{FM2O}\right)^{-1}$ Proposed	$\widehat{K}3_B^{FM2SR} = \left(\widehat{K}3_B^{FM2O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{FM2RSR} = \left(\widehat{K}3_B^{FM2O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{FM2PR} = \left(\widehat{K}3_B^{FM2O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{FM2RPR} = \left(\widehat{K}3_B^{FM2O}\right)^{-\frac{1}{P}}$ Proposed
FM3	$\widehat{K}3_B^{FM3O} = \sqrt{\widehat{K}1_B^{FM3O}}$ Proposed	$\widehat{K}3_B^{FM3R} = \left(\widehat{K}3_B^{FM3O}\right)^{-1}$ Proposed	$\widehat{K}3_B^{FM3SR} = \left(\widehat{K}3_B^{FM3O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{FM3RSR} = \left(\widehat{K}3_B^{FM3O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{FM3PR} = \left(\widehat{K}3_B^{FM3O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{FM3RPR} = \left(\widehat{K}3_B^{FM3O}\right)^{-\frac{1}{P}}$ Proposed
VM	$\widehat{K}3_B^{VMO} = \sqrt{\widehat{K}1_B^{VMO}}$ Proposed	$\widehat{K}3_B^{VMR} = \left(\widehat{K}3_B^{VMO}\right)^{-1}$ Proposed	$\widehat{K}3_B^{VMSR} = \left(\widehat{K}3_B^{VMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{VMRSR} = \left(\widehat{K}3_B^{VMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{VMPR} = \left(\widehat{K}3_B^{VMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{VMRPR} = \left(\widehat{K}3_B^{VMO}\right)^{-\frac{1}{P}}$ Proposed
AM	$\widehat{K}3_B^{AMO} = \sqrt{\widehat{K}1_B^{AMO}}$ Proposed	$\widehat{K}3_B^{AMR} = \left(\widehat{K}3_B^{AMO}\right)^{-1}$ Proposed	$\widehat{K}3_B^{AMSR} = \left(\widehat{K}3_B^{AMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{AMRSR} = \left(\widehat{K}3_B^{AMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{AMPR} = \left(\widehat{K}3_B^{AMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{AMRPR} = \left(\widehat{K}3_B^{AMO}\right)^{-\frac{1}{P}}$ Proposed
HM	$\widehat{K}3_B^{HMO} = \sqrt{\widehat{K}1_B^{HMO}}$ Proposed	$\widehat{K}3_B^{HMR} = \left(\widehat{K}3_B^{HMO}\right)^{-1}$ Proposed	$\widehat{K}3_B^{HMSR} = \left(\widehat{K}3_B^{HMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{HMRSR} = \left(\widehat{K}3_B^{HMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{HMPR} = \left(\widehat{K}3_B^{HMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{HMRPR} = \left(\widehat{K}3_B^{HMO}\right)^{-\frac{1}{P}}$ Proposed
GM	$\widehat{K}3_B^{GMO} = \sqrt{\widehat{K}1_B^{GMO}}$ Proposed	$\widehat{K}3_B^{GMR} = \left(\widehat{K}3_B^{GMO}\right)^{-1}$ Proposed	$\widehat{K}3_B^{GMSR} = \left(\widehat{K}3_B^{GMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{GMRSR} = \left(\widehat{K}3_B^{GMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{GMPR} = \left(\widehat{K}3_B^{GMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{GMRPR} = \left(\widehat{K}3_B^{GMO}\right)^{-\frac{1}{P}}$ Proposed
M	$\widehat{K}3_B^{MO} = \sqrt{\widehat{K}1_B^{MO}}$ Proposed	$\widehat{K}3_B^{MR} = \left(\widehat{K}3_B^{MO}\right)^{-1}$ Proposed	$\widehat{K}3_B^{MSR} = \left(\widehat{K}3_B^{MO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{MRSR} = \left(\widehat{K}3_B^{MO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{MPR} = \left(\widehat{K}3_B^{MO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{MRPR} = \left(\widehat{K}3_B^{MO}\right)^{-\frac{1}{P}}$ Proposed
MR	$\widehat{K}3_B^{MRO} = \sqrt{\widehat{K}1_B^{MRO}}$ Proposed	$\widehat{K}3_B^{MRR} = \left(\widehat{K}3_B^{MRO}\right)^{-1}$ Proposed	$\widehat{K}3_B^{MRSR} = \left(\widehat{K}3_B^{MRO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{MRRSR} = \left(\widehat{K}3_B^{MRO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}3_B^{MRPR} = \left(\widehat{K}3_B^{MRO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}3_B^{MRRPR} = \left(\widehat{K}3_B^{MRO}\right)^{-\frac{1}{P}}$ Proposed

Table 4. Summary of different forms and various types of Batach et al. (2008) for reciprocal of square root kind (RSR)

	Various Types of K					
FORMS	O	R	SR	RSR	PR	RPR
FM1	$\widehat{K}4_B^{FM1O} = \sqrt{\widehat{K}2_B^{FM1O}}$ Proposed	$\widehat{K}4_B^{FM1R} = \left(\widehat{K}4_B^{FM1O}\right)^{-1}$ Proposed	$\widehat{K}4_B^{FM1SR} = \left(\widehat{K}4_B^{FM1O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{FM1RSR} = \left(\widehat{K}4_B^{FM1O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{FM1PR} = \left(\widehat{K}4_B^{FM1O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{FM1RPR} = \left(\widehat{K}4_B^{FM1O}\right)^{-\frac{1}{P}}$ Proposed
FM2	$\widehat{K}4_B^{FM2O} = \sqrt{\widehat{K}2_B^{FM2O}}$ Proposed	$\widehat{K}4_B^{FM2R} = \left(\widehat{K}4_B^{FM2O}\right)^{-1}$ Proposed	$\widehat{K}4_B^{FM2SR} = \left(\widehat{K}4_B^{FM2O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{FM2RSR} = \left(\widehat{K}4_B^{FM2O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{FM2PR} = \left(\widehat{K}4_B^{FM2O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{FM2RPR} = \left(\widehat{K}4_B^{FM2O}\right)^{-\frac{1}{P}}$ Proposed
FM3	$\widehat{K}4_B^{FM3O} = \sqrt{\widehat{K}2_B^{FM3O}}$ Proposed	$\widehat{K}4_B^{FM3R} = \left(\widehat{K}4_B^{FM3O}\right)^{-1}$ Proposed	$\widehat{K}4_B^{FM3SR} = \left(\widehat{K}4_B^{FM3O}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{FM3RSR} = \left(\widehat{K}4_B^{FM3O}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{FM3PR} = \left(\widehat{K}4_B^{FM3O}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{FM3RPR} = \left(\widehat{K}4_B^{FM3O}\right)^{-\frac{1}{P}}$ Proposed
VM	$\widehat{K}4_B^{VMO} = \sqrt{\widehat{K}2_B^{VMO}}$ Proposed	$\widehat{K}4_B^{VMR} = \left(\widehat{K}4_B^{VMO}\right)^{-1}$ Proposed	$\widehat{K}4_B^{VMSR} = \left(\widehat{K}4_B^{VMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{VMRSR} = \left(\widehat{K}4_B^{VMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{VMPR} = \left(\widehat{K}4_B^{VMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{VMRPR} = \left(\widehat{K}4_B^{VMO}\right)^{-\frac{1}{P}}$ Proposed
AM	$\widehat{K}4_B^{AMO} = \sqrt{\widehat{K}2_B^{AMO}}$ Proposed	$\widehat{K}4_B^{AMR} = \left(\widehat{K}4_B^{AMO}\right)^{-1}$ Proposed	$\widehat{K}4_B^{AMSR} = \left(\widehat{K}4_B^{AMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{AMRSR} = \left(\widehat{K}4_B^{AMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{AMPR} = \left(\widehat{K}4_B^{AMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{AMRPR} = \left(\widehat{K}4_B^{AMO}\right)^{-\frac{1}{P}}$ Proposed
HM	$\widehat{K}4_B^{HMO} = \sqrt{\widehat{K}2_B^{HMO}}$ Proposed	$\widehat{K}4_B^{HMR} = \left(\widehat{K}4_B^{HMO}\right)^{-1}$ Proposed	$\widehat{K}4_B^{HMSR} = \left(\widehat{K}4_B^{HMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{HMRSR} = \left(\widehat{K}4_B^{HMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{HMPR} = \left(\widehat{K}4_B^{HMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{HMRPR} = \left(\widehat{K}4_B^{HMO}\right)^{-\frac{1}{P}}$ Proposed
GM	$\widehat{K}4_B^{GMO} = \sqrt{\widehat{K}2_B^{GMO}}$ Proposed	$\widehat{K}4_B^{GMR} = \left(\widehat{K}4_B^{GMO}\right)^{-1}$ Proposed	$\widehat{K}4_B^{GMSR} = \left(\widehat{K}4_B^{GMO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{GMRSR} = \left(\widehat{K}4_B^{GMO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{GMPR} = \left(\widehat{K}4_B^{GMO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{GMRPR} = \left(\widehat{K}4_B^{GMO}\right)^{-\frac{1}{P}}$ Proposed
M	$\widehat{K}4_B^{MO} = \sqrt{\widehat{K}2_B^{MO}}$ Proposed	$\widehat{K}4_B^{MR} = \left(\widehat{K}4_B^{MO}\right)^{-1}$ Proposed	$\widehat{K}4_B^{MSR} = \left(\widehat{K}4_B^{MO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{MRSR} = \left(\widehat{K}4_B^{MO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{MPR} = \left(\widehat{K}4_B^{MO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{MRPR} = \left(\widehat{K}4_B^{MO}\right)^{-\frac{1}{P}}$ Proposed
MR	$\widehat{K}4_B^{MRO} = \sqrt{\widehat{K}2_B^{MRO}}$ Proposed	$\widehat{K}4_B^{MRR} = \left(\widehat{K}4_B^{MRO}\right)^{-1}$ Proposed	$\widehat{K}4_B^{MRSR} = \left(\widehat{K}4_B^{MRO}\right)^{\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{MRRSR} = \left(\widehat{K}4_B^{MRO}\right)^{-\frac{1}{2}}$ Proposed	$\widehat{K}4_B^{MRPR} = \left(\widehat{K}4_B^{MRO}\right)^{\frac{1}{P}}$ Proposed	$\widehat{K}4_B^{MRRPR} = \left(\widehat{K}4_B^{MRO}\right)^{-\frac{1}{P}}$ Proposed

Table 5. Summary of different forms and various types of Batach et al. (2008) for P^{th} root kind (PR)

Various Types of K						
FORMS	O	R	SR	RSR	PR	RPR
FM1	$\hat{K}5_B^{FM1O} = (\hat{K}1_B^{FM1O})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{FM1R} = (\hat{K}5_B^{FM1O})^{-1}$ Proposed	$\hat{K}5_B^{FM1SR} = (\hat{K}5_B^{FM1O})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{FM1RSR} = (\hat{K}5_B^{FM1O})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{FM1PR} = (\hat{K}5_B^{FM1O})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{FM1RPR} = (\hat{K}5_B^{FM1O})^{-\frac{1}{P}}$ Proposed
FM2	$\hat{K}5_B^{FM2O} = (\hat{K}1_B^{FM2O})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{FM2R} = (\hat{K}5_B^{FM2O})^{-1}$ Proposed	$\hat{K}5_B^{FM2SR} = (\hat{K}5_B^{FM2O})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{FM2RSR} = (\hat{K}5_B^{FM2O})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{FM2PR} = (\hat{K}5_B^{FM2O})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{FM2RPR} = (\hat{K}5_B^{FM2O})^{-\frac{1}{P}}$ Proposed
FM3	$\hat{K}5_B^{FM3O} = (\hat{K}1_B^{FM3O})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{FM3R} = (\hat{K}5_B^{FM3O})^{-1}$ Proposed	$\hat{K}5_B^{FM3SR} = (\hat{K}5_B^{FM3O})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{FM3RSR} = (\hat{K}5_B^{FM3O})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{FM3PR} = (\hat{K}5_B^{FM3O})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{FM3RPR} = (\hat{K}5_B^{FM3O})^{-\frac{1}{P}}$ Proposed
VM	$\hat{K}5_B^{VMO} = (\hat{K}1_B^{VMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{VMR} = (\hat{K}5_B^{VMO})^{-1}$ Proposed	$\hat{K}5_B^{VMSR} = (\hat{K}5_B^{VMO})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{VMRSR} = (\hat{K}5_B^{VMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{VMPR} = (\hat{K}5_B^{VMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{VMRPR} = (\hat{K}5_B^{VMO})^{-\frac{1}{P}}$ Proposed
AM	$\hat{K}5_B^{AMO} = (\hat{K}1_B^{AMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{AMR} = (\hat{K}5_B^{AMO})^{-1}$ Proposed	$\hat{K}5_B^{AMSR} = (\hat{K}5_B^{AMO})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{AMRSR} = (\hat{K}5_B^{AMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{AMPR} = (\hat{K}5_B^{AMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{AMRPR} = (\hat{K}5_B^{AMO})^{-\frac{1}{P}}$ Proposed
HM	$\hat{K}5_B^{HMO} = (\hat{K}1_B^{HMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{HMR} = (\hat{K}5_B^{HMO})^{-1}$ Proposed	$\hat{K}5_B^{HMSR} = (\hat{K}5_B^{HMO})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{HMRSR} = (\hat{K}5_B^{HMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{HMPR} = (\hat{K}5_B^{HMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{HMRPR} = (\hat{K}5_B^{HMO})^{-\frac{1}{P}}$ Proposed
GM	$\hat{K}5_B^{GMO} = (\hat{K}1_B^{GMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{GMR} = (\hat{K}5_B^{GMO})^{-1}$ Proposed	$\hat{K}5_B^{GMSR} = (\hat{K}5_B^{GMO})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{GMRSR} = (\hat{K}5_B^{GMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{GMPR} = (\hat{K}5_B^{GMO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{GMRPR} = (\hat{K}5_B^{GMO})^{-\frac{1}{P}}$ Proposed
M	$\hat{K}5_B^{MO} = (\hat{K}1_B^{MO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{MR} = (\hat{K}5_B^{MO})^{-1}$ Proposed	$\hat{K}5_B^{MSR} = (\hat{K}5_B^{MO})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{MRSR} = (\hat{K}5_B^{MO})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{MPR} = (\hat{K}5_B^{MO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{MRPR} = (\hat{K}5_B^{MO})^{-\frac{1}{P}}$ Proposed
MR	$\hat{K}5_B^{MRO} = (\hat{K}1_B^{MRO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{MRR} = (\hat{K}5_B^{MRO})^{-1}$ Proposed	$\hat{K}5_B^{MRSR} = (\hat{K}5_B^{MRO})^{\frac{1}{2}}$ Proposed	$\hat{K}5_B^{MRRSR} = (\hat{K}5_B^{MRO})^{-\frac{1}{2}}$ Proposed	$\hat{K}5_B^{MRPR} = (\hat{K}5_B^{MRO})^{\frac{1}{P}}$ Proposed	$\hat{K}5_B^{MRRPR} = (\hat{K}5_B^{MRO})^{-\frac{1}{P}}$ Proposed

Table 6. Summary of different forms and various types of Batach et al. (2008) for reciprocal of P^{th} root kind (RPR).

Various Types of K						
FORMS	O	R	SR	RSR	PR	RPR
FM1	$\hat{K}6_B^{FM1O} = (\hat{K}1_B^{FM1O})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{FM1R} = (\hat{K}6_B^{FM1O})^{-1}$ Proposed	$\hat{K}6_B^{FM1SR} = (\hat{K}6_B^{FM1O})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{FM1RSR} = (\hat{K}6_B^{FM1O})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{FM1PR} = (\hat{K}6_B^{FM1O})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{FM1RPR} = (\hat{K}6_B^{FM1O})^{-\frac{1}{P}}$ Proposed
FM2	$\hat{K}6_B^{FM2O} = (\hat{K}1_B^{FM2O})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{FM2R} = (\hat{K}6_B^{FM2O})^{-1}$ Proposed	$\hat{K}6_B^{FM2SR} = (\hat{K}6_B^{FM2O})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{FM2RSR} = (\hat{K}6_B^{FM2O})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{FM2PR} = (\hat{K}6_B^{FM2O})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{FM2RPR} = (\hat{K}6_B^{FM2O})^{-\frac{1}{P}}$ Proposed
FM3	$\hat{K}6_B^{FM3O} = (\hat{K}1_B^{FM3O})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{FM3R} = (\hat{K}6_B^{FM3O})^{-1}$ Proposed	$\hat{K}6_B^{FM3SR} = (\hat{K}6_B^{FM3O})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{FM3RSR} = (\hat{K}6_B^{FM3O})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{FM3PR} = (\hat{K}6_B^{FM3O})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{FM3RPR} = (\hat{K}6_B^{FM3O})^{-\frac{1}{P}}$ Proposed
VM	$\hat{K}6_B^{VMO} = (\hat{K}1_B^{VMO})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{VMR} = (\hat{K}6_B^{VMO})^{-1}$ Proposed	$\hat{K}6_B^{VMSR} = (\hat{K}6_B^{VMO})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{VMRSR} = (\hat{K}6_B^{VMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{VMPR} = (\hat{K}6_B^{VMO})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{VMRPR} = (\hat{K}6_B^{VMO})^{-\frac{1}{P}}$ Proposed
AM	$\hat{K}6_B^{AMO} = (\hat{K}1_B^{AMO})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{AMR} = (\hat{K}6_B^{AMO})^{-1}$ Proposed	$\hat{K}6_B^{AMSR} = (\hat{K}6_B^{AMO})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{AMRSR} = (\hat{K}6_B^{AMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{AMPR} = (\hat{K}6_B^{AMO})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{AMRPR} = (\hat{K}6_B^{AMO})^{-\frac{1}{P}}$ Proposed
HM	$\hat{K}6_B^{HMO} = (\hat{K}1_B^{HMO})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{HMR} = (\hat{K}6_B^{HMO})^{-1}$ Proposed	$\hat{K}6_B^{HMSR} = (\hat{K}6_B^{HMO})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{HMRSR} = (\hat{K}6_B^{HMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{HMPR} = (\hat{K}6_B^{HMO})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{HMRPR} = (\hat{K}6_B^{HMO})^{-\frac{1}{P}}$ Proposed
GM	$\hat{K}6_B^{GMO} = (\hat{K}1_B^{GMO})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{GMR} = (\hat{K}6_B^{GMO})^{-1}$ Proposed	$\hat{K}6_B^{GMSR} = (\hat{K}6_B^{GMO})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{GMRSR} = (\hat{K}6_B^{GMO})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{GMPR} = (\hat{K}6_B^{GMO})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{GMRPR} = (\hat{K}6_B^{GMO})^{-\frac{1}{P}}$ Proposed
M	$\hat{K}6_B^{MO} = (\hat{K}1_B^{MO})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{MR} = (\hat{K}6_B^{MO})^{-1}$ Proposed	$\hat{K}6_B^{MSR} = (\hat{K}6_B^{MO})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{MRSR} = (\hat{K}6_B^{MO})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{MPR} = (\hat{K}6_B^{MO})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{MRPR} = (\hat{K}6_B^{MO})^{-\frac{1}{P}}$ Proposed
MR	$\hat{K}6_B^{MRO} = (\hat{K}1_B^{MRO})^{\frac{-1}{P}}$ Proposed	$\hat{K}6_B^{MRR} = (\hat{K}6_B^{MRO})^{-1}$ Proposed	$\hat{K}6_B^{MRSR} = (\hat{K}6_B^{MRO})^{\frac{1}{2}}$ Proposed	$\hat{K}6_B^{MRRSR} = (\hat{K}6_B^{MRO})^{-\frac{1}{2}}$ Proposed	$\hat{K}6_B^{MRPR} = (\hat{K}6_B^{MRO})^{\frac{1}{P}}$ Proposed	$\hat{K}6_B^{MRRPR} = (\hat{K}6_B^{MRO})^{-\frac{1}{P}}$ Proposed

2.2. Classification-Based Ridge Parameter of Fayose and Ayinde (2019)

Recently, Fayose and Ayinde (2019) proposed a modified version of Batach et al. (2008) ridge parameter given as:

$$\hat{K}_{FAi} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\left\{\left[\left(\frac{\hat{\alpha}_i^4 \min(\lambda_i)}{4\hat{\sigma}^2}\right)+\left(\frac{6\hat{\alpha}_i^4 \min(\lambda_i)}{\hat{\sigma}^2}\right)\right]^{\frac{1}{2}} - \left(\frac{\hat{\alpha}_i^2 \min(\lambda_i)}{2\hat{\sigma}^2}\right)\right\} \quad (8)$$

This is also classified into different forms, various types and diverse kinds.

2.3. Criterion for Investigation

The performances of these ridge parameters are compared using the mean squared error (MSE). The mean squared error of OLS and Ridge Regression are given respectively as:

$$MSE(\hat{\alpha})_{OLS} = E\left(\beta - \hat{\beta}_{OLS}\right)' E\left(\beta - \hat{\beta}_{OLS}\right) = \hat{\sigma}^2 Trace(X'X)^{-1} = \hat{\sigma}^2 \sum_{i=1}^{p} \frac{1}{\lambda_i} \quad (9)$$

$$MSE(\hat{\alpha})_{Ridge} = E\left(\beta - \hat{\beta}_{Ridge}\right)' E\left(\beta - \hat{\beta}_{Ridge}\right) = \hat{\sigma}^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i+\hat{k})^2} + \hat{k}^2 \sum_{i=1}^{p} \frac{\hat{\alpha}_i}{(\lambda_i+\hat{k})^2} \quad (10)$$

where $\lambda_1, \lambda_2, \dots, \lambda_p$ are the eigenvalues of $X'X$, $\hat{k}$ is the estimator of the ridge parameter *k*, $\hat{\alpha}_i^2$ is the i^{th} element of the vector $\hat{\alpha} = Q'\hat{\beta}$ where Q is an orthogonal matrix whose column constitute the eigenvectors of $X'X$ matrix. The mean square errors (MSE) of the existing and the proposed estimators are compared with Cross Validation, Algama (2018) discussed explicitly and also suggested a modified approach to Cross Validation in ridge regression, and Least Absolute Shrinkage and Selection Operator (LASSO) by Tibshirani (1996). The MSE produced by the Ridge parameters are ranked in ascending order and the ones with rank less than or equal to five estimators were counted over six (6) levels of multicollinearity, and four levels of error variances.

3. SIMULATION STUDY

A Monte Carlo simulation was carried out to investigate the performances of these estimators, in accordance with the simulation procedure used by McDonald and Galarneau (1975), Wichern and Churchill (1978), Gibbons (1981) and Kibria (2003), Dorugade and Kashid (2010), Lukman and Ayinde (2017) and Fayose and Ayinde (2019). The equation to generate the explanatory variables is given as:

$$X_{ij} = \left(1-\rho^2\right)^{1/2} z_{ij} + \rho z_{ip} i = 1,2,\ldots,n,\ j = 1,2,\ldots,p \tag{11}$$

where Z_{ij} is independent standard normal distribution with mean zero and unit variance, ρ is the correlation between any two explanatory variables and p is the number of explanatory variables. The values of ρ were taken as 0.8, 0.9, 0.95, 0.99, 0.999 and 0.9999, respectively. In this study, the number of explanatory variable (p) was taken to be three (3) and seven (7) respectively.

The response variable is defined as:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \ldots + \beta_p X_p + \varepsilon_i \tag{12}$$

where $\varepsilon_i \sim \left(0,\sigma^2\right)$. The values of β were chosen such that $\beta'\beta=1$. The sample sizes used are 10, 20, 30, 40 and 50. Four different values of σ used are 0.5, 1, 5 and 10. The experiment is repeated 1000 times. The estimated MSE is calculated as

$$MSE\left(\hat{\beta}\right) = \frac{1}{1000}\sum_{i=1}^{p}\sum_{j=1}^{1000}\left(\hat{\beta}_{ij} - \beta_i\right)^2 \tag{13}$$

where $\hat{\beta}_{ij}$ denotes the estimate of the i^{th} parameter in j^{th} replication and β_i is the true parameter values. The simulation results are presented in Table 7 and Table 8. This is also supported by Figures 1 and 2.

Table 7. Frequency of the efficiency of some best performing ridge parameters based on Batach et al. (2008) Estimator, $\hat{K}_{Bi} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\left\{\left[\left(\frac{\hat{\alpha}_i^4\lambda_i}{4\hat{\sigma}^2}\right)+\left(\frac{6\hat{\alpha}_i^4\lambda_i}{\hat{\sigma}^2}\right)\right]^{\frac{1}{2}} - \left(\frac{\hat{\alpha}_i^2\lambda_i}{2\hat{\sigma}^2}\right)\right\}$

Diverse Kinds	Different Forms	Various Types	Methods	P=3						P=7					
				10	20	30	40	50	Total	10	20	30	40	50	Total
Original	Arithmetic Mean	Pth Root	KOAMPR*	9	9	9	9	9	45	9	9	9	9	9	45
Original	Harmonic Mean	Pth Root	KOHMPR*	5	6	6	6	6	29	5	6	6	6	6	29
Original	Fixed Maximum 3	Square Root	KOFM3SR*	2	2	2	2	2	10	2	2	2	2	2	10

Table 7 (Continue). Frequency of the efficiency of some best performing ridge parameters based on Batach et al. (2008) Estimator, $\hat{K}_{Bi} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\left\{\left[\left(\frac{\hat{\alpha}_i^4\lambda_i}{4\hat{\sigma}^2}\right)+\left(\frac{6\hat{\alpha}_i^4\lambda_i}{\hat{\sigma}^2}\right)\right]^{\frac{1}{2}}-\left(\frac{\hat{\alpha}_i^2\lambda_i}{2\hat{\sigma}^2}\right)\right\}$

Original	Geometric Mean	Square Root	KOGMS*	5	6	6	6	6	29	5	6	6	6	6	29
Original	Harmonic Mean	Original	KOHMO*	6	7	7	7	7	34	7	7	7	7	7	35
Original	Fixed Maximum 1	Original	KOFM1O*	11	11	11	11	11	55	11	11	11	11	11	55
Original	Median	Pth Root	KOMEP*	5	7	7	6	6	31	5	7	7	7	7	33
Pth Root	Arithmetic Mean	Pth Root	KPRAMP*	6	7	7	7	7	34	7	7	7	7	7	35
Pth Root	Arithmetic Mean	Square Root	KPRAMSR*	3	3	3	3	3	15	3	3	3	3	3	15
Reciprocal	Arithmetic Mean	Square Root	KRAMS*	2	2	2	2	2	10	2	2	2	2	2	10
Reciprocal	Fixed Maximum 3	Square Root	KRFM3S*	3	3	3	3	3	15	3	3	3	3	3	15
Reciprocal	Geometric Mean	Pth Root	KRGMPR*	6	7	7	7	7	34	7	7	7	7	7	35
Reciprocal of Pth Root	Harmonic Mean	Reciprocal	KRPRHMR*	3	3	3	3	3	15	3	3	3	3	3	15
Reciprocal of Square Root	Fixed Maximum 2	Pth Root	KRSRFM2P*	5	6	6	6	6	29	5	6	6	6	6	29
Reciprocal of Square Root	Fixed Maximum 3	Pth Root	KRSRFM3P*	5	6	6	6	6	29	5	6	6	6	6	29
Reciprocal of Square Root	Median	Reciprocal	KRSRMER*	6	7	7	7	7	34	7	7	7	7	7	35
Square Root	Fixed Maximum 3	Original	KSRFM3O*	3	3	3	3	3	15	3	3	3	3	3	15
Square Root	Fixed Maximum 4	Square Root	KSRFM3S*	2	2	2	2	2	10	2	2	2	2	2	10
Square Root	Harmonic Mean	Original	KSRHMO*	2	2	2	2	2	10	2	2	2	2	2	10

Note: Estimators with asterisk are proposed estimators.

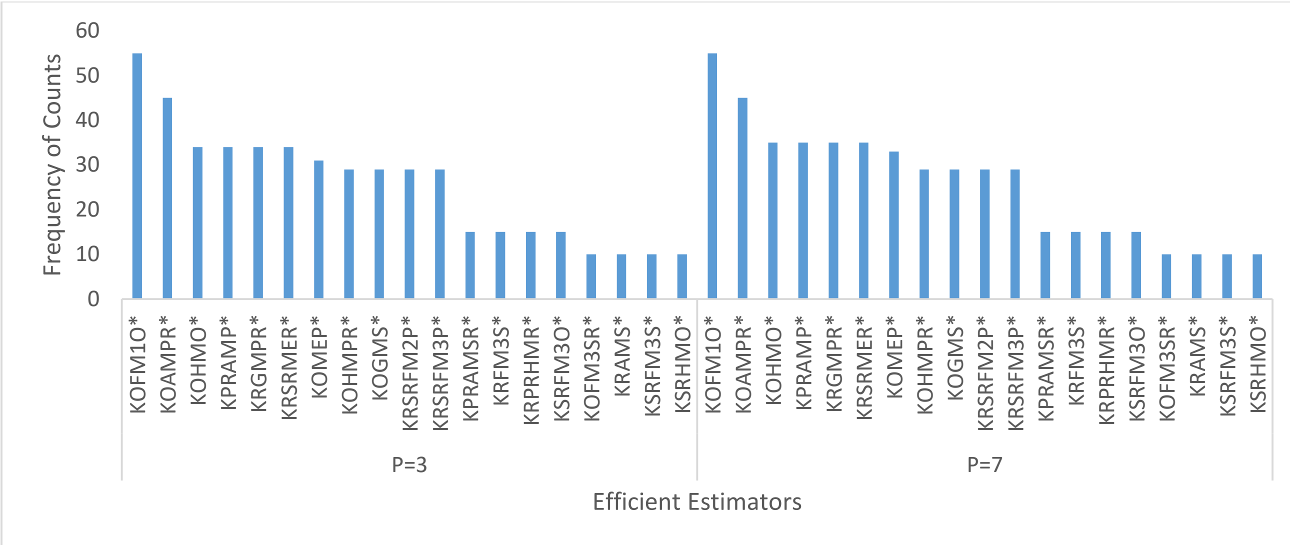

Figure 1. Number of counts at which MSE is minimum (Rank ≤ 5) for different estimators of diverse kinds, various types and different forms of estimators based on Batach et al. (2008).

The three best estimators are expressed mathematically as follows:

$$\hat{K}_B^{OFM1O} = \frac{\hat{\sigma}^2}{\max(\hat{\alpha}_i^2)} \left\{ \left[\left(\frac{\max(\hat{\alpha}_i^4)\max(\lambda_i)}{4\hat{\sigma}^2} \right) + \left(\frac{6\max(\hat{\alpha}_i^4)\max(\lambda_i)}{\hat{\sigma}^2} \right) \right]^{\frac{1}{2}} - \left(\frac{\max(\hat{\alpha}_i^2)\max(\lambda_i)}{2\hat{\sigma}^2} \right) \right\} \quad (14)$$

$$\hat{K}_B^{OAMPR} = \left(\frac{1}{p} \sum_{i=1}^{p} \hat{k}_{B_i} \right)^{\frac{1}{p}} \quad (15)$$

$$\hat{K}_B^{OHMO} = \frac{p}{\sum_{i=1}^{p} \hat{k}_{B_i}^{-1}} \quad (16)$$

where $\hat{k}_{Bi}$ is given in equation (3).

Table 8. Frequency of the efficiency of some best performing ridge parameters based on Fayose and Ayinde (2019), $\hat{K}_{FAi} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\left\{\left[\left(\frac{\hat{\alpha}_i^4 \min(\lambda_i)}{4\hat{\sigma}^2}\right)+\left(\frac{6\hat{\alpha}_i^4 \min(\lambda_i)}{\hat{\sigma}^2}\right)\right]^{\frac{1}{2}} - \left(\frac{\hat{\alpha}_i^2 \min(\lambda_i)}{2\hat{\sigma}^2}\right)\right\}$

Diverse Kinds	Different Forms	Various Types	Methods	P=3						P=7					
				10	20	30	40	50	Total	10	20	30	40	50	Total
Original	Arithmetic Mean	Pth Root	KOAMPR*	2	1	2	2	2	9	2	1	2	2	2	9
Pth Root	Arithmetic Mean	Original	KPRAMO*	4	6	6	6	6	28	4	6	6	6	6	28
Pth Root	Geometric Mean	Original	KPRMRO*	4	5	7	7	7	30	4	5	7	7	7	30
Reciprocal of Pth Root	Geometric Mean	Pth Root	KRPRGMP*	2	3	3	3	3	14	4	5	2	2	2	15
Reciprocal of Pth Root	Harmonic Mean	Pth Root	KRPRMEP*	4	5	2	2	2	15	4	5	2	2	2	15
Reciprocal of Pth Root	Varying Maximum	Original	KRPRVMO*	3	3	2	2	2	12	3	3	2	2	2	12
Reciprocal of Square Root	Arithmetic Mean	Original	KRSRAMO*	8	9	9	9	9	44	7	10	10	10	10	47
Reciprocal of Square Root	Arithmetic Mean	Square Root	KRSRAMS*	4	10	10	10	10	44	8	9	9	9	9	44
Reciprocal of Square Root	Mid-Range	Original	KRSRMRO*	7	8	8	8	8	39	7	8	8	8	8	39
Reciprocal of Square Root	Varying Maximum	Original	KRSRVMO*	7	7	7	7	7	35	7	7	7	7	7	35
Reciprocal of Square Root	Varying Maximum	Pth Root	KRSRVMP*	3	3	2	2	2	12	3	3	2	2	2	12
Square Root	Arithmetic Mean	Reciprocal of Pth Root	KSRAMRP*	1	4	2	2	2	11	1	4	2	2	2	11
Square Root	Geometric Mean	Reciprocal of Pth Root	KSRGMRP*	2	3	3	3	3	14	3	3	3	3	3	15
Square Root	Median	Reciprocal of Pth Root	KSRMERP*	4	5	2	2	2	15	3	3	3	3	3	15
Square Root	Mid-Range	Reciprocal of Pth Root	KSRMRRP*	1	2	2	2	2	9	1	2	2	2	2	9
SquareRoot	Varying Maximum	Reciprocal of Pth Root	KSRVMRP*	2	3	2	2	2	11	2	3	2	2	2	11

Note: Estimators with asterisk are proposed estimators.

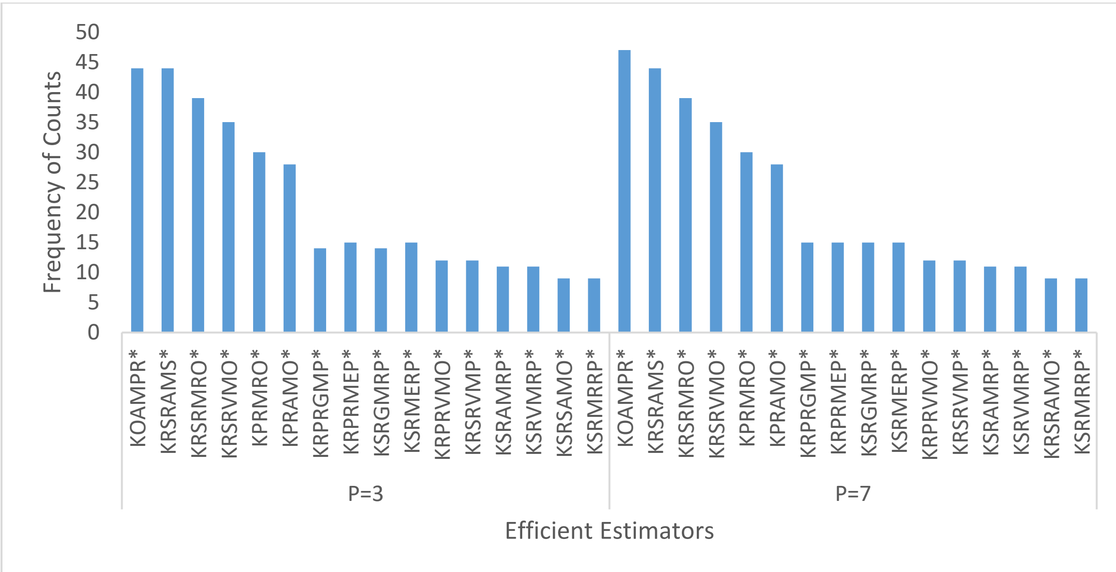

Figure 2. Number of counts at which MSE is minimum (Rank ≤ 5) for different estimators of diverse kinds, various types and different forms of estimators based on Fayose and Ayinde (2019)

Three best estimators are expressed mathematically as follows:

$$\hat{K}_{FA}^{OAMPR} = \left[\frac{1}{p}\sum_{i=1}^{p}\hat{k}_{FA_i}\right]^{\frac{1}{P}} \tag{17}$$

$$\hat{K}_{FA}^{RSRAMS} = \left[\frac{1}{p}\sum_{i=1}^{p}\hat{k}_{FA_i}^{\frac{1}{2}}\right]^{-\frac{1}{2}} \tag{18}$$

$$\hat{K}_{FA}^{RSRMRO} = \left[\frac{\max\left(\hat{k}_{FA_i}\right)+\min\left(\hat{k}_{FA_i}\right)}{2}\right]^{-\frac{1}{2}} \tag{19}$$

where $\hat{k}_{FAi}$ is as expressed in (4).

The frequency of the efficiency of ridge parameters over the levels of multicollinearity and error variance is summarized in Table 7 and 8. From Table 7, all the best methods in p=7 are also best in p=3. As it is seen, KOFM1O is best in both p=3 and p=7. Other best techniques are: KOAMPR, KOHMO, KRGMPR, KRSRMER, KPRAMP, KOMEP, KOHMPR, KOGMS and KRSRFM2P in their order.

However, from Table 8, the best methods based on Fayose and Ayinde (2019) are KOAMPR, KRSRAMS, KRSRMRO, KRSRVMO, KPRMRO, KPRAMO, KSRMERP, KRPRMEP, KSRGMRP and KRPRGMP in that order. All these are newly proposed techniques.

Examining their overall performances, all the proposed techniques of estimating biasing parameters are compared with ones in existence, including cross validation and LASSO. The performance of the best estimator is summarized in Table 9.

Table 9. Overall performance of the ridge parameter estimators

Ridge Parameter	Kind	Form	Type	Method	P=3						P=7						Total	Rank
					10	20	30	40	50	Total	10	20	30	40	50	Total		
Cross Validation				CV	24	12	12	11	10	69	24	10	11	12	11	68	137	1
Fayose and Ayinde (2019)	Generalized			FAYOSE AND AYINDE	24	7	11	6	5	53	8	11	11	11	11	52	105	2
Fayose and Ayinde (2019)	Original	Arithmetic Mean	Pth Root	KOAMPR	6	11	11	11	11	50	0	5	9	16	19	49	99	3
Batach *et.al(2008)*	Original	Fixed Maximum 1	Original	KOFM1O	24	0	7	6	8	45	5	10	10	10	10	45	90	4
Batach et.al(2008)	Original	Harmonic Mean	Pth Root	KOHMPR	24	1	6	6	5	42	6	9	9	9	9	42	84	5
Batach et.al(2008)	Original	Median	Pth Root	KOMEP	2	5	6	6	6	25	24	0	0	0	0	24	49	27
Fayose and Ayinde (2019)	Pth Root	Geometric Mean	Original	KPRMRO	0	3	6	10	15	34	4	8	7	7	7	33	67	9
Fayose and Ayinde (2019)	Reciprocal of Pth Root	Geometric Mean	Pth Root	KRPRGMP	1	8	8	8	8	33	4	7	7	7	7	32	65	10
Fayose and Ayinde (2019)	Reciprocal of Pth Root	Harmonic Mean	Pth Root	KRPRMEP	4	7	7	7	7	32	3	7	7	7	7	31	63	11.5
Batach et.al(2008)	Reciprocal of Pth Root	Varying Maximum	Original	KRPRVMO	2	7	7	7	7	30	3	7	7	7	7	31	61	13.5
Fayose and Ayinde (2019)	Reciprocal of Square Root	Mid - Range	Original	KRSRMRO	1	9	10	10	10	40	24	5	6	4	2	41	81	6

Table 9 (Continue). Overall performance of the ridge parameter estimators

Fayose and Ayinde (2019)	Reciprocal of Square Root	Mid - Range	Square Root	KRSRMRS	14	1	3	4	4	26	2	5	6	6	6	25	51	24.5
Fayose and Ayinde (2019)	Reciprocal of Square Root	Varying Maximum	Original	KRSRVMO	4	9	9	9	9	40	2	8	8	8	8	34	74	7
Fayose and Ayinde (2019)	Reciprocal of Square Root	Varying Maximum	Original	KRSRVMO	1	6	6	6	6	25	24	0	0	0	0	24	49	27
LASSO				LASSO	24	3	2	1	0	30	3	7	7	7	7	31	61	13.5

Note: Bold font indicates proposed ridge parameter.

From Table 9, Cross Validation Performed best, especially, at small sample size. Generalized Fayose and Ayinde (2019) follows having its peak performance also at small sample size. Original kind of Fayose and Ayinde (2019) and Batach *et. al* (2008) outperformed other kinds, while the Reciprocal of Square Root Kind follows. Lasso also performed well when the sample size is small but when P=7. Hence, from Table 9, the overall best seven (7) Ridge parameter estimators are: Cross Validation, Fayose and Ayinde Generalized, Original Kind of the Arithmetic Mean Form of Pth Root Type (KOAMPR) of Fayose and Ayinde (2019), Original Kind of the Fixed Maxinmum 1 Form of Original Type (KOFM1O) of Batach *et. al* (2008), Original Kind of Harmonic Mean Form of Pth Root Type (KOHMPR) of Batach *et. al* (2008), Reciprocal of Square Root Kind of the Mid – Range Form of Original Type (KRSRMRO) of Fayose and Ayinde (2019) and Reciprocal of Square Root Kind of Varying Maximum Form of Original Type (KRSRVMO) of Fayose and Ayinde (2019).

However, the graphs that follow summarize the performances of the best Seven (7) performing estimators, Cross Validation, Fayose and Ayinde Generalized, Original Kind of the Arithmetic Mean Form of Pth Root Type (KOAMPR) of Fayose and Ayinde (2019), Original Kind of the Fixed Maximum 1 Form of Original Type (KOFM1O) of Batach *et. al* (2008), Original Kind of Harmonic Mean Form of Pth Root Type (KOHMPR) of Batach *et. al* (2008), Reciprocal of Square Root Kind of the Mid – Range Form of Original Type (KRSRMRO) of Fayose and Ayinde (2019) and Reciprocal of Square Root Kind of Varying Maximum Form of Original Type (KRSRVMO) of Fayose and Ayinde (2019) Ridge Parameter Estimators, as sample size, increase. It is classified based on the levels of multicollinearity (low and high) and at different values of error variances when p = 3 (since the graphs repeated themselves when p = 7).

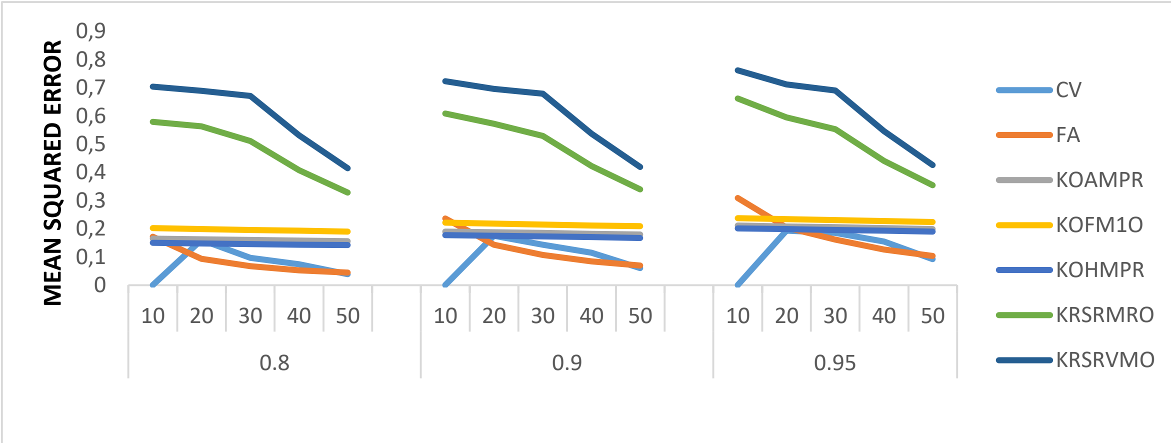

Figure 3. Performance of preferred estimators at low level of multicollinearity when error variance is 0.25 and p = 3

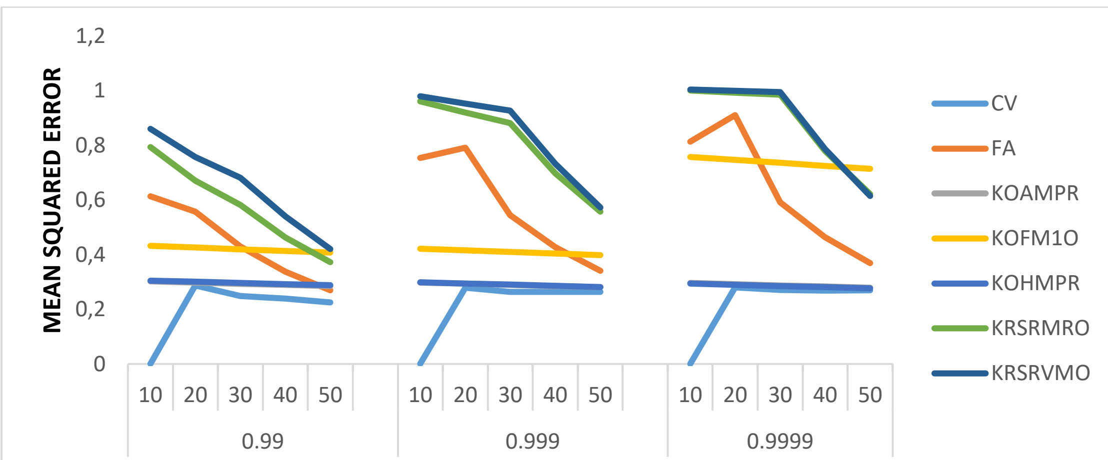

Figure 4. Performance of Preferred Estimators at high level of Multicollinearity when error variance is 0.25 and p = 3

For the two numerical examples, Table 10 shows the coefficients and MSE produce by best 7 estimators.

4. NUMERICAL EXAMPLE

In this section, we apply the new classification based biasing parameter to two real life datasets to support our findings. The computations were performed using R statistical software.

Example 1 (Longley Data)

To investigate the theoretical properties of the biasing parameters, we consider Longley (1967) dataset. The data are time series for the year 1947 to 1962 and consist of Y (number of people employed in thousands); X1 (Gross national product: implicit price deflator, making 1954 the reference year); X2 (Gross National Product in millions of Dollars); X3 (number of people unemployed in thousands); X4 (number of armed forces); X5 (noninstitutionalized population over 14 years of age); and X6 (year). The dataset has been used by several authors, examples of which are Gujarati (1995), Faraway (2002) and Ajiboye et al. (2016) and reported that the dataset suffers multicollinearity problem. It can be gotten as well from MASS library on R. The eigenvalues of X'X matrix are 666652990, 209073, 105355, 18039.76, 24.557 and 2.015117. this make the condition index to be 33076481 indicating severe multicollinearity

Example 2 (Portland Cement)

In this example, we used Portland Cement data used by Woods et al. (1932), Hald (1952), Hamaker (1962), Gorman and Toman (1966), Daniel and Wood (1980) and Nomura (1988). It has 5 variables, viz.Y is the heat evolved after 180 days of curing measured in calories per gram of cement, X_1 represents tricalcium aluminate, X_2 represent tricalcium silicate, X_3 represent tetracalcium aluminoferrite and X_4 represent β-dicalcium silicate. All these researchers made it clear that the dataset suffers multicollinearity. The eigenvalues of X'X matrix are: 164633.8587, 6020.2350, 707.5887 and 166.3175. Then its condition Index is 989.8769 which indicates multicollinearity.

Table 10. Mean square errors of seven (7) of the best ridge estimators for real life datasets

Ridge Parameter			Batach	Fayose and Ayinde (2019)	Batach et al. (2008)	*Batach et al (2008)*	Fayose and Ayinde (2019)	Fayose and Ayinde (2019)
Kinds, Forms and Types		CV	FA	KOAMPR	KOFM1O	KOHMPR	KRSRMRO	KRSRMRO
Longley Data	X1	-0.15477	-0.154777078	-0.154777078	-0.000650098	-0.154777078	-0.154777078	-0.234086401
	X2	-0.54223	-0.549383892	-0.549384304	-5.89E-06	-0.549384273	-0.549384304	-0.830894315
	X3	0.096899	0.845146003	0.845530625	1.50E-08	0.845501975	0.845530625	1.278788972
	X4	0.027831	1.011667255	1.013801119	3.87E-09	1.013641929	1.013801119	1.533282951
	X5	0.000896	10.54325606	6.50942999	1.13E-10	34.68049237	42.50942999	42.7542514
	X6	-2.08E-05	-0.145004189	-0.96516256	-1.17E-12	-1.870806219	-43.96516256	-57.70031899
	MSE	15.25483	13.16942	10.76854	12.0789	14.09808	19.4089	21.0146
Woods Data	X1	-0.06908	-0.069796197	-0.069798303	-0.069495217	-0.069795826	-0.069798303	-0.072125543
	X2	0.026441	0.03398772	0.03401576	0.030391962	0.033982789	0.03401576	0.03812515
	X3	0.187069	0.636538111	0.640997233	0.318516518	0.635759732	0.640997233	0.612214412
	X4	-0.03629	-0.399935179	-0.411767734	-0.078045724	-0.397922709	-0.411767734	-0.411767734
	MSE	5.753178	4.965087	7.544133	10.12542	13.88474	15.4321	16.95525

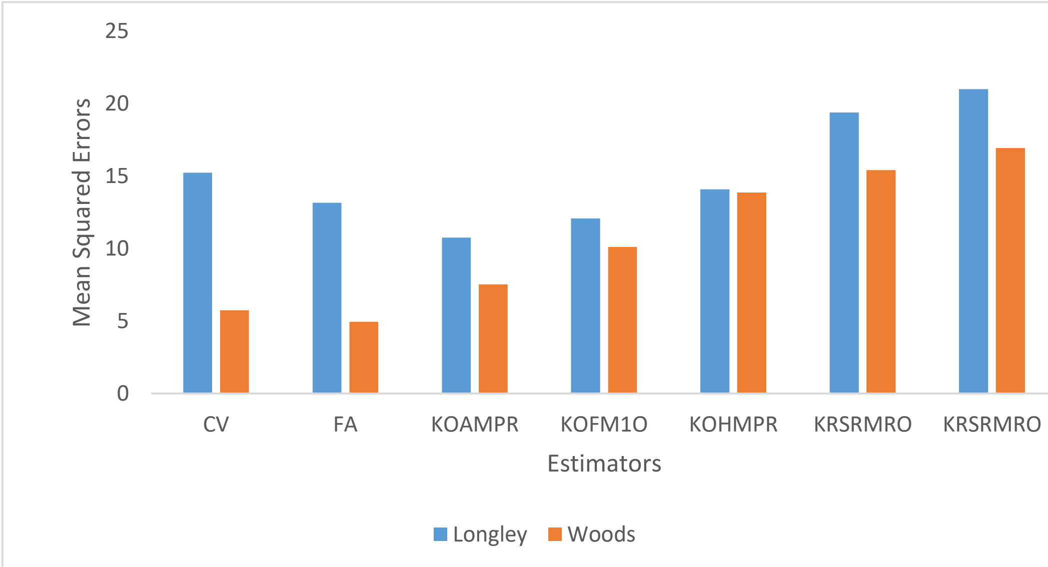

Figure 5. Performance of most efficient estimators in reallife datasets

The performances of the best performing biasing parameters are summarized in table 10. From Figure 5, Cross Validation and Fayose and Ayinde (2019) seem to perform best in the Real life situation for woods et al (1932) dataset, otherwise known as Portland Cement Dataset, but the estimates produced by KOAMPR has minimum MSE with Longley dataset. Therefore, some proposed biasing parameters are among the best ones and, their performances vary with respect to the sample sizes, number of coefficients and levels of multicollinearity.

5. CONCLUSION

In this study, ridge parameters proposed by Batach et al. (2008) and Fayose and Ayinde (2019) are classified into different forms, various types and diverse kinds following the idea of Lukman and Ayinde (2015), and some new ridge parameters are proposed. The performances of these estimators are evaluated through Monte Carlo Simulation, where levels of multicollinearity, sample sizes, number of regressors and error variances have been varied. The performance evaluation was done using the mean square error. Numerical examples were also used to demonstrate the theoretical properties of the biasing parameters. Some proposed estimators are among those that have the least minimum square error when compared to others.

ETHICAL DECLARATION

In the writing process of the study titled “Alternative Ridge Parameters in Linear Model”, there were followed the scientific, ethical and the citation rules; was not made any falsification on the collected data and this study was not sent to any other academic media for evaluation.

REFERENCES

Ajiboye S. A., Adewuyi, E., Ayinde, K. and Lukman, A. F. (2016), A comparative study of some robust ridge and liu estimators, *Science World Journal*, 11(4),16-20.

Alkhamisi, M. and Shukur, G. (2008), Developing ridge parameters for SUR model, *Communications in Statistics - Theory and Methods*, 37(4), 544-564.

Alkhamisi, M., Khalaf, G. and Shukur, G. (2006), Some modifications for choosing ridge parameters, *Communications in Statistics - Theory and Methods*, 35(11), 2005-2020.

Bhat, S. S. (2016), A comparative study on the performance of new ridge estimators, *Pakistan Journal of Statistics and Operation Research*, 12(2), 317-325.

Batach, F., Gore, S. and Verma, M. (2008), Effect of jackknifing on various ridge type estimators, *Model Assisted Statistics and Applications*, 3, 121-130.

Daniel, C. and Wood, F. S. (1980), *Fitting equations to data: Computer analysis of multifactor data*, 2*th* Edition, Wiley, Newyork.

Dempster, A. P., Schatzoff, M. and Wermuth, N. (1977), A Simulation study of alternatives to ordinary least squares, *Journal of the American Statistical Association*, 72(357), 77-91.

Dorugade, A. V. and Kashid, D. N. (2010), Alternative method for choosing ridge parameter for regression, *International Journal of Applied Mathematical Sciences*, 4(9), 447-456.

Faraway, J. (2002), *Practical Regression and Anova using R*, 109 – 113.

Fayose, T. S. and Ayinde, K. (2019), Different forms biasing parameter for generalized ridge regression estimator. *International Journal of Computer Applications*, 181(37), 21-29.

Ghadban, K. and Mohamed, I. (2014), Ridge regression and Ill-conditioning, *Journal of Modern Applied Statistical Methods*, 13(2), 355-363.

Gibbons, D. G. (1981), A Simulation study of some ridge estimators, *Journal of the American Statistical Association*, 76, 131-139.

Gorman, J. W. and Toman, R. J. (1966), Selection of variables for fitting equations to data, *Technometrics*, 8, 27–51.

Gujarati, D.N. (1995), Basic econometrics, (3rd Edition), Mcgraw-Hill, New York.

Hald, A. (1952), *Statistical theory with engineering applications*, Wiley, New York.

Hamaker, H. C. (1962), On multiple regression analysis, *Statistica Neerlandica*, 16, 31–56.

Hoerl, A.E. and Kennard, R.W. (1970), Ridge regression: Biased estimation for non-orthogonal problems, *Technometrics*, 12, 55-67.

Longley, J. W. (1967), An appraisal of least-squares programs from the point of view of the user, *Journal of the American Statistical Association*, 62, 819–841.

Khalaf, G. (2013), A Comparison between biased and unbiased estimators, *Journal of Modern Applied Statistical Methods*, 12(2), 293-303.

Khalaf, G. and Shukur, G. (2005), Choosing ridge parameters for regression problems, *Communications in Statistics - Theory and Methods*, 34, 1177-1182.

Kibria, B. M. G. (2003), Performance of some new ridge regression estimators, *Communications in Statistics - Simulation and Computation*, 32, 419-435.

Kibria, B. M. G. and Shipra, B. (2016), Some ridge regression estimators and their performances, *Journal of Modern Applied Statistical Methods*, 15(1), 206-231.

Lawless, J. F. and Wang, P. (1976), A simulation study of ridge and other regression estimators, *Communications in Statistics* A, 5, 307-323.

Lukman A. F. and Ayinde K. (2017), Review and classifications of the ridge parameter estimation techniques, *Hacettepe Journal of Mathematics and Statistics*, 46(5), 953-967.

Lukman, A. F (2015), *Review and classification of the ridge parameter estimation techniques*, Unpublished Thesis. PhD. Thesis, Ladoke Akintola University of Technology, Ogbomoso, Oyo State, Nigeria.

Mansson, K., Shukur, G. and Kibria, B. M. G. (2010), A simulation study of some ridge regression estimators under different distributional assumptions, *Communications in Statistics - Simulations and Computations*, 39(8), 1639 –1670.

McDonald, G. C. and Galarneau, D. I. (1975), A monte carlo evaluation of some ridge-type estimators, *Journal of the American Statistical Association*, 70, 407-416.

Muniz, G. and Kibria, B. M. G. (2009), On some ridge regression estimators: An empirical comparison, *Communications in Statistics-Simulation and Computation*, 38, 621-630.

Nomura, M. (1988), On the almost unbiased ridge regression estimator, *Communications in Statistics–Simulation and Computation*, 17, 729–743.

Wichern, D. and Churchill, G. (1978), A comparison of ridge estimators, *Technometrics*, 20, 301–311.

Woods, H., Steinour, H. H. and Starke, H. R. (1932), Effect of composition of Portland cement on heat evolved during hardening, *Industrial and Engineering Chemistry*, 24, 1207–1214.

Algamal, Z.Y. (2018), Shrinkage estimators for gamma regression model, *Electronic Journal of Applied Statistical Analysis*, 11(1), 253-268.

Algamal, Z.Y. (2018), Shrinkage parameter selection via modified cross-validation approach for ridge regression model, *Communications in Statistics - Simulation and Computation, doi: 10.1080/03610918.2018.1508704.*

Algamal, Z.Y. (2018), A new method for choosing the biasing parameter in ridge estimator for generalized linear model, *Chemometrics and Intelligent Laboratory Systems*, 183, 96-101.