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The Almost Hilbert-Smith Matrices on Gcd-closed Sets[1]

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Abstract: *Let* $S=\{x_1,x_2,...,x_n\}$ *be a set of positive integers and let* (x_i,x_j) *denote the greatest common divisor of* x_i *and* x_j. *The* $n\times n$ *matrix* $[S]=(s_{ij})$, *where* $s_{ij}=(x_i,x_j)/x_ix_j$, *is called the almost Hilbert-Smith matrix on S. In this paper we obtain the value of the determinant* $[S]=(s_{ij})$, *and calculate the inverse of* $[S]=(s_{ij})$ *when S is gcd-closed.*

Key Words: **The almost Hilbert-Smith matrix, the GCD matrix, gcd-closed set, factor closed set.**

En Büyük Ortak Bölen Kapalı Kümeler Üzerinde Hemen Hemen Hilbert-Smith Matrisleri

Özet: $S=\{x_1,x_2,...,x_n\}$ *elemanları pozitif tamsayılar olan bir küme olsun ve* (x_i,x_j), x_i *ve* x_j *tamsayılarının en büyük ortak bölenini göstersin. ij-yinci elemanı* $s_{ij}=(x_i,x_j)/x_ix_j$ *olan* $n\times n$ *tipinde* $[S]=(s_{ij})$ *matrisine, S kümesi üzerinde hemen hemen Hilbert-Smith matrisi denir. Bu çalışmada* $[S]=(s_{ij})$ *matrisinin determinantının değeri elde edilmiş ve S, en büyük ortak bölen kapalı olduğunda* $[S]=(s_{ij})$ *matrisinin tersi hesaplanmıştır.*

Anahtar Kelimeler: **Hemen hemen Hilbert-Smith matrisi, GCD matrisi, en büyük ortak bölen kapalı küme, çarpan kapalı küme.**

1. Introduction

Let $S=\{x_1,x_2,...,x_n\}$ be a set of distinct positive integers. The matrix $(S)=(s_{ij})$, where $s_{ij}=(x_i,x_j)$, the greatest common divisor of x_i and x_j, is called the greatest common divisor (GCD) matrix on S [1]. Beslin and Ligh initiated the study of GCD matrices in the direction of their

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structure, determinant, arithmetic in Z_n. Also they showed that $\det(S) = \phi(x_1)\phi(x_2)..\phi(x_n)$, where ϕ is Euler's totient function, if S is factor closed. A set S of positive integers is said to be factor closed (FC) if all positive factors of any element of S belong to S. In [2] Li calculated the determinant of the GCD matrix on S when S is not factor closed.

Then Beslin and Ligh [3] showed that the determinant of the GCD matrix on a gcd-closed set $S = \{x_1, x_2, \dots, x_n\}$ is $B(x_1)B(x_2)..B(x_n)$, where B is an arithmetical function defined on S as

$$B(x_i) = \sum_{\substack{d|x_i \\ d\nmid x_j \\ j<i}} \phi(d).$$

A set $S = \{x_1, x_2, \dots, x_n\}$ of positive integers is greatest common divisor closed (gcd-closed) if for every $i, j = 1,2,\dots,n$, (x_i, x_j) is in S. Also, Beslin and Ligh calculated the determinant of the GCD matrix on S when S is not gcd-closed. Furthermore, Bourque and Ligh calculated the inverse of the GCD matrix on S if S is gcd-closed [4].

Let f be a multiplicative function and let $S = \{x_1, x_2, \dots, x_n\}$ be factor closed. An arithmetical function f is called multiplicative if f is not identically zero and if $f(ab) = f(a)f(b)$ whenever $(a,b) = 1$. Denote by $f([x_i, x_j])$ the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij-entry. In [6] Bourque and Ligh calculated the determinant of $f([x_i, x_j])$. Also they obtained the inverse of $f([x_i, x_j])$ if $f([x_i, x_j])$ is invertible.

In this paper, we give a structure theorem for the almost Hilbert-Smith matrix and calculate the determinant of the almost Hilbert-Smith matrix on S whether S is gcd-closed or not. Also we show that the almost Hilbert-Smith matrix is positive definite. Furthermore we calculate the inverse of the almost Hilbert-Smith matrix on S if S is gcd-closed. In the last section we compare our results with the results presented by Bourque and Ligh [6].

2. The Value of the Determinant of the Almost Hilbert-Smith Matrix

Definition 1. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers and let (x_i, x_j) denote the greatest common divisor of x_i and x_j. The $n \times n$ matrix $[S] = (s_{ij})$, where $s_{ij} = (x_i, x_j)/x_i x_j$, is called the almost Hilbert-Smith matrix on S.

It is obvious that the almost Hilbert-Smith matrix on $S = \{x_1, x_2, \dots, x_n\}$ is symmetric and rearrangements of the elements of S yield similar matrices. Hence, we may assume $x_1 < x_2 < \dots < x_n$. Throughout this paper, $S = \{x_1, x_2, \dots, x_n\}$ denotes an ordered set of distinct positive integers such that $x_1 < x_2 < \dots < x_n$.

Definition 2. A set S of positive integers is said to be factor closed (FC) if all positive factors of any element of S belong to S.

Definition 3. A set $S = \{x_1, x_2, \dots, x_n\}$ of positive integers is greatest common divisor closed (gcd-closed) if for every $i, j = 1,2,\dots,n$, (x_i, x_j) is in S.

Every factor closed set is gcd-closed, but not conversely.

It is clear that any set $S = \{x_1, x_2, \dots, x_n\}$ of positive integers is contained in a gcd-closed set. By $\overline{S}$ we mean the minimal such gcd-closed set, or gcd-closure of S. It is obvious that $S \subseteq \overline{S}$, and $S = \overline{S}$ if and only if S is gcd-closed.

Let B be an arithmetical function on a set $S=\{x_1,x_2,\ldots,x_n\}$ of positive integers with $x_1<x_2<\ldots<x_n$ defined as

$$B(x_i)=\sum_{\substack{d|x_i\\ d\nmid x_j\\ j<i}}\phi(d), \tag{1}$$

where ϕ is Euler's totient function. For every $i,j=1,2,\ldots,n$,

$$(x_i,x_j)=\sum_{x_k|(x_i,x_j)}B(x_k) \tag{2}$$

if $S=\{x_1,x_2,\ldots,x_n\}$ is gcd-closed [3].

The following theorem describes the structure of the almost Hilbert-Smith matrix.

Theorem 1. Let $\overline{S}=\{y_1,y_2,\ldots,y_m\}$ be the gcd-closure of $S=\{x_1,x_2,\ldots,x_n\}$ with $x_1<x_2<\ldots<x_n$ and $y_1<y_2<\ldots<y_m$. Then the almost Hilbert-Smith matrix on S is the product of an $n\times m$ matrix R and an $m\times n$ matrix Q.

Proof: Let the $n\times m$ matrix $R=(r_{ij})$ and the matrix $Q=(q_{ij})$ defined as follows:

$$r_{ij}=\begin{cases}\dfrac{B(y_j)}{x_i} & \text{if } y_j|x_i,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$q_{ij}=\begin{cases}\dfrac{1}{x_j} & \text{if } y_i|x_j,\\ 0 & \text{otherwise.}\end{cases}$$

By (2) the ij-entry of RQ is equal to

$$(RQ)_{ij}=\sum_{k=1}^{m}r_{ik}q_{kj}=\sum_{\substack{y_k|x_i\\ y_k|x_j}}\frac{B(y_k)}{x_ix_j}=\frac{1}{x_ix_j}\sum_{y_k|(x_i,x_j)}B(y_k)=\frac{(x_i,x_j)}{x_ix_j}.$$

Then $[S]=RQ$. Thus the proof is complete. ∎

Let $R=(r_{ij})$ and $Q=(q_{ij})$ be as in Theorem 1. It is clear that $r_{ij}=q_{ji}B(y_j)$. If $\Delta=\mathrm{diag}(\delta_1,\delta_2,\ldots,\delta_m)$, where $\delta_i=B(y_i)$ for $i=1,2,\ldots,m$, is an $m\times m$ diagonal matrix, then the almost Hilbert-Smith matrix on S is written as $[S]=Q^T\Delta Q$. Also we define the $n\times m$ matrix $E=(e_{ij})$, where

$$e_{ij}=\begin{cases}1 & \text{if } y_j|x_i,\\ 0 & \text{otherwise,}\end{cases} \tag{3}$$

and the $n\times n$ matrix $D=\mathrm{diag}\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)$. It is clear that $Q^T=DE$. Then

$$[S]=RQ=Q^T\Delta Q=DE\Delta E^TD.$$

Theorem 2. Let S and $\overline{S}$ be as in Theorem 1. Then the determinant of the almost Hilbert-Smith matrix on S is

$$\det[S] = \sum_{1\le k_1<k_2<\ldots<k_n\le m} \left(\det Q^T_{(k_1,k_2,\ldots,k_n)}\right)^2 B(y_{k_1})B(y_{k_2})\ldots B(y_{k_n}),$$

where $Q^T_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of Q^T consisting of k_1th, k_2th, ..., k_nth columns of Q^T.

Proof: From Theorem 1 $[S]=RQ$. Now apply the Cauchy-Binet formula (see [5], p. 9) to obtain

$$\det[S] = \det(RQ) = \sum_{1\le k_1<k_2<\ldots<k_n\le m} \det R_{(k_1,k_2,\ldots,k_n)} \det Q^T_{(k_1,k_2,\ldots,k_n)}.$$

It is clear that

$$\det R_{(k_1,k_2,\ldots,k_n)} = \det Q^T_{(k_1,k_2,\ldots,k_n)} \det \Delta_{(k_1,k_2,\ldots,k_n)} = \det Q^T_{(k_1,k_2,\ldots,k_n)} B(y_{k_1})B(y_{k_2})\ldots B(y_{k_n}).$$

Then

$$\det[S] = \sum_{1\le k_1<k_2<\ldots<k_n\le m} \left(\det Q^T_{(k_1,k_2,\ldots,k_n)}\right)^2 B(y_{k_1})B(y_{k_2})\ldots B(y_{k_n}).$$

Thus the proof is complete. ∎

Corollary 1. Let S and $\overline{S}$ be as in Theorem 1. Then the determinant of the almost Hilbert-Smith matrix on S is

$$\det[S] = \frac{1}{x_1^2 x_2^2 \ldots x_n^2} \sum_{1\le k_1<k_2<\ldots<k_n\le m} \left(\det E_{(k_1,k_2,\ldots,k_n)}\right)^2 B(y_{k_1})B(y_{k_2})\ldots B(y_{k_n}),$$

where $E_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of $E=(e_{ij})$ consisting of k_1th, k_2th, ..., k_nth columns of $E=(e_{ij})$ given in (3).

Proof: By Theorem 2,

$$\det[S] = \sum_{1\le k_1<k_2<\ldots<k_n\le m} \left(\det Q^T_{(k_1,k_2,\ldots,k_n)}\right)^2 B(y_{k_1})B(y_{k_2})\ldots B(y_{k_n}).$$

It is clear that

$$\det Q^T_{(k_1,k_2,\ldots,k_n)} = \det D \det E_{(k_1,k_2,\ldots,k_n)} = \frac{1}{x_1 x_2 \ldots x_n} \det E_{(k_1,k_2,\ldots,k_n)},$$

since $Q^T = DE$. The result is immediate. ∎

Example 1. The almost Hilbert-Smith matrix on $S=\{4,6,8\}$ is

$$[S] = \begin{bmatrix} \frac{1}{4} & \frac{1}{12} & \frac{1}{8} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{24} \\ \frac{1}{8} & \frac{1}{24} & \frac{1}{8} \end{bmatrix}.$$

Since gcd-closure of S is $\overline{S}=\{2,4,6,8\}$, $E=(e_{ij})$ given in (3) is

$$E=\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&1&0&1\end{bmatrix}.$$

By Corollary 1,

$$\det[S]=\frac{1}{4^2.6^2.8^2}\left(\begin{vmatrix}1&1&0\\1&0&1\\1&1&0\end{vmatrix}^2 B(2)B(4)B(6)+\begin{vmatrix}1&1&0\\1&0&0\\1&1&1\end{vmatrix}^2 B(2)B(4)B(8)\right.$$

$$\left.+\begin{vmatrix}1&0&0\\1&1&0\\1&0&1\end{vmatrix}^2 B(2)B(6)B(8)+\begin{vmatrix}1&0&0\\0&1&0\\1&0&1\end{vmatrix}^2 B(4)B(6)B(8)\right).$$

Since

$B(2)=\phi(1)+\phi(2)=2$, $B(4)=\phi(4)=2$, $B(6)=\phi(3)+\phi(6)=4$, and $B(8)=\phi(8)=4$,

we have

$$\det[S]=\frac{5}{2304}.$$

Corollary 2. Let $[S]=(s_{ij})$ be the $n\times n$ almost Hilbert-Smith matrix on a set $S=\{x_1,x_2,\ldots,x_n\}$ of positive integers. Then $[S]=(s_{ij})$ is positive definite and invertible.

Proof: Let S and $\overline{S}$ be as in Theorem 1, and let $[S]=(s_{ij})$ be the $n\times n$ almost Hilbert-Smith matrix on S. Consider the matrix $[S_t]=(s_{ij})_{i,j=1}^{t}$, which is a submatrix of $[S]=(s_{ij})$ for every $t=1,2,\ldots,n$. It is clear that $[S_t]$ is the $t\times t$ almost Hilbert-Smith matrix on the set $S_t=\{x_1,x_2,\ldots,x_t\}\subset S$. $\overline{S}_t$, the gcd-closure of S_t, is a subset of $\overline{S}$ since $S_t\subset S$. Let $\overline{S}_t=\{y_{\alpha_1},y_{\alpha_2},\ldots,y_{\alpha_r}\}$, where $\{\alpha_1,\alpha_2,\ldots,\alpha_r\}\subset\{1,2,\ldots,m\}$ with $\alpha_1<\alpha_2<\ldots<\alpha_r$. By Corollary 1,

$$\det[S_t]=\frac{1}{x_1^2x_2^2\ldots x_t^2}\sum_{1\le k_1<k_2<\ldots<k_t\le r}\left(\det E_{(\alpha_{k_1},\alpha_{k_2},\ldots,\alpha_{k_t})}\right)^2 B\left(y_{\alpha_{k_1}}\right)B\left(x_{\alpha_{k_2}}\right)..B\left(x_{\alpha_{k_t}}\right)\tag{4}$$

for every $t=1,2,\ldots,n$. Since each summand in the right hand side of (4) is positive, $\det[S_t]>0$ for every $t=1,2,\ldots,n$. Thus $[S]=(s_{ij})$ is positive definite, and hence invertible. ■

3. The Inverse of the Almost Hilbert-Smith Matrix

In this section we calculate the inverse of the almost Hilbert-Smith matrix on S when S is gcd-closed.

Theorem 3. Let $S=\{x_1,x_2,\ldots,x_n\}$ be gcd-closed. Then the inverse of the almost Hilbert-Smith matrix $[S]=(s_{ij})$ is the matrix $B=(b_{ij})$ such that

$$b_{ij} = x_i x_j \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{B(x_k)} \sum_{\substack{dx_i | x_k \\ dx_i \nmid x_t \\ x_t < x_k}} \mu(d) \sum_{\substack{dx_j | x_k \\ dx_j \nmid x_t \\ x_t < x_k}} \mu(d),$$

where μ is Möbius function.

Proof: Let $Q = (q_{ij})$ be the $n \times n$ matrix defined in Theorem 1 and the $n \times n$ matrix $N = (n_{ij})$ be defined as follows:

$$n_{ij} = x_i \sum_{\substack{dx_i | x_j \\ dx_i \nmid x_t \\ x_t < x_j}} \mu(d).$$

Calculating the ij-entry of the product NQ gives

$$(NQ)_{ij} = \sum_{k=1}^{n} n_{ik} q_{kj} = \sum_{x_k | x_j} \frac{x_i}{x_j} \sum_{\substack{dx_i | x_k \\ dx_i \nmid x_t \\ x_t < x_k}} \mu(d) = \frac{x_i}{x_j} \sum_{d | \frac{x_j}{x_i}} \mu(d) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence $Q^{-1} = N$. If $\Delta = \text{diag}(B(x_1), B(x_2), \ldots, B(x_n))$ then $[S] = Q^T \Delta Q$. Therefore $[S]^{-1} = N\Delta^{-1}N^T = (b_{ij})$, where

$$b_{ij} = \left(N\Delta^{-1}N^T\right)_{ij} = \sum_{k=1}^{n} \frac{1}{B(x_k)} n_{ik} n_{jk} = x_i x_j \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{B(x_k)} \sum_{\substack{dx_i | x_k \\ dx_i \nmid x_t \\ x_t < x_k}} \mu(d) \sum_{\substack{dx_j | x_k \\ dx_j \nmid x_t \\ x_t < x_k}} \mu(d).$$

The proof is complete. ■

Example 2. The almost Hilbert-Smith matrix on $S = \{2,4,6\}$ is

$$[S] = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \end{bmatrix}.$$

$[S]$ is invertible, since $S = \{2,4,6\}$ is gcd-closed. Moreover, by Theorem 3

$b_{11} = 2.2\left(\frac{\mu(1)\mu(1)}{B(2)} + \frac{\mu(2)\mu(2)}{B(4)} + \frac{\mu(3)\mu(3)}{B(6)}\right) = 5$, $b_{12} = 2.4\left(\frac{\mu(2)\mu(1)}{B(4)}\right) = -4$,

$b_{13} = 2.6.\frac{\mu(3)\mu(1)}{B(6)} = -3$, $b_{22} = 4.4\left(\frac{\mu(1)\mu(1)}{B(4)}\right) = 8$, $b_{23} = 0$, $b_{33} = 6.6.\frac{\mu(1)\mu(1)}{B(6)} = 9$

Therefore, since $[S]^{-1} = B = (b_{ij})$ is symmetric we have

$$[S]^{-1} = \begin{bmatrix} 5 & -4 & -3 \\ -4 & 8 & 0 \\ -3 & 0 & 9 \end{bmatrix}.$$

4. Discussion

In this section, we compare our results with the results presented by Bourque and Ligh in [6].

Let f be a multiplicative function, and let $S = \{x_1, x_2, \ldots, x_n\}$ be factor closed. Denote by $f([x_i, x_j])$ the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij-entry. In [6] Bourque and Ligh calculated the determinant of $f([x_i, x_j])$ and also they obtained the inverse of $f([x_i, x_j])$ if S is invertible. If f is defined as $f(n) = 1/n$ for all $n \in Z^+$ then $f([x_i, x_j])$ becomes the $n \times n$ almost Hilbert-Smith matrix on S. For $f(n) = 1/n$, the statements of Theorem 2 in [6] are special cases of our results since every factor closed set is gcd-closed.

Let [S] be the $n \times n$ almost Hilbert-Smith matrix on $S = \{x_1, x_2, \ldots, x_n\}$. If S is factor closed then $B(x_i) = \phi(x_i)$ for every $i = 1, 2, \ldots, n$, and the matrix $E = (e_{ij})$ given in (3) is an $n \times n$ lower triangular matrix with diagonal $(1, 1, \ldots, 1)$. Thus, by Corollary 1,

$$\det[S] = \prod_{i=1}^{n} \frac{\phi(x_i)}{x_i^2}, \tag{5}$$

and by Theorem 3, the inverse of [S] is the matrix $B = (b_{ij})$, where

$$b_{ij} = x_i x_j \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{\phi(x_k)} \mu\left(\frac{x_k}{x_i}\right) \mu\left(\frac{x_k}{x_j}\right). \tag{6}$$

It should be noted that one can obtain (5) and (6) by taking $f(n) = 1/n$ in Theorem 2 of [6].

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