S.Ü. Fen-Edebiyat Fakültesi Fen Dergisi Sayı:19 (2002) 7-10 KONYA

On C-Continuous Functions

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Abstract: *E. Hatır and et. al introduced a new decomposition of continuity called C-continuity. S. Jafari investigated further this type of continuity. In this paper, we obtain some properties of C-sets and C-continuity.*

Key Words: **C-set, C-continuous function, C-irresolute function, strongly C-closed graph.**

Özet: *[Eşref Hatır ve arkadaşları, C-süreklilik adlı sürekliliğin yeni bir ayrışımını tanımladılar. S. Jafari, bu süreklilik çeşidini daha ayrıntılı inceledi. Bu makalede biz, C-kümelerin ve C-sürekliliğin bazı özelliklerini elde ettik.*

Anahtar Kelimeler: **C-küme, C-sürekli fonksiyon, C-kararsız fonksiyon, kuvvetli C-kapalı grafik.**

Introduction

E. Hatır, T. Noiri and Ş. Yüksel [1] introduced the notions of α^* -set, C-set and C-continuity in topological spaces and established a decomposition of continuity. In [3], S. Jafari investigated further this type of continuity and introduced notion of strongly C-closed. He also proved that for f : X $\to$ Y is a function if Y is a Hausdorff space, C-continuity necessary to strongly C-closed. Recently, E. Hatır [4] defined C-irresolute function. In this paper, we obtain some properties of C-sets and C-continuity. We also compare with the notions of C-continuity and strongly C-closed.

Preliminaries

Throughout this paper X and Y indicate topological spaces on which no seperation axiom is pressumed. Let A be a subset of a space X. The closure of A and interior of A are denoted by Cl(A) and Int(A), respectively.

We will recall some definitions used in the sequel.

Definition 2.1. A subset A of a space X is said to be

(a) α^* -set [1], if Int(Cl(Int(A))) = Int(A),

(b) C-set [1], if A = O$\cap$F, where O is open and F is an α^* -set.

We used the notion of C-open set instead of C-set and taken the notion of

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C-closed set as complement of C-open set.

Definition 2.2. A function $f:X \to Y$ is said to be C-continuous [1](resp. α-continuous [2]) if for each open set V of Y, $f^{-1}(V)$ is a C-set (resp. α-set) in X.

Definition 2.3. A function $f : X \to Y$ is C-continuous at $x \in X$, if for each open set $V \subset Y$ containing f (x), there exists a C-set U in X containing x such that $f(U) \subset V$. The function f is called C-continuous on X if it has this property for each point x in X [3].

It is clear that Definition 2.2 equivalent to Definition 2.3.

Definition 2.4. For a function $f : (X,\tau) \to (Y, \varphi)$, the graph

$$G(f) = \{ (x, f(x)) : x \in X \}$$

is called strongly C-closed if for each $(x,y) \in ((X \times Y) \setminus G(f))$, there exists a C-set U and open set V containing x and y respectively such that $[U \times V] \cap G(f) = \varnothing$ by ([3], Definition 2.4).

In [1], the following decompositions of continuity have been established.

Theorem 2.1. The following are equivalent for a function $f : X \to Y$:

(a) f is continuos,

(b) f is α-continuous and C-continuos.

In [4], E. Hatır given a new strengthen type of C-continuity called C-irresolute. This notion is given as similar to following.

Definition 2.5. A function $f : X \to Y$ is said to be C-irresolute, if for every C-set of A in Y, its inverse image $f^{-1}(A)$ is C-set in X ([4], Definition 4.1).

Definition 2.6. A point x in X called C-cluster point of $A \subset X$ if $A \cap C \neq \varnothing$ for every C-set C containing x. The set of all C-cluster points of A is called C-closure of A ([3], Definition 2.1). He also denoted C-closure of A by $[A]_C$ and said that A is a C-closed if $[A]_C = A$.

Some new properties of C-sets

The family of all C-sets of a space (X, τ) will be denoted by $C(X, \tau)$ or $C(X)$.

Remark 3.1. The union of two C-sets need not be an C-set.

Example 3.1. Let (X,τ) be the same topological space as in [1], Example 3 that is, $X = \{ a, b, c, d \}$ and $\tau = \{ \varnothing, X, \{a\}, \{a, d\}, \{a, b, d\}, \{a, c, d\} \}$. Then $\{a\}$ and $\{b\}$ are C-sets, but $\{a\} \cup \{b\} = \{a, b\}$ is not a C-set. It is known in [5, Theorem 2.11] that the intersection of any members of $\alpha^*(X,\tau)$ belongs to $\alpha^*(X,\tau)$ .Furthermore, the intersection of two open-sets is always an open set from [6], Chapter 1, Definition 1. It is clear that , the intersection of finite members of $C(X,\tau)$ belongs to $C(X,\tau)$.

We can also give the following two lemma.

Lemma 3.1. If A is an open and C is a C-set in a space X, then $(A \cap C)$ is a C-set in a space X.

Proff. Since C is a C-set, there exists an open set O and a C-set F such that $C = O \cap F$, from Definition 2.1.a). It folows that $B = A \cap O$ is an open set, because the intersection of two open-sets is always an open set from [1], Chapter 1, Definition 1. Therefore;

$$A \cap C = A \cap (O \cap F) = (A \cap O) \cap F = B \cap F = C_1$$

is a C-set from Definition 2.1.b).

Lemma 3.2. If F_1 is an α *-set and C is a C-set in a space X, then $C \cap F_1$ is a C-set in a space X.

Proff. Since C is a C-set, there exists an open set O and C-set F such that $C = O \cap F$, Definition 2.1.b). It folows that $F_2 = (F \cap F_1)$ is an α *-set, because the intersection of two α *-sets is always an α *-set from [5], Theorem 2.11. Therefore;

$$C \cap F_1 = (O \cap F) \cap F_1 = O \cap (F \cap F_1) = O \cap F_2 = C_2$$

is a C-set from Definition 2.1.b).

We can ask this question ourselves: "Are there else sets such that its intersection with a C-set is a C-set?" This question is a clear problem with this subject.

4. Some new properties with C-continuity

S.Jafari [3] given following theorem without proof. We prove its.

Theorem 4.1. The following are equivalent for a function $f : X \to Y$.

(a) f is C-continuous,

(b) $f([A]_C) \subset Cl(f(A))$ for every subset A of X.

(c) $[f^{-1}(B)]_C \subset f^{-1}[Cl(B)]$ for every subset B of Y.

Proof. a) $\Rightarrow$ b). Let $x \in [A]_C$ and V be any open set in Y such that containing f(x). By hypothesis, there exists a C-continuous function f such that $f : X \to Y$. Therefore $f^{-1}(V)$ is a C-set in a space X such that containing x from Definition 2.2. In this condition, before $x \in [A]_C$ and $(A \cap f^{-1}(V)) \neq \varnothing$ by Definition 2.6. Hence, $(f(A) \cap V) \neq \varnothing$ and $f(x) \in Cl(f(A))$. It follows that $f([A]_C) \subset Cl(f(A))$ for every subset A of X.

b) $\Rightarrow$ c). If we take $A = f^{-1}(B)$, we could obtain

$$f([f^{-1}(B)]_C) \subset Cl(f(f^{-1}(B))) \subset Cl(B)$$

by b). Since

$$f([f^{-1}(B)]_C) \subset Cl(B),$$

it follows that

$$[f^{-1}(B)]_C \subset f^{-1}(Cl(B)).$$

c) $\Rightarrow$ a). F be any closed set in Y. Therefore,

$$[f^{-1}(B)]_C \subset f^{-1}(Cl(F)) = f^{-1}(F)$$

by c). Hence, $f^{-1}(F)$ is a C-closed set in X, so f function is a C-continuous.

Example 4.1 shows the well-known fact that even C-continuous functions may not have strongly C-closed. Example 4.2 shows the equally well-known fact that a function having strongly C-closed graph need not be C-continuous.

Example 4.1. Let (X, τ) be a topological space such that X = { a, b, c, d } and τ = { X, $\varnothing$, { d }, { b, c }, { b, c, d } }. Let (Y, φ) be a topological space such that Y = { x, y } and φ = { Y, $\varnothing$, { x } }. Let $f : (X, \tau) \to (Y, \varphi)$ be a function defined as follows: f (a) = f (b) = x and f (c) = f (d) = y. Then f is a C-continuous functions (This example is given and f is a denoted C-continuous functions in [1], Example 4.2), but G(f) is not strongly C-closed. In fact; for $(a, y) \notin G(f)$, there exist a C-set { a, b } and an open set Y containing a and y, respectively. In this case, we obtain that

$$(f(\{a, b\}) \cap Y) = (\{x\} \cap Y) \neq \varnothing.$$

It follows that, G (f) is not strongly C-closed from Definition 2.4.

Example 4.2. Let (X, τ) be the same topological space as in Example 3.1 that is, X = { a, b, c, d } and τ = { $\varnothing$, X, { a }, { a, d }, { a, b, d }, {a, c, d } }. Let (Y, φ) be a topological space such that Y = { x, y, z } and $\varphi = \wp(Y)$. Let $f : (X, \tau) \to (Y, \varphi)$ be a function defined as follows: f (a) = f (b) = x, f (c) = y and f (d) =z. Then G (f) is strongly C-closed, but f is not a C-continuous. In fact, for an open set { x } of φ containing x, $f^{-1}(\{x\})$ = { a, b } $\notin C(X, \tau)$. It follows that, f is not a C-continuous functions from Definition 2.2.

Remark 4.1. Although the composition of two continuous functions is a always continuous function, the composition of two C-continuous functions is not a always C-continuous function as the following example shows.

Example 4.3. Let (X, τ) be the same topological space as in Example 4.1. that is; X = { a, b, c, d } and τ = { X, $\varnothing$, { d }, { b, c }, { b, c, d } }. Let $f : (X, \tau) \to (X, \tau)$ be identity function. Let g be the same function as f in Example 4.1. Since f, is a identity function, it is always continuous. According to [1] , Theorem 4.1, it also C-continuous. We denoted that g is a C-continuous function in Example 4.1. But gof : (X, τ) $\to$ (Y, φ) is not a C-continuous function. Actually; for $\{x\} \subset Y$ open set such that containing x, $g^{-1}(\{x\}) = \{a, b\}$ is a C-set in X by Definition 2.2 but not an open set in X.

We will obtain some conditions for the composition of two functions to be C-continuous.

Theorem 4.2. If f : X $\to$ Y is C-continuous and g : Y $\to$ Z is continuous, then gof : X $\to$ Z is C-continuous.

Proof. Let V be an open set of Z. Since g is continuous, $g^{-1}(V)$ is an open set in Y. In addition since f is C-continuous by Definition 2.2, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is a C-set in X. It follows from Definition 2.2 that gof is C-continuous.

Theorem 4.3. If f:X$\to$Y C-irresolute and g:Y$\to$Z is C-continuous, then gof:X$\to$Z is a C-continuous.

Proof. Let V be an open set of Z. Since g is C-continuous by Definition 2.2, $g^{-1}(V)$ is an C-set in Y. In addition since f is C-irresolute by Definition 2.5, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is a C-set in X. It follows from Definition 2.5 that gof is a C-continuous.

Remark 4.2. If f : X $\to$ Y is C-continuous and $A \subset X$ an arbitrary subset, then the restriction $f|A : A \to Y$ is not C-continuous function as the following example shows.

Example 4.4. Let (X,τ) and (Y,φ) be same topological spaces as in Example 4.1. Let f : (X, τ) $\to$ (Y, φ) be a function defined as follows: f (a) = f (b) = f (d) = x and f (c) = y. Then f is a C-continuous functions. But; for A = { a, b, c } is a subset of X, $f|A : (A, \tau_A) \to (Y, \varphi)$ is not C-continuous. Actually, for an arbitrary subset A = { a, b, c }, $\tau_A = \{A, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. We take an open set { x } in Y such that containing x. In this case, $(f|A)^{-1}(\{x\}) = \{a, b, d\}$ is not a C-set in (A, τ_A).

We will obtain some conditions for the restriction of a C-continuous functions to be C-continuous. We recall that; " Let f : X $\to$ Y is an arbitrary function and A is a subset in a space X. Then $f|A : A \to Y$ is called restriction".

Theorem 4.4. If f : X $\to$ Y is C-continuous function and A is an open set in a space X, then the restrictions $f|A : A \to Y$ is C-continuous function.

Proof. Let V be an open set of Y. Then $f^{-1}(V)$ is a C-set in X. It follows from Lemma 3.1 that $(f|A)^{-1}(V) = (f^{-1}(V) \cap A)$ is a C-set in the subspace A. Therefore, f|A is a C-continuous by Definition 3.2.

It is clear that; C-continuityis not a heredity property.

Theorem 4.4. If f : X $\to$ Y is C-continuous function and A is an α *-set in a space X, then the restrictions $f|A : A \to Y$ is C-continuous function.

Proof. Let V be an open set of Y. Then $f^{-1}(V)$ is a C-set in X. It follows from Lemma 3.2 that $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is a C-set in the subspace A. Therefore, f|A is a C-continuous by Definition 3.2.

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