INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 15 NO. 2 PAGE 178–182 (2022)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1046707

Convex Hull of Extreme Points in Flat Riemannian Manifolds

Reza Mirzaie* and Omid Rezaie

(Communicated by Vitor Balestro)

ABSTRACT

We show that convex hull of extreme points of a closed strongly convex subset of a compact flat Riemannian manifold is equal to the subset itself.

Keywords: *Flat Reimannian manifold, convex subset, convex hull, extreme point.*
AMS Subject Classification (2020): *Primary: 53C50 ; Secondary: 52A99.*

1. Introduction

Let A be a subset of a Riemannian manifold M. It is interesting to find relations between global (geometric or topological) properties of A and its boundary points. In special case when A is considered to be a convex subset, the boundary points set can be replaced by a usually smaller subset containing extreme points. A point is called extreme if it is not included in the interior of a geodesic segment with endpoints in A. One of the important results in this direction is the Krein-Milman theorem which states that if $M = E^n$ and A is a compact and convex subset, then A is equal to the convex hull of its extreme points [8]. Thus, one only needs the extreme points of A to recover its shape. The Krein-Milman theorem has been generalized to convex noncompact submanifolds of E^n in [3]. After that, the author of [9] studied similar problems, when M is a complete simply connected Riemannian manifold without conjugate points. As far as we know, there is no explicit result about relations between A and its extreme points when M is not simply connected.
In the present article, we consider the problem under the condition that M is a compact flat Riemannian manifold (nonsimply connected) and A is a closed strongly convex subset of M. We replaced the convexity condition of A by strong convexity. Because, when M is compact, the convex hull of a closed subset is equal to M itself, and the problem is trivial. As a consequence of our main result, we also consider a noncompact case where M is equal to the product of a compact flat Riemannian manifold and the Euclidean space, and A is a subset with the geodesic decomposition property.

2. Preliminaries

Let M be a complete Riemannian manifold. A subset C of M is called (strongly) convex, if for each pair of points a, b in C, all points of each (minimal) geodesic segment joining a to b is contained in C. It is clear that each convex subset is strongly convex, but the converse is not true. For instance, S^{2+} is a strongly convex subset of S^2 which is not a convex subset. If $B \subset M$, then the (strong) convex hull of B, which we denote by ($C_s(B)$) $C(B)$, is by definition, the smallest (strongly) convex set containing B, that is the intersection of all (strongly) convex subsets containing B. A point e in a (strongly) convex subset C is called an extreme point if it does not lie in the interior of any geodesic joining two points of C. That is for each geodesic segment $\gamma : [0, 1] \to M$, with $\gamma(0), \gamma(1)$ in C, $e \notin \gamma(0, 1)$. The union of all extreme points of C is called the extreme subset of C which we denote by $E(C)$. Note that $E(C)$ is the same for convex and strongly convex sets.

Received : *25-12-2021,* ***Accepted :*** *30-05-2022*
** Corresponding author*

In what follows, the domain of all geodesic segments are considered to be $[0,1]$.

Definition 2.1. Let B be a subset of a Riemannian manifold M. Put $G_0(B) = B$ and

$$G_1(B) = \{\alpha(t) : \alpha \ is \ a \ \ geodesic \ joining \ two \ points \ of \ B\}$$

$$G_{m+1}(B) = G_1(G_m(B))$$

If $e \in G_{m+1}(B)$ then a sequence of geodesic segments $\alpha_1, \alpha_2, ..., \alpha_{m+1}$ is called an spanning geodesic sequence for e from the set B, if α_i is a geodesic with end points in $G_i(B)$ and $e = \alpha_{m+1}(t)$ for some $t \in [0,1]$.
In this case we write:

$$\alpha_1 \to \alpha_2 \to ... \to \alpha_{m+1} \to e.$$

The sequence is called a spanning minimal geodesic sequence for e from B, if all geodesic segments α_i are minimal.

Remark 2.1. It is easy to show that $C(B)$ ($C_s(B)$) is the collection of all points $e \in M$ with the property that there is a spanning (minimal) geodesic sequence for e from B.

If M is a compact Riemannian manifold and A is a closed subset of M, then $C(A) = M$. Thus, convex hull of closed sets in compact Riemannian manifolds are not interesting and we consider strong convex hulls in this case.

3. Results

Definition 3.1. Let M be a complete Riemannian manifold, $B \subset M$ and $b \in B$. A convex component of B containing b is a convex subset C of B which contains b and is maximal. That is, if $C \subset D$ and D is a convex subset of B, then $D = C$. The strongly convex component is defined similarly.

Remark 3.1. If $A \subset R^n$ and $b \in R^n$, then the cone on A with the vertex b is defined by

$$cone(A, b) = \{ta + (1-t)b : t \in [0,1], \ \ a \in A\}.$$

It is clear that if A is convex then $cone(A, b)$ is convex. Note that convexity and strong convexity are the same in R^n.

Recall 1. *Let M be a Riemannian manifold and $\tilde{M}$ be its universal covering space with the covering map $\pi : \tilde{M} \to M$. If $a \in M$ then there is a neighbourhood V for a and disjoint neighbourhoods V_b for each $b \in \pi^{-1}(a)$ such that $\pi : V_b \to V$ is an isometry. V is called an admissable neighbourhood of a. If $\alpha : [0,1] \to M$ is a curve with initial point a ($\alpha(0) = a$), then there is a unique curve $\tilde{\alpha} : [0,1] \to \tilde{M}$ with initial point b ($\tilde{\alpha}(0) = b$) such that $\pi \circ \tilde{\alpha} = \alpha$. $\tilde{\alpha}$ is called the lift of α to the point b and it is a geodesic if α is a geodesic.*

Theorem 3.1. *Let M be a complete flat Riemannian manifold and $\pi : R^n \to M$ be a covering map. If A is a closed strongly convex subset of M, $a \in A$ and $b \in \pi^{-1}(a)$, then there is a closed and convex subset $\tilde{A}$ of R^n such that $\tilde{A}$ with the following properties is maximal.*
$b \in \tilde{A}$, $\pi(\tilde{A}) = A$ and $\pi(E(\tilde{A})) = E(A)$.

Proof. Denote by $\tilde{A}$ the convex component of $\pi^{-1}(A)$ containing b. We show that $\pi(\tilde{A}) = A$. Clearly, $\pi(\tilde{A}) \subset A$. Let $c \in A$ and let γ be the minimal geodesic in A joining a to c ($\gamma(0) = a, \gamma(1) = c$). Suppose that $\tilde{\gamma}$ is the lift of γ to the point b. Then, $\tilde{\gamma}([0,1])$ is a subset of $\pi^{-1}(A)$. Since $b \in \tilde{A} \cap \tilde{\gamma}([0,1])$, then from the definition of convex component, $cone(\tilde{A}, \tilde{\gamma}(1)) = \tilde{A}$. Thus, $\tilde{\gamma}(1) \in \tilde{A}$, and $c = \pi(\tilde{\gamma}(1)) \in \pi(\tilde{A})$. Therefore, $A \subset \pi(\tilde{A})$.
Now, we show that $\pi(E(\tilde{A})) = E(A)$.
Let $c \in E(A)$ and $\tilde{c} \in \tilde{A}$ with $\pi(\tilde{c}) = c$. We show $\tilde{c} \in E(\tilde{A})$. If not, then there is a geodesic α in $\tilde{M}$ such that $\alpha(0), \alpha(1) \in \tilde{A}$ and for some $t \in (0,1)$, $\alpha(t) = \tilde{c}$. Then, $\pi \circ \alpha$ is a geodesic in M with end points in A which contains c in the interior. Thus, $c \notin E(A)$ which is a contradiction. Thus, $E(A) \subset \pi(E(\tilde{A}))$.
Now, suppose that $d \in \pi(E(\tilde{A}))$. We have $d = \pi(\tilde{d})$ for some $\tilde{d} \in E(\tilde{A})$. If $d \notin E(A)$, then there is a minimal geodesic β in M such that $\beta(0), \beta(1) \in A$ and for some $t \in (0,1)$, $d = \beta(t)$. Since A is strongly convex, then $\beta([0,1]) \subset A$. Consider an admissable strongly convex neigbourhoods V of d and $\tilde{V}$ of $\tilde{d}$ such that $\pi_{|\tilde{V}} : \tilde{V} \to V$

be isometry. For sufficiently small positive number ϵ we have $\beta([t-\epsilon, t+\epsilon]) \subset V$. Put $\tilde{\beta} = (\pi_{|\tilde{V}})^{-1} \circ \beta_{|[t-\epsilon,t+\epsilon]}$. We have $\tilde{\beta}(t) = \tilde{d}$. We show that the endpoints of $\tilde{\beta}$ belong to $\tilde{A}$. Then we get $\tilde{d} \notin E(\tilde{A})$ which is a contradiction and we get that $d \in E(A)$, so $\pi(E(\tilde{A})) \subset E(A)$.
Consider the endpoints of $\tilde{\beta}$, $b_1 = \tilde{\beta}(t+\epsilon)$ and $b_2 = \tilde{\beta}(t-\epsilon)$. We have $\tilde{A} \cup \tilde{\beta}([t-\epsilon, t+\epsilon]) \subset \pi^{-1}(A)$ and $\tilde{d} \in \tilde{A} \cap \tilde{\beta}([t-\epsilon, t+\epsilon])$. Thus, from the fact that $\tilde{A}$ is a convex component of $\pi^{-1}(A)$, we have $cone(\tilde{A}, b_1) = \tilde{A}$. So, $b_1 \in \tilde{A}$. In similar way, $b_2 \in \tilde{A}$. □

Remark 3.2. A geodesic loop in a Riemannian manifold M is a curve $\alpha : [0,1] \to M$ such that $\alpha(0) = \alpha(1)$ and α is geodesic on interior points of its domain (in $(0,1)$). Note that a closed geodesic is a geodesic loop.

Remark 3.3. We will use the flat torus T^n, $n \geq 2$, in the proof of following theorem. The n-dimensional torus T^n is the product of n circles. T^n can also be described as a quotient of R^n under integer shifts in any coordinate. That is, we consider the action of Z^n on R^n defined by

$$Z^n \times R^n \to R^n, \ \ (a, x) = a + x,$$

then T^n is the quotient R^n/Z^n.

Theorem 3.2. *If A is a closed strongly convex subset of a compact and complete flat Riemannian manifold M and there is no geodesic loop in A, then $C_s(E(A)) = A$.*

Proof. If $dimM = n$, then by theorem of Bieberbach, T^n is a covering space for M [4]. Consider the following maps: $\pi_1 : R^n \to T^n$, the universal covering map, $\pi_2 : T^n \to M$ a covering map and $\pi = \pi_2 \circ \pi_1 : R^n \to M$.
Without loss of generality consider a point $a \in A$ and $b \in \pi^{-1}(a)$ such that $b \in I^n$ (where I is [0,1]). By Theorem 3.1, there exists a closed and convex subset A_1 of R^n such that $b \in A_1$ and $\pi(A_1) = A$, and A_1 is maximal with the mentioned properties. Put $\pi_1(A_1) = A_2$. Clearly, $\pi_2(A_2) = A$. Since there is no geodesic loop in A, then there is no geodesic loop in A_2 (if γ is a geodesic loop contained in A_2 then $\pi_2 \circ \gamma$ is a geodesic loop in A).
Consider T^n as quotient of R^n under the action of Z^n. We show that $A_1 \subset I^n$. If not, then there is a point $a_1 \in A_1$ and a nonidentity element $\delta \in Z^n$ such that $\delta(a_1) \in A_1$. Consider the line segment

$$\gamma(t) = (1-t)a_1 + t\delta(a_1).$$

Since A_1 is convex, then for all $t \in [0,1]$, $\gamma(t) \in A_1$. Now, put $\alpha = \pi_1 \circ \gamma$. Then for all $t \in [0,1]$, $\alpha(t) \in A_2$. Since $\delta(\gamma(0)) = \gamma(1)$, then

$$\alpha(0) = \pi_1 \circ \gamma(0) = \pi_1 \circ \gamma(1) = \alpha(1).$$

This means that α is a geodesic loop in T^n contained in A_2, which is a contradiction. Thus, $A_1 \subset I^n$. Therefore, A_1 is compact and by Krein-Milman theorem, $C(E(A_1)) = A_1$. By Theorem 3.1, $\pi(E(A_1)) = E(A)$.
To complete the proof of the theorem, we prove the following claim:

(*) Claim: For each minimal geodesic segment $\gamma : [0,1] \to M$ contained in A, there is a geodesic $\tilde{\gamma}$ in A_1 such that $\pi(\tilde{\gamma}) = \gamma$.

Proof of the claim. Let e in A_1 such that $\pi(e) = \gamma(0)$ and let $\tilde{\gamma}$ be the lift of γ to the point e. If $\tilde{\gamma}(1) \in A_1$ then we have done, if not then the convex cone $cone(A_1, \tilde{\gamma}(1))$ is a convex set containing A_1 which is in contrast with the maximality of A_1. By Definition 2.1, Remark 2.1 and Claim (*), it is easy to show that $\pi(C(E(A_1)) = C(\pi(E(A_1)))$. Now, from $C(E(A_1)) = A_1$ and $\pi(E(A_1)) = E(A)$ we get that $C(E(A)) = A$. □

4. A remark on strongly convex subsets of product flat manifolds

Definition 4.1. Let M_1 and M_2 be Riemannian manifolds and A be an strongly convex subset of $M_1 \times M_2$. We say that A has geodesic decomposition property, if the following assertion is true:
Let $(a, b) \in A$, $a \in M_1, b \in M_2$ and

$$A_{a-} = \{y : (a, y) \in A\}, \quad A_{-b} = \{x : (x, b) \in A\}.$$

If $\beta = (\beta_1, \beta_2) : [0,1] \to M_1 \times M_2$, is a geodesic contained in A and $\beta(t_0) = (a, b)$ for some $0 < t_0 < 1$, then there is a positive number ϵ such that $(\beta_1(t), b) \in A_{-b}$, $t_0 - \epsilon \leq t \leq t_0 + \epsilon$ and $(a, \beta_2(t)) \in A_{a-}$, $t_0 - \epsilon \leq t \leq t_0 + \epsilon$.

Example 4.1. If A_1 and A_2 are strongly convex in M_1 and M_2, then $A = A_1 \times A_2$ has geodesic decomposition property.

Corollary 4.1. *Let M_1 and M_2 be complete flat Riemannian manifolds such that for each strongly convex subset A_i of M_i, $C_s(E(A_i)) = A_i$, $i = 1, 2$. If A is a closed and strongly convex subset of $M_1 \times M_2$ with geodesic decomposition property, then $C_s(E(A)) = A$.*

Proof. For all (a, b) in A consider A_{a-} and A_{-b} as Definition 4.1. Since A is closed and strongly convex, it is easy to show that A_{a-} and A_{-b} are closed and strongly convex in M_2 and M_1, respectively, and by assumption of the corollary,

$$C_s(E(A_{a-})) = A_{a-}, \ \ C_s(E(A_{-b})) = A_{-b}. \quad (1)$$

We show that

$$E(A) = \{(a, b) \in A : b \in E(A_{a-}) \ \ and \ \ a \in E(A_{-b})\}.$$

Let $(a, b) \in E(A)$. If $b \notin E(A_{a-})$, then there is geodesic $\gamma : [0, 1] \to M_2$ such that $\gamma(0), \gamma(1) \in A_{a-}$ and for some $t \in (0, 1)$, $b = \gamma(t)$. Put $\tilde{\gamma}(t) = (a, \gamma(t))$. $\tilde{\gamma}$ is a geodesic in $M_1 \times M_2$ and $\tilde{\gamma}(0), \tilde{\gamma}(1) \in A$, $\tilde{\gamma}(t) = (a, b)$, which contradicts $(a, b) \in E(A)$. Thus, $b \in E(A_{a-})$. In similar way, we can show that $a \in E(A_{-b})$.
Conversely, let $a \in E(A_{-b})$ and $b \in E(A_{a-})$. We show $(a, b) \in E(A)$. If $(a, b) \notin E(A)$, then there is a geodesic $\beta = (\beta_1, \beta_2)$ in $M_1 \times M_2$ such that $\beta(0), \beta(1) \in A$ and for some $t_0 \in (0, 1)$, $\beta(t_0) = (a, b)$. Consider the geodesics $\gamma_2 = (a, \beta_2)$ and $\gamma_1 = (\beta_1, b)$ in $M_1 \times M_2$. Put

$$I_1 = \{t \in [0, 1] : \gamma_1(t) \in A_{-b} \times \{b\}\}.$$

Clearly, $t_0 \in I_1$. If for some small number $\epsilon > 0$, $[t_0 - \epsilon, t_0 + \epsilon] \subset I_1$ then $\beta_1 : [t_0 - \epsilon, t_0 + \epsilon] \to M_1$ is a geodesic with end points in A_{-b} which contains the point $a (= \beta_1(t_0))$ as interior point. Then $a \notin E(A_{-b})$ which is a contradiction. Then, for all small positive numbers ϵ, $[t_0 - \epsilon, t_0 + \epsilon]$ is not a subset of I_1. Thus, there is a sequence of decreasing positive numbers ϵ_n such that $\epsilon_n \to 0$ and either $\gamma_1(t_0 - \epsilon_n) \notin A_{-b} \times \{b\}$ for all n or $\gamma_1(t_0 + \epsilon_n) \notin A_{-b} \times \{b\}$ for all n.
We get from convexity of A_{-b} that for sufficiently large n,
(1) $\gamma_1(t) \notin A_{-b} \times \{b\}$ for all $t \in [t_0 - \epsilon_n, t_0)$
or
(2) $\gamma_1(t) \notin A_{-b} \times \{b\}$ for all $t \in (t_0, t_0 + \epsilon_n]$.
In a similar way, we can find a small positive number δ such that
(3) for all $t \in [t_0 - \delta, t_0)$, $\gamma_2(t) \notin \{a\} \times A_{a-}$
or
(4) for all $t \in (t_0, t_0 + \delta]$, $\gamma_2(t) \notin \{a\} \times A_{a-}$.

Put $\eta = min\{\epsilon_n, \delta\}$. If (1), (3) are true then for all $t \in [t_0 - \eta, t_0)$, we have:

$$\gamma_1(t) \notin A_{-b} \times \{b\}, \gamma_2(t) \notin \{a\} \times A_{a-} \Rightarrow \beta_1(t) \notin A_{-b}, \beta_2(t) \notin A_{a-},$$

which contradicts the geodesic decomposibility of A. Similarly we have contradiction if (1),(4) or (2), (3) or (2), (4) are true. Then $(a, b) \in E(A)$.

Now, we show that $A \subset C_s(E(A))$. Suppose, $(a, b) \in A$. Since $C_s(E(A_{a-})) = A_{a-}$ and $C_s(E(A_{-b})) = A_{-b}$, then there are spanning geodesic sequence $\alpha_1 \to \alpha_2 \to .... \to \alpha_m \to a$ for a from $E(A_{-b})$ and $\beta_1 \to \beta_2 \to ... \to \beta_k \to b$ for b from $E(A_{a-})$. Without loss of generality suppose $k \leq m$. Now, it is easy to show that

$$(\alpha_1, \beta_1) \to (\alpha_2, \beta_2) \to ... \to (\alpha_k, \beta_k) \to (\alpha_{k+1}, \beta_k) \to ... \to (\alpha_m, \beta_k)$$

is an spanning geodesic sequence for (a, b). Thus, $(a, b) \in C_s(E(A))$. Clearly, $C_s(E(A)) \subset A$, then $C_s(E(A)) = A$. □

Now, from Theorem 3.2 and Corollary 4.1, we get the following theorem.

Theorem 4.1. *Let M be a compact and complete flat Riemannian manifold. If A is a closed, compact and strongly convex subset of $M \times R^n$ and A has geodesic decomposition property, then $C_s(E(A)) = A$.*

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Ballmann, W.: Lectures on Spaces of Nonpositive curvature. Brikhauser, Boston, Basel, Berlin, Stuttgart (1985).

[2] Bangert, V.: *Totally convex sets in complete Riemannian manifolds*. J. Differential Geometry. **16**, 333-345 (1981). https://doi.org/10.4310/jdg/1214436108

[3] Beltagy, M. Shenawy, S.: *On the boundary of closed convex sets in* $\mathbb{E}^n$. arxiv:1301.0688v1 [math.MG] 4 Jan (2013).

[4] Bieberbach, L.: *Über die Bewegungsgruppen der Euklidischen Räume II: Die Gruppen mit einem endlichen Fundamentalbereich*. Mathematische Annalen. **72** 400-412 (1912). https://doi.org/10.1007/BF01456724

[5] Bredon, B.: Introduction to compact transformation groups. Acad Press. New york, London (1972).

[6] do Carmo, M. P.: Riemannian Geometry. Brikhauser, Boston, Basel, Berlin (1992).

[7] Munkres, J. R.: Topology; a First course. Prentic-Hall (1974).

[8] Lay, S. R.: Convex sets and their applications. John Wiley and Sons. Dekker, New York (1982).

[9] Shenawy, S.: *Convex and Starshaped Sets in Manifolds without Conjugate Points*. International Electronic Journal Of Geometry. Volume 12, no. 2, 223-228 (2019). https://doi.org/10.36890/iejg.628087

Affiliations

REZA MIRZAIE
ADDRESS: Imam Khomeini International University (IKIU), Department of Pure Mathematics, Faculty of Science, Qazvin, Iran.
E-MAIL: Email: r.mirzaei@sci.ikiu.ac.ir
ORCID ID: 0000-0002-9480-0222

OMID REZAIE
ADDRESS: Imam Khomeini International University (IKIU), Department of Pure Mathematics, Faculty of Science, Qazvin, Iran.
E-MAIL: omid.rezaie61@gmail.com
ORCID ID: 0000-0002-9101-1092