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# On $(q^2+q+1)$–Sets of Plane–Type $(m,n,r)_2$ in $\mathrm{PG}(3,q)$

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**ABSTRACT**

**In this paper $(q^2+q+1)$–sets of points in $\mathrm{PG}(3,q)$ of type $(m,n,r)$ with respect to planes are studied, and as a by–product for $q$ odd a characterization of quadratic cones is obtained.**

***Keywords:*** *Projective spaces, intersection numbers, oval, quadratic cone.*
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## 1. Introduction

Let $\mathbb{P}=\mathrm{PG}(r,q)$ be the $r$–dimensional (desarguesian) finite projective space of order $q$, let $1\leq h\leq r-1$ and $0\leq m_1\leq\cdots\leq m_s$ be $s+1$ integers with $m_s\leq\theta_{r-1}=\frac{q^r-1}{q-1}=q^{r-1}+\cdots+q+1$, and let $\mathcal{P}_h$ be the collection of all the $h$–dimensional subspaces of $\mathbb{P}$. A set $\mathcal{K}$ of points of $\mathbb{P}$ is of *type* $(m_1,\ldots,m_s)_h$ *with respect to the h–subspaces of* $\mathbb{P}$ if $|\pi\cap\mathcal{K}|\in\{m_1,\ldots,m_s\}$ for every $\pi\in\mathcal{P}_h$ and each of the non–negative integers $m_i$ (called *intersection numbers*) occurs as the size of the intersection of $\mathcal{K}$ with at least one member of $\mathcal{P}_h$. If $h=1,2$ or $r-1$, one says that $\mathcal{K}$ is of *line–type* $(m_1,\ldots,m_s)_1$, of *plane–plane type* $(m_1,\ldots,m_s)_2$ or of *hyperplane–type* $(m_1,\ldots,m_s)_{r-1}$, respectively [6]. As usual, if the size of $\mathcal{K}$ is $k$, $\mathcal{K}$ is called a *k–set*, and a line (plane) intersecting $\mathcal{K}$ in exactly $j$ points is called a *j–line* (*j–plane*). Also, if $j=0$ the line (plane) is also called *external* line (plane) and if $j=1$ the line (plane) is also called *tangent* line (plane), and if $2\leq j\leq q$ a $j$–line is called a *secant* line.

The interest in the study of $k$–sets of $\mathbb{P}$ with respect to the intersections with the members of one (or more than one) family of subspaces is motivated not only by the fact that most of the classical and beautiful geometric object of $\mathbb{P}$ have few intersection numbers with respect to the lines and/or to the hyperplanes of $\mathbb{P}$ but also because they are related with other combinatorial objects such as for example graphs, some classes of difference sets and linear codes with few weights (cf e.g. [2]). The study of such sets goes back to the famous B. Segre theorem characterizing the (non–degenerate) conics of $\mathrm{PG}(2,q)$, $q$ odd, as sets of points of the plane of line–type $(0,1,2)_1$ [9]. Since then there has been a wide literature devoted to the study of sets of points of $\mathbb{P}$ with respect to their intersection numbers with all the members of one (or more than one) prescribed family of subspaces of $\mathbb{P}$. Most of the papers devoted to these sets considers the following issues

• the determination of the *admissible* $s+1$ *tuples* $(k,m_1,\ldots,m_s)_h$ with respect to the family of all the $h$–subspaces of $\mathbb{P}$ , and so the corresponding existence problem of sets of points of $\mathbb{P}$ associated with an admissible $s+1$ *t*uple;

• reconstruct a classical subset of points of $\mathbb{P}$ (such as for example a quadric or a subgeometry) starting from its intersection numbers with respect to a prescribed subspace dimension $h$, possibly with some extra arithmetic and/or geometric condition.

In [3] the authors call *quasi*–quadric a set of points of $\mathbb{P}$ having the same intersction numbers as a quadric with respect to the hyperplanes, and give examples of quasi–quadrics which are not quadrics. Thus, as already B. Segre theorem suggested, the knownledge of the intersection numbers is not enough to recognize a quadric. Other examples of quasi–quadrics, for $r=3$, may be found in [5, 8]. After the publication of [3] it became customary to precede the name of a classical subset of points of $\mathbb{P}$ by *quasi* in characterizing it with respect

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to its intersection numbers. Thus, one may find papers whose title contains *quasi*–hermitian varieties (cf e.g. [1, 4, 7]). Note that in general, having the intersection numbers does not give the size of the set, for example in $\mathrm{PG}(3,q)$ there are $(q^2+q+1)$–sets of points of plane–type $(1,q+1,2q+1)_2$ which are not cones projecting an oval of plane $\pi$ from a point outside $\pi$ (cf e.g. [8] examples 1 and 2).

In addition, in the literature devoted to the characterization problem of such sets, there are also results assuming weaker conditions on the intersection numbers of the set, that is for example only some of the $s(\geq 2)$ intersection numbers are known, or the size of the set is given and it has the same number of intersection numbers such as a classical object, or the distribution of the intersection numbers "resembles" those of a classical object, that is for example the differences between any pair of the intersection numbers are the same as those of a classical subset of $\mathbb{P}$.

Let $s \geq 2$ be an integer and $X$ be a subset of $q+1$ points of $\mathrm{PG}(3,q)$ contained in a plane $\pi$ such that $X$ is of line–type $(0,1,s)_1$ with respect to the lines of $\pi$ and with $(s-1)|q$. So, for $s=2$ $X$ is a $(q+1)$–arc of $\pi$ and if $q$ odd it is a conic. Let $V$ be a point of $\mathrm{PG}(3,q)$ not in $\pi$ and let $\mathcal{K}$ be the set consisting of all the points of the $q+1$ lines passing through $V$ and a point of $X$. $\mathcal{K}$ is a $(q^2+q+1)$–set, and for $s=2$ and $q$ odd it is a quadratic quasi–cone. The set $\mathcal{K}$ is of plane–type $(1,q+1,2q+1)_2$ and of line–type $(0,1,s,q+1)_1$ and any external line belongs to exactly one tangent plane.

The punctured* 3–dimensional affine space of order 2 has $2^3-1=2^2+2+1$ points, and when embedded in its projective closure it is intersected by the planes of $\mathrm{PG}(3,2)$ in 0, 3 or 4 points, and any external line belongs to exactly one external plane.

Note that a $(q^2+q+1)$–set of plane–type $(m,n,r)_2$ admits at least one external line. Indeed, if there would be no external line, then all the lines passing through a point outside of $\mathcal{K}$ are tangent ones. Thus, any line connecting two points of $\mathcal{K}$ is contained in $\mathcal{K}$ and so $\mathcal{K}$ is a $(q^2+q+1)$–set of line–type $(1,q+1)_1$, that is a plane, which is a contradiction.

In [10] the author gives a characterization of a quadratic cone of $\mathrm{PG}(3,q)$, $q$ odd, as a $(q^2+q+1)$–set of plane–type $(m,m+q,m+2q)_2$ with an extra arithmetic condition and the extra (geometric) assumption on the external lines: *any external line belong to exactly one $m$–plane*. This result, has been recently improved by Zuanni [11] which showhs that the extra arithmetic condiction may be dropped.

There is no $(q^2+q+1)$–set $\mathcal{K}$ of plane–type $(0,q,2q)_2$ such that on any external line there is exactly one external plane. Indeed, if $\ell$ is an external line and $x_\ell$ and $y_\ell$ are the numbers of $q$–planes and $2q$–planes on $\ell$, respectively, then counting the number of points of $\mathcal{K}$ trough the planes on an external line gives $q^2+q+1 = x_\ell \cdot q + y_\ell \cdot 2q$, which is not possible since $q \geq 2$. Thus, the sets studied in [10] fulfils the condition $m \geq 1$, and so the second intersection number is at least $q+1$.

In this paper, we show that this result may be generalized by weakening the assumptions on the intersection numbers, that is not assuming that they are in arithmetic progression. We will prove the following result.

**Theorem 1.1.** *Let $\mathcal{K}$ be a $(q^2+q+1)$–set of plane type $(m,n,r)_2$, with $n \geq q+1$ and such that each external line is on exactly one $m$–plane. Then either $m=0$ and $\mathcal{K}$ is $\mathrm{AG}(3,2)$ less a point or $m=1, n=q+1$ and $r=sq+1$ for some integer $s \geq 2$ with $(s-1)|q$ and $\mathcal{K}$ is a cone with base a $(q+1)$–set $\Omega$ of a plane $\pi$ intersected by any line of $\pi$ in 0, 1 or $s$ points and vertex a point $V$ not in $\pi$. In particular, if $s=2$ and $q$ is odd $\mathcal{K}$ is a quadratic cone.*

## 2. The proof

Let $0 \leq m < n < n \leq q^2+q+1$ be three integers and $\mathcal{K}$ be a $(q^2+q+1)$–set of points of $\mathrm{PG}(3,q)$ of plane–type $(m,n,r)_2$ fulfilling the assumptions of Theorem 1.1.

Since by assumptions, any external line lies in exactly one $m$–plane, counting the number of points of $\mathcal{K}$ via the planes through an external line gives

$$q^2+q+1 \geq m+qn \geq m+q^2+q$$

and so $n=q+1$ and $m \leq 1$.

**Proposition 2.1.** *Either $m=1$ or $\mathcal{K}$ is $\mathrm{AG}(3,2)$ less a point.*

*Proof.* Assume to the contrary that $m=0$. So, the $m$–planes are external to $\mathcal{K}$ and by assumptions it follows that there is a single 0–plane, say $\pi_0$, and all the external lines to $\mathcal{K}$ are in $\pi_0$. If $p$ is a point not in $\mathcal{K} \cup \pi_0$ then

*That is, $\mathrm{AG}(3,2)$ less one of its points.

there is no external line containing $p$ and so all the lines passing through $p$ are tangent lines. Thus, through $p$ there is no $r$–plane. If $\pi$ is an $r$-plane, all its points not in $\mathcal{K}$ are in $\pi_0$ and so $r = q^2$. Let $\pi$ be a $q^2$–plane and $\ell_0$ be the line that $\pi$ shares with $\pi_0$. Counting the number of points via the planes through $\ell_0$ one has:

$$q^2 + q + 1 \geq q^2 + (q-1)(q+1) = 2q^2 - 1$$

and so $q = 2$. So, $\mathcal{K}$ is a 7–cap of $\mathrm{PG}(3,2)$, that is $\mathrm{AG}(3,2)$ less a point. □

From now on, assume that $m = 1$. It follows that $\mathcal{K}$ is a $(q^2+q+1)$–set of plane–type $(1, q+1, r)_2$ and that any external line belongs to exactly one tangent plane and to $q$ $(q+1)$–planes.

Let $\pi$ be a tangent plane and $V = \pi \cap \mathcal{K}$. Since an external line lies in exactly one tangent plane, it follows that all the tangent planes pass through $V$.

**Proposition 2.2.** *If $\pi$ is a $(q+1)$–plane through $V$, then $\pi \cap \mathcal{K}$ is a line.*

*Proof.* Let $\pi$ be a $(q+1)$–plane containing $V$. Since any tangent plane contains $V$ one has that $\pi$ does not contain an external line to $\mathcal{K}$. So, $\pi$ intersects $\mathcal{K}$ in a line. □

**Proposition 2.3.** *Any $(q+1)$–plane not passing through $V$ contains an external line.*

*Proof.* Let $\pi$ be a $(q+1)$–plane not containing $V$. If $\pi$ has no external line, then it intersects $\mathcal{K}$ in a line and so there is no tangent plane through $V$, which is not possible. □

If $\ell$ is a tangent line not through $V$, then all the planes containing $\ell$ are $(q+1)$–planes.

Since an $r$–plane has no external line, it has to contain $V$. So, on any line not passing through $V$ there is at most one $r$–plane.

**Proposition 2.4.** *Any line intersecting $\mathcal{K}$ in at least two points belongs to at least one $r$–plane.*

*Proof.* Let $\ell$ be a line with at least two points in $\mathcal{K}$ on which there is no $r$–plane, and let $(2 \leq) s = |\ell \cap \mathcal{K}| \leq q+1$. Then

$$q^2 + q + 1 = s + (q+1)(q+1-s) = (q+1)^2 - sq,$$

which is a contradiction. □

**Proposition 2.5.** *Any line intersecting $\mathcal{K}$ in at least two points and not containing $V$ is not contained in $\mathcal{K}$.*

*Proof.* Let $p$ and $q$ be two distinct points of $\mathcal{K}$ and different from $V$, such that the line $\ell$ connecting them does not contain $V$. If $\ell$ is contained in $\mathcal{K}$, then by Proposition 2.4 and the statement preceding that proposition it follows that on $\ell$ there is exactly one $r$–plane, so

$$q^2 + q + 1 = r$$

which is not possible, since a plane has non–tangent ones. So, $\ell$ meets $\mathcal{K}$ in at most $q$ points. □

Note that, from the proof of the previous proposition it follows that if $L$ is a line contained in $\mathcal{K}$, then $L$ contains $V$.

**Proposition 2.6.** *The secant lines not containing $V$ intersect $\mathcal{K}$ in the same number of points.*

*Proof.* Let $\ell$ be a secant line not passing through $V$, counting the number of points of $\mathcal{K}$ via the planes on $\ell$ gives

$$q^2 + q + 1 = r + q(q + 1 - |\ell \cap \mathcal{K}|)$$

and so

$$|\ell \cap \mathcal{K}| = \frac{r-1}{q}.$$

It follows that all lines not containing $V$ and intersecting $\mathcal{K}$ in at least two points intersect $\mathcal{K}$ in a constant number of points, $s := \dfrac{r-1}{q}$. □

The next step completes the proof of Theorem 1.1.

Let $L$ be a line through $V$ with at least two points in $\mathcal{K}$, $\pi$ be an $r$–plane through $L$ existing by Proposition 2.4, and let $y$ be a point of $L$ different from $V$. Since the lines of $\pi$ not containing $V$ cannot be tangent lines, counting the number of points of $\pi$ via the lines on $y$ gives

$$r = |L \cap \mathcal{K}| + q(s-1)$$

and so $q$ divides $|L \cap \mathcal{K}| - 1$. Thus, $|L \cap \mathcal{K}| = q+1$ and $r = sq+1$. It follows that $V$ belongs to exactly $q+1$ lines contained in $\mathcal{K}$, that is the lines connecting $V$ with the points of $\mathcal{K}$ in a $(q+1)$–plane not through $V$. Thus, $\mathcal{K}$ is the cone projecting from $V$ a $(q+1)$–set of line–type $(0,1,s)_1$ of a plane not through $V$. So, if $s=2$ $\mathcal{K}$ is an oval cone, and so if $q$ is odd a quadratic cone. Finally, if $L$ is a $(q+1)$–line, then the usual counting argument gives $q^2+q+1 = x(sq+1-q-1)+q+1$ and so $(s-1)|q$.

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