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Direct Product of Bitonic Algebras

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Abstract

The purpose of this study is to construct the concept of direct product of bitonic algebras, and investigate some respective features. Also, the concept of direct product of commutative bitonic algebras, bitonic homomorphism are studied. Then the notion of direct product of bitonic algebras is expanded to finite family of bitonic algebras and their qualifications are practised.

Keywords: Bitonic algebras; Direct product; Homomorphisms; Filters.

Bitonic Cebirlerin Direkt Çarpımları

Öz

Bu çalışmanın amacı bitonic cebirlerin direkt çarpımları olup bitonic cebirlerin direkt çarpımlarının ilgili özelliklerini çalışmaktır. Ayrıca, değişmeli bitonic cebirlerinin direkt çarpımları, bitonic homomorfizmalar incelenmiş ve değişmeli bitonic cebirlerin direkt çarpımlarının da değişmeli olduğu elde edilmiş ve direkt çarpımların homomorfizmaları da çalışılmıştır.

Anahtar Kelimeler: Bitonic cebirleri; Direk çarpım; Homomorfizmalar; Süzgeçler.

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1. Introduction

In 1984, the form of BCC-algebras was presented by Komori [1] and Dudek [2] as a generalization of BCK algebra that was introduced by Iseki [3] in 1966 and studied by him and Tanaka [4] in 1978. A dual BCC-algebra is an algebraic system $(X, *, 1)$ satisfies the following axioms: (D1) $(x * y) * ((y * z) * (x * z)) = 1$, (D2) $1 * x = x$, (D3) $x * 1 = 1$, (D4) $x * x = 1$, (D5) $x * y = 1$ and $y * x = 1$ imply $x = y$. The notion of dual BCC-algebra is a generalization of DBCK-algebras [5-7], Hilbert algebras [8-11], Heyting algebras [12, 13], implications algebras [14] and lattice implication algebras [15, 16]. The property (P): $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$ is satisfied by all such algebras. Indeed, it can be said that these are the algebras that have the axiom (P). The notion of bitonic algebra as a generalization of dual BCC-algebra was introduced by Yong Ho Yon and Şule Ayar Özbal in 2018 [17]. The notion of direct product was firstly studied in group and some of their generalizations were obtained, such as the direct product of the group is a group and the direct product of the abelian group is again an abelian group are the ones that can be given as properties that are obtained in these studies. In 2016, the notion of direct product of B-algebra, 0-commutative B-algebra and B-homomorphism were studied by Lingcong and Endham [18]. In 2020, the concept of direct product of BP-algebras was given by Setani, Gemawati and Deswita [19]. The purpose of this study is to construct the concept of direct product of bitonic algebras, and investigate commutative direct product of commutative bitonic algebras and also homomorphisms on direct product of bitonic algebras are studied.

2. Preliminaries

Definition 1. [17] A bitonic algebra is an algebraic systems $(\mathrm{A}, *, 1)$ where A is a set, 1 is an element in A and $*$ is a binary operation on A, satisfying the following axioms for every $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{A}$,

(B1) $a * 1 = 1$,

(B2) $1 * a = a$,

(B3) $a * b = 1$ and $b * a = 1$ implies $a = b$,

(B4) $a * b = 1$ implies $(c * a) * (c * b) = 1$ and $(b * c) * (a * c) = 1$.

Example 1. [17] Let $N = \{1, x, y, z, w\}$ be a set. If we define a binary operation $*$ on N by the following table:

Table 1: Cayley table of binary operation $*$ on N

$*$	1	x	y	z	w
1	1	x	y	z	w
x	1	1	y	z	w
y	1	x	1	z	w
z	1	1	1	1	x
w	1	1	1	z	1

Then $(N,*,1)$ is a bitonic algebra with Hasse diagram given below.

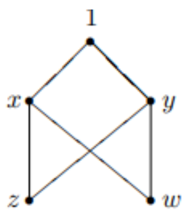

Diagram 1: Hasse diagram of the bitonic algebra N in Example 1

Definition 2. [17] Let A be a bitonic algebra a nonempty subset S of A is labeled a bitonic subalgebra of A if $x * y \in S$ for every $x, y \in S$ and F as a nonempty subset of A is labeled a filter of A if it performs:

(F1) $1 \in F$

(F2) $\mathfrak{e} \in F$ and $\mathfrak{e} * \mathfrak{f} \in F$ imply $\mathfrak{f} \in F$ for any $\mathfrak{e}, \mathfrak{f} \in F$.

Definition 3. [17] A bitonic algebra $(A,*,1)$ is said to be commutative if $(a * b) * b = (b * a) * a$ for all $a, b \in A$.

3. Direct product of Bitonic algebras

Definition 4. Let $(A; *, 1_A)$ and $(B; *', 1_B)$ be bitonic algebras. The direct product of A and B is an algebraic nature $A \times B = (A \times B; \otimes, (1_A, 1_B))$ where $A \times B$ is the set $\{(a, b) | a \in A, b \in B\}$ and the binary operation $\otimes$ is given by $(a_1, b_1) \otimes (a_2, b_2) = (a_1 * a_2, b_1 *', b_2)$.

Example 2. Let $A = \{1_A, a, b, c\}$ be a set. If a binary relation $*$ on A is illustrated by the following table:

Table 2: Cayley table of binary relation $*$ on A in Example 1

$*$	1_A	a	b	c
1_A	1_A	a	b	c
a	1_A	1_A	b	c
b	1_A	a	1_A	c
c	1_A	1_A	1_A	1_A

then $(A; *, 1_A)$ is a bitonic algebra.

Let $B = \{1_B, x, y, 0\}$ be a set. If we define a binary relation $*'$ on B by the following table:

Table 3: Cayley table of binary relation $*'$ on A in Example 1

$*'$	1_B	x	y	0
1_B	1_B	x	y	0
x	1_B	1_B	y	y
y	1_B	x	1_B	0
0	1_B	1_B	1_B	1_B

then $(B; *', 1_B)$ is a bitonic algebra. It is clear that the direct product of A and B is a bitonic algebra $A \times B = (A \times B; \otimes, (1_A, 1_B))$ whose Cayley table is given below

Table 4: Cayley table of binary relation $\otimes$ on $A \times B$ given in Example 1

$\otimes$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(a,1_B)$	(a,x)	(a,y)	$(a,0)$	$(b,1_B)$	(b,x)	(b,y)	$(b,0)$	$(c,1_B)$	(c,x)	(c,y)	$(c,0)$
$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(a,1_B)$	(a,x)	(a,y)	$(a,0)$	$(b,1_B)$	(b,x)	(b,y)	$(b,0)$	$(c,1_B)$	(c,x)	(c,y)	$(c,0)$
$(1_A,x)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,y)$	$(1_A,y)$	$(a,1_B)$	$(a,1_B)$	(a,y)	(a,y)	$(b,1_B)$	$(b,1_B)$	(b,y)	(b,y)	$(c,1_B)$	$(c,1_B)$	(c,y)	(c,y)
$(1_A,y)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,1_B)$	$(1_A,0)$	$(a,1_B)$	$(a,1_B)$	(a,x)	$(a,0)$	$(b,1_B)$	(b,x)	$(b,1_B)$	$(b,0)$	$(c,1_B)$	(c,x)	$(c,1_B)$	$(c,0)$
$(1_A,0)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(a,1_B)$	$(a,1_B)$	$(a,1_B)$	$(a,1_B)$	$(b,1_B)$	$(b,1_B)$	$(b,1_B)$	$(b,1_B)$	$(c,1_B)$	$(c,1_B)$	$(c,1_B)$	$(c,1_B)$
$(a,1_B)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(b,1_B)$	(b,x)	(b,y)	$(b,0)$	$(c,1_B)$	(c,x)	(c,y)	$(c,0)$
(a,x)	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,y)$	$(1_A,y)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,y)$	$(1_A,y)$	$(b,1_B)$	$(b,1_B)$	(b,y)	(b,y)	$(c,1_B)$	$(c,1_B)$	(c,y)	(c,y)
(a,y)	$(1_A,1_B)$	$(1_A,x)$	$(1_A,1_B)$	$(1_A,0)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,1_B)$	$(1_A,0)$	$(b,1_B)$	(b,x)	$(b,1_B)$	$(b,0)$	$(c,1_B)$	(c,x)	$(c,1_B)$	(c,y)
$(a,0)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(b,1_B)$	$(b,1_B)$	$(b,1_B)$	$(b,1_B)$	$(c,1_B)$	$(c,1_B)$	$(c,1_B)$	$(c,1_B)$
$(b,1_B)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(a,1_B)$	(a,x)	(a,y)	$(a,0)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(c,1_B)$	(c,x)	(c,y)	$(c,0)$
(b,x)	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,y)$	$(1_A,y)$	$(a,1_B)$	$(a,1_B)$	(a,y)	(a,y)	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,y)$	$(1_A,y)$	$(c,1_B)$	$(c,1_B)$	(c,y)	(c,y)
(b,y)	$(1_A,1_B)$	$(1_A,x)$	$(1_A,1_B)$	$(1_A,0)$	$(a,1_B)$	(a,x)	$(a,1_B)$	$(a,0)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,0)$	$(c,1_B)$	(c,x)	$(c,1_B)$	$(c,0)$
$(b,0)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(a,1_B)$	$(a,1_B)$	$(a,1_B)$	$(a,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(c,1_B)$	$(c,1_B)$	$(c,1_B)$	$(c,1_B)$
$(c,1_B)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,y)$	$(1_A,0)$
(c,x)	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,y)$	$(1_A,y)$	$(1_A,y)$	$(1_A,y)$	$(1_A,y)$	$(1_A,y)$	$(1_A,y)$	$(1_A,y)$
(c,y)	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,x)$	$(1_A,x)$	$(1_A,x)$	$(1_A,x)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,y)$	$(1_A,1_B)$	$(1_A,0)$	$(1_A,0)$	$(1_A,0)$	$(1_A,0)$
$(c,0)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$	$(1_A,1_B)$

The next theorem is one of the main theorems of this study.

Theorem 1. $(A; *, 1_A)$ and $(B; *', 1_B)$ are bitonic algebras if and only if $(A \times B; \otimes, (1_A, 1_B))$ is a bitonic algebra.

Proof. Let $(A; *, 1_A)$ and $(B; *', 1_B)$ be bitonic algebras, then we have $\mathfrak{z} * 1_A = 1_A$ and $\mathfrak{f} *' 1_B = 1_B$, $1_A * \mathfrak{z} = \mathfrak{z}$ and $1_B *' \mathfrak{f} = \mathfrak{f}$ for ant elements $\mathfrak{z} \in A$ and $\mathfrak{f} \in B$. The direct product of *A* and *B* is an algebraic nature $(A \times B; \otimes, (1_A, 1_B))$.

Thus, for all $(\mathfrak{z}, \mathfrak{f}) \in A \times B$ we have

$$(\mathfrak{z}, \mathfrak{f}) \otimes (1_A, 1_B) = (\mathfrak{z} * 1_A, \mathfrak{f} *' 1_B) = (1_A, 1_B). \tag{1}$$

Then axiom (B1) is satisfied.

We have

$$(1_A, 1_B) \otimes (\mathfrak{z}, \mathfrak{f}) = (1_A * \mathfrak{z}, 1_B *' \mathfrak{f}) = (\mathfrak{z}, \mathfrak{f}). \tag{2}$$

Then the axiom (B2) is satisfied.

Let $(\mathfrak{z}, \mathfrak{f})$, $(\mathfrak{K}, \mathfrak{X}) \in A \times B$. Then $(\mathfrak{z}, \mathfrak{f}) \otimes (\mathfrak{K}, \mathfrak{X}) = (1_A, 1_B)$ and $(\mathfrak{K}, \mathfrak{X}) \otimes (\mathfrak{z}, \mathfrak{f}) = (1_A, 1_B)$, that is $(\mathfrak{z} * \mathfrak{K}, \mathfrak{f} *' \mathfrak{X}) = (1_A, 1_B)$ and $(\mathfrak{K} * \mathfrak{z}, \mathfrak{X} *' \mathfrak{f}) = (1_A, 1_B)$. Since $\mathfrak{z}, \mathfrak{K} \in A$, $\mathfrak{f}, \mathfrak{X} \in B$ we have $\mathfrak{z} * \mathfrak{K} = 1_A$ and $\mathfrak{K} * \mathfrak{z} = 1_A$ imply $\mathfrak{z} = \mathfrak{K}$ and $\mathfrak{f} *' \mathfrak{X} = 1_B$ and $\mathfrak{X} *' \mathfrak{f} = 1_B$ imply $\mathfrak{f} = \mathfrak{X}$ we get $(\mathfrak{z}, \mathfrak{f}) = (\mathfrak{K}, \mathfrak{X})$. So, axiom (B3) is satisfied.

Let $(\mathfrak{z}, \mathfrak{f}), (\mathfrak{K}, \mathfrak{X}), (\beth, q) \in A \times B$ and $(\mathfrak{z}, \mathfrak{f}) \otimes (\mathfrak{K}, \mathfrak{X}) = (1_A, 1_B)$. Hence, we get $\mathfrak{z} * \mathfrak{K} = 1_A$, $\mathfrak{f} *' \mathfrak{X} = 1_B$ for all $\mathfrak{z}, \mathfrak{K} \in A$, $\mathfrak{f}, \mathfrak{X} \in B$. Since *A* and *B* are bitonic algebras, we have $(\beth * \mathfrak{z}) * (\beth * \mathfrak{K}) = 1_A$ and $(q *' \mathfrak{f}) *' (q *' \mathfrak{X}) = 1_B$ and $(\mathfrak{K} * \beth) * (\mathfrak{z} * \beth) = 1_A$ and $(\mathfrak{X} *' q) *' (\mathfrak{f} *' q) = 1_B$. Then we have

$$\begin{aligned}
\big((\beth, q) \otimes (\mathfrak{z}, \mathfrak{f})\big) \otimes \big((\beth, q) \otimes (\mathfrak{K}, \mathfrak{X})\big) &= (\beth * \mathfrak{z}, q *' \mathfrak{f}) \otimes (\beth * \mathfrak{K}, q *' \mathfrak{X}) \\
&= ((\beth * \mathfrak{z}) * (\beth * \mathfrak{K}), (q *' \mathfrak{X}) *' (q *' \mathfrak{X})) \\
&= (1_A, 1_B)
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
((x, y) \otimes (p, q)) \otimes ((a, b) \otimes (p, q)) &= (x * p, y *' q) \otimes (a * p, b *' q) \\
&= ((x * p) * (a * p), (y *' q) *' (b *' q))
\end{aligned}$$

$$= (1_A, 1_B). \quad (4)$$

So, the axiom (B4) is satisfied. Finally, it is obvious that (B1), (B2), (B3), (B4) are satisfied for bitonic algebras. Hence, $A \times B$ is a bitonic algebra.

Conversely, let $A \times B$ be bitonic algebras and let $(a, b), (x, y), (p, q)$ be in $A \times B$. Then we have $(a, b) \otimes (1_A, 1_B) = (1_A, 1_B)$ that is $a * 1_A = 1_A$ and $b *' 1_B = 1_B$. So, axiom (B1) is satisfied for *A* and *B*.

Also, $(1_A, 1_B) \otimes (a, b) = (a, b)$, that is $1_A * a = a$ and $1_B *' = b$. This means that axiom (B2) is satisfied for *A* and *B*.

We also have $(\mathfrak{z}, \mathfrak{f}) \otimes (\mathfrak{K}, \mathfrak{X}) = (1_A, 1_B)$ and $(\mathfrak{K}, \mathfrak{X}) \otimes (\mathfrak{z}, \mathfrak{f}) = (1_A, 1_B)$ implying that $(\mathfrak{K}, \mathfrak{X}) = (\mathfrak{z}, \mathfrak{f})$. That is $\mathfrak{z} * \mathfrak{K} = 1_A$ and $\mathfrak{K} * \mathfrak{z} = 1_A$ implying $\mathfrak{z} = \mathfrak{K}$, and $\mathfrak{f} *' \mathfrak{X} = 1_B$ and $\mathfrak{X} *' \mathfrak{f} = 1_B$ implying $\mathfrak{f} = \mathfrak{X}$. Hence, axiom (B3) is satisfied for *A* and *B*.

Additionally,$(a, b) \otimes (x, y) = (1_A, 1_B)$ implies that $((p, q) \otimes (a, b)) \otimes ((p, q) \otimes (x, y)) = (1_A, 1_B)$ and $((x, y) \otimes (p, q)) \otimes ((a, b) \otimes (p, q)) = (1_A, 1_B)$. So, $a * x = 1_A$ implies that $(p * a) * (p * x) = 1_A$ and $(x * p) * (a * p) = 1_A$ and $b *' y = 1_B$ implies that $(q *' b) *' (q *' y) = 1_B$ and $(y *' q) *' (y *' b) = 1_B$. Thus, axiom (B4) is satisfied for *A* and *B*. Therefore, *A* and *B* are bitonic algebras.

Also, we can generalize this product to any finite family of bitonic algebras.

Definition 5. Let $(A_i, *^i, 1_i)$ be a finite family of bitonic algebras for each $i \in \{1, \ldots, s\}$. Then we can define direct product of A_i to be the structure $(\prod_{i=1}^{s} A_i; \otimes, (1_{A_1}, \ldots, 1_{A_s}))$ whose operation is $(a_1, \ldots, a_s) \otimes (x_1, \ldots, x_s) = (a_1 *^1 x_1, \ldots, a_s *^s x_s)$ for all $a_i, x_i \in A_i$, $i \in \{1, \ldots, s\}$.

Then we have the following corollary.

Corollary 1. $(A_1, *^1, 1_1)$, $(A_2, *^2, 1_2), \ldots$, $(A_s, *^s, 1_s)$ are bitonic algebras if and only if $(\prod_{i=1}^{s} A_i; \otimes, (1_{A_1}, \ldots, 1_{A_s}))$ is a bitonic algebra for $i \in \{1, \ldots, s\}$.

Proof. Clear.

Corollary 2. Let $(A_i, *^i, 1_i)$ be a finite family of bitonic algebras for each $i \in \{1, \ldots, s\}$. Then each A_i is commutative bitonic algebras if and only if $(\prod_{i=1}^{s} A_i; \otimes, (1_{A_1}, \ldots, 1_{A_s}))$ is commutative.

Proof. Let each of $(A_i,*^i,1_i)$ be commutative for all $i \in \{1,\dots,s\}$. If $(a_1,\dots,a_s),(b_1,\dots,b_s) \in \prod_{i=1}^{s} A_i$, then $(a_i *^i b_i) *^i b_i = (b_i *^i a_i) *^i a_i$ for all $a_i, b_i \in A_i$ and $i \in \{1,\dots,s\}$. Then we have

$$((a_1,\dots,a_s) \otimes (b_1,..,b_s)) \otimes ((b_1,..,b_s)) = ((a_1 *^1 b_1),\dots,(a_s *^s b_s)) \otimes (b_1,..,b_s)$$

$$= ((a_1 *^1 b_1) *^1 b_1,\dots,(a_s *^s b_s) *^s b_s)$$

$$= ((b_1 *^1 a_1) *^1 a^1,\dots,(b_s *^s a_s) *^s a_s)$$

$$= ((b_1 *^1 a_1),\dots,(b_s *^s a_s)) \otimes (a_1,..,a_s)$$

$$= ((b_1,\dots,b_s) \otimes (a_1,..,a_s)) \otimes ((a_1,..,a_s). \quad (5)$$

This implies $\prod_{i=1}^{s} A_i$ is commutative.

Conversely, let $\prod_{i=1}^{s} A_i$ be commutative. This is to say, if $a_i, b_i \in A_i$ for all $i \in \{1,\dots,s\}$ then $(a_1,\dots,a_s),(b_1,..,b_s) \in \prod_{i=1}^{s} A_i$. We have

$$\big((a_1,\dots,a_s) \otimes (b_1,..,b_s)\big) \otimes \big((b_1,..,b_s)\big)$$

$$= ((b_1,\dots,b_s) \otimes (a_1,..,a_s)) \otimes ((a_1,..,a_s)). \quad (6)$$

That is

$$((a_1 *^1 b_1) *^1 b_1,\dots,(a_s *^s b_s) *^s b_s) = ((b_1 *^1 a_1),\dots,(b_s *^s a_s)) \otimes ((a_1,..,a_s)$$

$$= ((b_1 *^1 a_1) *^1 a^1,\dots,(b_s *^s a_s) *^s a_s). \quad (7)$$

Hence, we get $(a_i *^i b_i) *^i b_i = (b_i *^i a_i) *^i a_i$ for all $a_i, b_i \in A_i$ and $i \in \{1,\dots,s\}$. Therefore, each A_i is commutative.

2. Homomorphisms of direct product of Bitonic algebras

Definition 6. Let $(X;*,1_X)$ and $(Y;*',1_Y)$ be bitonic algebras. An assignment $\beta: X \to Y$ is labeled a bitonic homorphism if $\beta(x * y) = \beta(x) *' \beta(y)$ for any $x, y \in X$.

Theorem 2. Let $(A_i,*^i,1_i)$ and $(B_i,*^i,1_i)$ be a finite family of bitonic algebras and $\beta_i: A_i \to B_i$ be bitonic homomorphisms for each $i \in \{1,\dots,s\}$. If the mapping $\beta: \prod_{i=1}^{s} A_i \to \prod_{i=1}^{s} B_i$ given by $\beta(a_1,\dots,a_s) = (\beta(a_1),\dots,\beta(a_s))$, then β is a bitonic homomorphism with $ker_\beta = \prod_{i=1}^{s} ker_{\beta_i}$, $\beta(\prod_{i=1}^{s} A_i) = \prod_{i=1}^{s} \beta_i(A_i)$.

Proof. Let $(A_i,*^i,1_i)$ and $(B_i,*^i,1_i)$ be a finite family of bitonic algebras and $\beta_i: A_i \rightarrow B_i$ be bitonic homomorphisms for each $i \in \{1,\dots,s\}$ and let β be the mapping $\prod_{i=1}^{s} A_i \rightarrow \prod_{i=1}^{s} B_i$ given by $(a_1,\dots,a_s) \mapsto (\beta(a_1),\dots,\beta(a_s))$.

Let $(a_1,\dots,a_s),(b_1,\dots,b_s) \in \prod_{i=1}^{s} A_i$,then

$$\begin{aligned}\beta\big((a_1,\dots,a_s)\otimes(b_1,\dots,b_s)\big) &= \beta(a_1 *^1 b_1,\dots,a_s *^s b_s)\\ &= (\beta_1(a_1 *^1 b_1),\dots,\beta_s(a_s *^s b_s))\\ &= (\beta_1(a_1) *^1 \beta_1(b_1),\dots,\beta_s(a_s) *^s \beta_s b_s))\\ &=(\beta_1(a_1),\dots,\beta_s(a_s))\otimes(\beta_1(b_1),\dots,\beta_s(b_s))\\ &= \beta\big((a_1,\dots,a_s)\big)\otimes\beta\big((b_1,\dots,b_s)\big). \qquad (8)\end{aligned}$$

Thus, we have that β is a bitonic homomorphism. Also, if β is a bitonic homomorphism, then each β_i is a bitonic homomorphism.

Let $(a_1,\dots,a_s) \in ker_\beta$. Then

$$\begin{aligned}(a_1,\dots,a_s) \in ker_\beta &\Leftrightarrow \beta((a_1,\dots,a_s)) = (1_1,\dots,1_s)\\ &\Leftrightarrow (\beta_1(a_1),\dots,\beta_s(a_s)) = (1_1,\dots,1_s)\\ &\Leftrightarrow \beta_i(a_i) = 1_i \text{ for each } i \in \{1,\dots,s\}\\ &\Leftrightarrow a_i \in ker_{\beta_i} \text{ for each } i \in \{1,\dots,s\}\\ &\Leftrightarrow (a_1,\dots,a_s) \in \prod_{i=1}^{s} ker_{\beta_i}. \qquad (9)\end{aligned}$$

That is to say $ker_\beta = \prod_{i=1}^{s} ker_{\beta_i}$.

Finally, let β be one-to-one. If $\beta_i(a_i) = \beta_i(b_i)$ for each $i \in \{1,\dots,s\}$, then

$$\begin{aligned}\beta((a_1,\dots,a_s)) &= (\beta_1(a_1),\dots,\beta_s(a_s))\\ &= (\beta_1(b_1),\dots,\beta_s(b_s))\\ &= \beta((b_1,\dots,b_s)). \qquad (10)\end{aligned}$$

We have that β is one-to-one, therefore $(a_1,\dots,a_s) = (b_1,\dots,b_s)$. Hence, $a_i = b_i$ for each $i \in \{1,\dots,s\}$. That is β_i is one-to-one for each $i \in \{1,\dots,s\}$.

Conversely, let β_i be one-to-one for each $i \in \{1,\dots,s\}$. If $\beta((a_1,\dots,a_s)) = \beta((b_1,\dots,b_s))$, then

$$\begin{aligned}(\beta_1(a_1),\dots,\beta_s(a_s)) &= \beta((a_1,\dots,a_s))\\ &= \beta((b_1,\dots,b_s))\\ &= (\beta_1(b_1),\dots,\beta_s(b_s)).\end{aligned} \quad (11)$$

Since $\beta_i(a_i) = \beta_i(b_i)$ for each $i \in \{1,\dots,s\}$ and all β_i is one-to-one, we get $a_i = b_i$ for each $i \in \{1,\dots,s\}$ and hence $(a_1,\dots,a_s) = (b_1,\dots,b_s)$. So, β is one-to-one.

Finally, let β be onto. If $b_i \in B_i$ for each $i \in \{1,\dots,s\}$ then $(b_1,\dots,b_s) \in \prod_{i=1}^{s} B_i$. Since β is onto, there exist $(a_1,\dots,a_s) \in \prod_{i=1}^{s} A_i$ such that $(b_1,\dots,b_s) = \beta((a_1,\dots,a_s)) = (\beta_1(a_1),\dots,\beta_s(a_s))$. Hence $b_i = \beta_i(a_i)$ for some $i \in \{1,\dots,s\}$. Therefore, β_i is onto for all $i \in \{1,\dots,s\}$.

Conversely, let β_i be onto for all $i \in \{1,\dots,s\}$. If $(b_1,\dots,b_s) \in \prod_{i=1}^{s} B_i$ then $b_i \in B_i$ for all $i \in \{1,\dots,s\}$. So, there exists $a_i \in A_i$ such that $b_i = \beta_i(a_i)$ for some $i \in \{1,\dots,s\}$ since β_i is onto. Therefore, $(b_1,\dots,b_s) = (\beta_1(a_1),\dots,\beta_s(a_s)) = \beta((a_1,\dots,a_s))$. Hence, β is onto.

Theorem 3. Let $(A_i,*^i,1_i)$ and $(B_i,*^i,1_i)$ be a finite family of bitonic algebras and $\beta_i\colon A_i \to B_i$ be bitonic homomorphisms for $i \in \{1,\dots,s\}$ and let let β be given by $\prod_{i=1}^{s} A_i \to \prod_{i=1}^{s} B_i$ given by $(a_1,\dots,a_s) \mapsto (\beta(a_1),\dots,\beta(a_s))$, then $ker_\beta = \prod_{i=1}^{s} ker_{\beta_i}$ is a filter.

Proof. Let $(A_i,*^i,1_i)$ and $(B_i,*^i,1_i)$ be a finite family of bitonic algebras and $\beta_i\colon X_i \to B_i$ be bitonic homomorphisms for $i \in \{1,\dots,s\}$. Then

$$\begin{aligned}\beta((1_1,\dots,1_s)) = \beta(a_1 *^1 1_1,\dots,a_s *^s 1_s) &= (\beta_1(a_1 *^1 1_1),\dots,\beta_s(a_s *^s 1_s))\\ &= ((\beta_1(a_1),\dots,\beta_s(a_s)) \otimes (\beta_1(1_1),\dots,\beta_s(1_s))\\ &= (1_1,\dots,1_s).\end{aligned} \quad (12)$$

So, $(1_1,\dots,1_s) \in ker_\beta$, $ker_\beta \neq \emptyset$.

Let $(a_1,\dots,a_s) \in ker_\beta$ and $(a_1,\dots,a_s) \otimes (b_1,\dots,b_s) \in ker_\beta$. Consider

$$\begin{aligned}(1_1,\dots,1_s) &= \beta((a_1,\dots,a_s) \otimes (b_1,\dots,b_s))\\ &= \beta(a_1 *^1 b_1,\dots,a_s *^s b_s)\end{aligned}$$

$$= (\beta_1(a_1) *^1 \beta_1(b_1), \ldots, \beta_s(a_s) *^s \beta_s(b_s))$$

$$= \beta((a_1, \ldots, a_s)) \otimes \beta((b_1, \ldots, b_s))$$

$$= (1_1, \ldots, 1_s) \otimes \beta((b_1, \ldots, b_s))$$

$$= \beta((b_1, \ldots, b_s)). \quad (13)$$

This implies $(b_1, \ldots, b_s) \in ker_\beta$. Therefore, ker_β is a filter.

Theorem 4. Let $(A_i, *^i, 1_i)$ and $(B_i, *^i, 1_i)$ be a finite family of bitonic algebras and $\beta_i: A_i \to B_i$ be bitonic homomorphisms for $i \in \{1, \ldots, s\}$ and let $\beta: \prod_{i=1}^{s} A_i \to \prod_{i=1}^{s} B_i$ given by $\beta(a_1, \ldots, a_s) = (\beta(a_1), \ldots, \beta(a_s))$ then

i) β is a bitonic monomorphism if and only if β_i is a bitonic monomorphism.

ii) β is a bitonic onto homomorphism if and only if β_i is a onto homomorphism.

Proof. Let $(A_i, *^i, 1_i)$ and $(B_i, *^i, 1_i)$ be a finite family of bitonic algebras and $\beta_i: A_i \to B_i$ be bitonic homomorphisms for $i \in \{1, \ldots, s\}$ and let $\beta: \prod_{i=1}^{s} A_i \to \prod_{i=1}^{s} B_i$ given by $\beta(a_1, \ldots, a_s) = (\beta(a_1), \ldots, \beta(a_s))$. Then

i) Let β be a bitonic monomorphism and $\beta_i(a_i) = \beta_i(b_i)$ for $i \in \{1, \ldots, s\}$. Then,

$$(\beta_1(a_1), \ldots, \beta_s(a_s)) = (\beta_1(b_1), \ldots, \beta_s(b_s)) \Rightarrow \beta((a_1, \ldots, a_s)) = \beta((b_1, \ldots, b_s)). \quad (14)$$

Since β is a bitonic monomorphism we have $(a_1, \ldots, a_s) = (b_1, \ldots, b_s)$, that is $a_i = b_i$ for $i \in \{1, \ldots, s\}$. Hence, we get β_i is a bitonic monomorphism.

Conversely, let β_i be bitonic monomorphisms. And consider, $\beta((a_1, \ldots, a_s)) = \beta((b_1, \ldots, b_s))$ for $a_i, b_i \in \prod_{i=1}^{s} A_i$. Then $(\beta_1(a_1), \ldots, \beta_s(a_s)) = (\beta_1(b_1), \ldots, \beta_s(b_s)) \Longrightarrow \beta_i(a_i) = \beta_i(b_i)$. Since β_i is bitonic monomorphism we have $a_i = b_i$ for $i \in \{1, \ldots, s\}$. Therefore, $(a_1, \ldots, a_s) = (b_1, \ldots, b_s)$ and β is a bitonic monomorphism.

ii) Let β be a bitonic onto homomorphism and let $b_i \in B_i$ for $i \in \{1, \ldots, s\}$ then we have

$(b_1, \ldots, b_s) \in \prod_{i=1}^{s} B_i$. Since β is a onto homomorphism, then $(a_1, \ldots, a_s) \in \prod_{i=1}^{s} A_i$ for all a_i for $i \in \{1, \ldots, s\}$, so $(b_1, \ldots, b_s) = \beta(a_1, \ldots, a_s) = (\beta_1(a_1), \ldots, \beta_s(a_s))$ implying that $b_i = \beta_i(a_i)$ for $i \in \{1, \ldots, s\}$. Therefore, it is proved that β_i is an onto homomorphism.

Conversely, let β_i be a bitonic epimorphism for all $i \in \{1, \ldots, s\}$ and $(b_1, \ldots, b_s) \in \prod_{i=1}^{s} B_i$, then $b_i \in B_i$. Since β_i is an onto function, then there exists $a_i \in \prod_{i=1}^{s} A_i$ for all $i \in$

$\{1,\ldots,s\}$ such that $b_i = \beta_i(a_i)$ implying that $(b_1,\ldots,b_s) = (\beta_1(a_1),\ldots,\beta_s(a_s)) = \beta(a_1,\ldots,a_s)$. Hence, it is proved that β is a bitonic epimorphism.

Theorem 5. Let $\{A_i = (A_i,*^i,1_i) | i \in \{1,\ldots,s\}\}$ be a family of bitonic algebras and let J_i be a filter of A_i. Then $\prod_{i=1}^{s} J_i$ is a filter of $\prod_{i=1}^{s} A_i$ and $\prod_{i=1}^{s} A_i \big/ \prod_{i=1}^{s} J_i \cong \prod_{i=1}^{s} \left(A_i / J_i \right)$.

Proof. Let $\{A_i = (A_i,*^i,1_i) | i \in \{1,\ldots,s\}\}$ be a family of bitonic algebras and let J_i be a filter of A_i. Then $(1_1,\ldots,1_s) \in \prod_{i=1}^{s} J_i$ since $1_i \in J_i$ for all $i \in \{1,\ldots,s\}$ and so $\prod_{i=1}^{s} J_i$ is not empty. Let $(a_1,\ldots,a_s) \in \prod_{i=1}^{s} J_i$ and $(a_1,\ldots,a_s) \otimes (b_1,\ldots,b_s) \in \prod_{i=1}^{s} J_i$. Then $(a_1 *^1 b_1,\ldots,a_s *^s b_s) \in \prod_{i=1}^{s} J_i$. This is to say that $(a_i *^i b_i) \in J_i$ for $i \in \{1,\ldots,s\}$. Since J_i is a filter of A_i we have $b_i \in J_i$. Hence, $(b_1,\ldots,b_s) \in \prod_{i=1}^{s} J_i$. Therefore, $\prod_{i=1}^{s} J_i$ is a filter.

Let $J = \prod_{i=1}^{s} J_i$ and $A = \prod_{i=1}^{s} A_i$. Define $\varpi\colon A/J \colon \longrightarrow \prod_{i=1}^{s} \left(A_i / J_i \right)$ given by $\varpi((a_1,\ldots,a_s)J) = (a_1J_1,\ldots,a_sJ_s) \in A/J$ for all $(a_1,\ldots,a_s)J \in A/J$.

Let $(a_1,\ldots,a_s)J, (b_1,\ldots,b_s)J \in A/J$. If $(a_1,\ldots,a_s)J = (b_1,\ldots,b_s)J$, then $(a_1,\ldots,a_s) \sim_J (b_1,\ldots,b_s)$, that is $(a_1 *^1 b_1,\ldots,a_s *^s b_s) = (a_1,\ldots,a_s) \otimes (b_1,\ldots,b_s) \in J$. Thus, $a_i *^i b_i \in J_i$ for all $i \in \{1,\ldots,s\}$, that is $a_i \sim_{J_i} b_i$ so that $a_iJ_i = b_iJ_i$. Therefore, $\varpi((a_1,\ldots,a_s)J) = (a_1J_1,\ldots,a_sJ_s) = (b_1J_1,\ldots,b_sJ_s) = \varpi((b_1,\ldots,b_s)J)$. Hence, ϖ is well - defined.

If $(a_1,\ldots,a_s)J, (b_1,\ldots,b_s) \in A/J$, then

$$\begin{aligned}
\varpi((a_1,\ldots,a_s)J *^i (b_1,\ldots,b_s)J) &= \varpi(((a_1,\ldots,a_s) \otimes (b_1,\ldots,b_s))J) \\
&= \varpi((a_1 *^1 b_1,\ldots,a_s *^s b_s)J) \\
&= ((a_1 *^1 b_1)J_1,\ldots,((a_s *^s b_s)J_s)) \\
&= (a_1J_1 *^1 b_1J_1,\ldots,a_sJ_s *^s b_sJ_s) \\
&= (a_1J_1,\ldots,a_sJ_s) \otimes (b_1J_1,\ldots,b_sJ_s) \\
&= \varpi((a_1,\ldots,a_s)J) \otimes \varpi((b_1,\ldots,b_s)J). \qquad (15)
\end{aligned}$$

This gives us that ϖ is a homomorphism.

If $\varpi((a_1,\ldots,a_s)J) = \varpi((b_1,\ldots,b_s)J)$, then

$(a_1J_1,\dots,a_sJ_s)=\varpi\,((a_1,\dots,a_s)J)$

$$= \varpi((b_1,\dots,b_s)J) = (b_1J_1,\dots,b_sJ_s). \tag{16}$$

Therefore, $a_iJ_i = {}^{b_i}/_{J_i}$ for all $i \in \{1,\dots,s\}$. Hence, $a_i \sim_{J_i} b_i$ that is $a_i *^i b_i \in J_i$ for all $i \in \{1,\dots,s\}$ so that $(a_1,\dots,a_s) \otimes (b_1,\dots,b_s) = (a_1 *^1 b_1,\dots,a_s *^s b_s) \in J$. Therefore, $(a_1,\dots,a_s) \sim_J (b_1,\dots,b_s)$ and so $(a_1,\dots,a_s)J = (b_1,\dots,b_s)J$. This implies ϖ is one-to-one.

If $(a_1J_1,\dots,a_sJ_s) \in \prod_{i=1}^{s}\left({}^{A_i}/_{J_i}\right)$, then $a_i \in A_i$ for all $i \in \{1,\dots,s\}$, that is $(a_1,\dots,a_s) \in A$. It gives us that $(a_1J_1,\dots,a_sJ_s) = \varpi\,((a_1,\dots,a_s)J)$, where $(a_1,\dots,a_s)J \in {}^{A}/_{J}$. This follows that ϖ is onto. Therefore, ϖ is a bitonic isomorphism that is ${}^{\prod_{i=1}^{s} A_i}/_{\prod_{i=1}^{s} J_i} \cong \prod_{i=1}^{s}\left({}^{A_i}/_{J_i}\right)$.

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