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Research Article

Inverse problem for differential systems having a singularity and turning point of even or odd order

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Abstract

In this paper, the canonical property of the solutions and the inverse problem for a system of differential equations having a singularity and turning point of even or odd order are investigated. First, we study the infinite product representation of the solutions of the system in turning case, and derive the corresponding dual equations. Then, by a replacement, we transform the system of differential equations to a second-order differential equation with a singularity and find the canonical product representation of its solution, and provide a procedure for constructing the solution of the inverse problem. We present a new approach to solve the inverse problems having a singularity inside the interval.

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1. Introduction

We consider the following system of differential equations

$$\frac{dy_1}{dx} = i\rho R(x) y_2, \; \frac{dy_2}{dx} = i\rho \frac{1}{R(x)} y_1, \; x \in [0, L] \tag{1.1}$$

with the initial conditions $y_1(0,\rho) = r_1$, $y_2(0,\rho) = -\frac{1}{\rho} r_2 i$, where r_1 and r_2 are real or complex numbers, $\rho = \sigma + i\tau$ is the spectral parameter and R is a real function which is called the wave resistance.

System (1.1) appears in spectroscopy, acoustic problems, optics and many problems in natural sciences. For example, Maxwell's equations can be reduced to the canonical form (1.1), where t is the variable in the direction of stratification, y_1 and y_2 are the components of the electromagnetic field, $R(t)$ is the wave resistance which describes the refractive properties of the medium and ρ is the wave number in a vacuum (see [18, 25]). In the case when $R > 0$, some aspects of synthesis problems for system (1.1) were studied in [16] and other works.

The fundamental system of solutions (FSS) of second-order differential equations with multiple turning points was realized in [4, 23]. In [6], the authors studied the inverse

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problem for the system with multiple turning points. The asymptotic estimates for a special FSS of the corresponding differential equation and the asymptotic distribution of the eigenvalues with several singularities or/and turning points inside $[0,1]$ were studied in [5]. Asymptotic approximation of the solution of second-order differential equations with two turning points was investigated in [11,14,23]. Note that, the canonical solution of the equation with one turning point of odd order was studied in [12], and the existence and the uniqueness of the solution for corresponding dual equations were investigated. For boundary value problems with singular points, see the works [2,3,6,8,13,19,20] and the references therein. Also, in the case when the problem has two turning points inside a finite interval, see [21]. In [28], the authors considered a singular Sturm-Liouville problem with eigenparameter dependent boundary conditions and two singular endpoints. They approximated the spectrum, and the strongly resolvent convergence and norm resolvent convergence of a sequence of the inherited restriction operators were studied. We mention that, in [1,24,26,27], inverse Sturm-Liouville problems with discontinuity (or jump) conditions were investigated.

To study the inverse problem, it is more complicated and practically is not convenient to use the asymptotic solutions. So, we cannot study reconstructing the coefficients of differential equation from given spectral information and corresponding dual equation by using the asymptotic forms, and we need to use the closed form of the solution. Hence, the infinite product representation of the solution plays an important role for studying the inverse problem.

Consider the system of differential equations (1.1) with

$$R(x) = |x - x_1|^{p-1} R_0(x), \tag{1.2}$$

where $0 < x_1 < L$, p is a real number, $R_0(x) \in W_2^1[0,L], R_0(x) > 0,\ R(0) = 1$ and $R'(0) = 0$.

By means of the replacement

$$y_1(x,\rho) = \sqrt{R(x)}u(x,\rho), \qquad y_2(x,\rho) = \frac{1}{\sqrt{R(x)}}v(x,\rho), \tag{1.3}$$

the system (1.1) can be transformed to the system

$$u' + h(x)u = i\rho v, \qquad v' + h(x)v = i\rho u, \qquad x \in [0,L], \tag{1.4}$$

with the conditions $u(0,\rho) = r_1$, $v(0,\rho) = -\frac{r_2}{\rho}i$, where

$$h(x) = (2R(x))^{-1}R'(x). \tag{1.5}$$

System (1.4) after elimination of v reduces to the equation

$$-u'' + q(x)u = \lambda u, \qquad x \in [0,L], \tag{1.6}$$

and the conditions

$$u(0,\rho) = r_1, \qquad u'(0,\rho) = r_2, \tag{1.7}$$

where $\lambda = \rho^2$ and

$$q(x) = h^2(x) - h'(x). \tag{1.8}$$

It follows from (1.2), (1.5) and (1.8) that $q(x)$ has the form

$$q(x) = \frac{a_1}{(x-x_1)^2} + q_0(x),$$

where $a_1 := (\frac{p}{2})^2 - \frac{1}{4}$. For definiteness, we assume that $\frac{p}{2} \neq 2k,\ k \in Z$. We also assume that $q_0(x)(x-x_1)^{1-|p|} \in L^1(0,L)$.

From [7], we know that the Equation (1.6) has a FSS $\{\varphi_k(x,\lambda)\},\ k = 1,2$, such that

$$\varphi_k^{(m-1)}(0,\lambda) = \delta_{k,m}, \quad k,m = 1,2, \tag{1.9}$$

($\delta_{k,m}$ is the Kronecker delta). Moreover,

$$<\varphi_1(x,\lambda), \varphi_2(x,\lambda)>= 1. \tag{1.10}$$

Lemma 1.1 ([7]). *For $(\rho, x) \in \Omega$, $x \in \omega_s$, $s = 0, 1$, $k, m = 1, 2$,*

$$\varphi_k^{(m-1)}(x,\lambda) = \frac{1}{2}(i\rho)^{m-k}\{\exp(i\rho x)[1]_\gamma + (-1)^{m-k}\exp(-i\rho x)[1]_\gamma$$
$$+(-1)^k 2si\cos\pi\mu \ \exp(i\rho(x-2x_1))[1]_\gamma \}, \quad |\rho| \to \infty, \tag{1.11}$$

where $[1]_\gamma = 1 + O\Big((\rho(x-x_1))^{-1}\Big)$.

From (1.7), (1.9) and using the preceding results, we have

$$u(x,\rho) = r_1\varphi_1(x,\lambda) + r_2\varphi_2(x,\lambda). \tag{1.12}$$

According to (1.11) and (1.12), we have the following theorem.

Theorem 1.2. *For $x \in \omega_s$, $s = 0, 1$, $(\rho, x) \in \Omega$, $|\rho| \to \infty$, $Im\rho \geq 0$, $m = 0, 1$:*

$$u^{(m)}(x,\rho) = \frac{1}{2}(i\rho)^{m-1}(i\rho r_1 + r_2)\exp(i\rho x)[1]_\gamma + \frac{1}{2}(-i\rho)^{m-1}(-i\rho r_1 + r_2)\exp(-i\rho x)[1]_\gamma$$
$$+s(i\rho)^{m-1}(\rho r_1 + ir_2)\cos\pi\mu \ \exp(i\rho(x-2x_1))[1]_\gamma. \tag{1.13}$$

In this paper, first, we study the FSS in the case when the system (1.1) has one turning point of even or odd order (Section 2), and derive the asymptotic form of the solution (Section 3). Then, by using the infinite product representation of the solutions, we derive the corresponding dual equations (Section 4). In Section 5, we transform the system (1.1) to a second-order differential equation with a singularity and present the canonical representation of its solution. In Section 6, we provide a procedure for constructing the solution of the inverse problem. Therefore, we present a new approach to solve the inverse problems having a singularity inside the interval.

2. FSS of differential equation with turning point

In this section, we transform (1.6) by a replacement to a differential equation with one turning point and study the asymptotic behavior of its FSS.

Let $0 < t_1 < 1$ and $\phi(t)$ be a real valued continuous function on $[0, 1]$ such that

$$\int_0^{t_1} |\phi(\zeta)|d\zeta = x_1, \ \int_0^1 |\phi(\zeta)|d\zeta = L.$$

We transform (1.6) by means of the replacement

$$\int_0^t |\phi(\zeta)|d\zeta = x, \ |\phi(t)|^{\frac{1}{2}}y(t) = u(x) \tag{2.1}$$

to the differential equation

$$-y'' + \chi(t)y = \lambda\phi^2(t)y, \qquad t \in [0, 1], \tag{2.2}$$

with initial conditions

$$y(0,\rho) = r_1|\phi(0)|^{\frac{-1}{2}}, \ y'(0,\rho) = r_2|\phi(0)|^{\frac{1}{2}} - \frac{1}{2}r_1|\phi(0)|^{\frac{-3}{2}}\phi'(0), \tag{2.3}$$

where $\chi(t)$ is a bounded integrable function on $[0, 1]$, $\phi^2(t) = A(t-t_1)^\ell$ where A is a real constant and $\ell = \mu^{-1} - 2$. More precisely, we will assume that $\ell \in N$, so $\phi^2(t)$ has the zero t_1 in $(0, 1)$, so called *turning point*. Also, we assume that $r_1 = |\phi(0)|^{\frac{1}{2}}$, $r_2 = \frac{1}{2}|\phi(0)|^{\frac{-3}{2}}\phi'(0)$. Hence, according to the initial conditions (2.3), $y(0,\rho) = 1$ and $y'(0,\rho) = 0$.

Definition 2.1. i) We introduce

$$\sigma \;:=\; \begin{cases} 1 & \mu > \frac{1}{4}, \\ 1-\delta_0 & \mu = \frac{1}{4} \;\; (with\ \delta_0 > 0\ arbitrary\ small), \\ 4\mu_\nu & \mu < \frac{1}{4}, \end{cases}$$

$$[1] \;:=\; 1 + O(\rho^{-\sigma}), \quad as\ \rho \to \infty.$$

ii) For $k \in Z$, we consider the sectors

$$S_k := \{\ \rho \mid \frac{k\pi}{4} \le arg\ \rho \le \frac{(k+1)\pi}{4}\ \}.$$

iii) We distinguish four different types of turning point t_1 as

$$T := \begin{cases} I, & \text{if } \ell \text{ is even and } \phi^2(t)(t-t_1)^{-\ell} < 0 \text{ in } [0,1], \\ II, & \text{if } \ell \text{ is even and } \phi^2(t)(t-t_1)^{-\ell} > 0 \text{ in } [0,1], \\ III, & \text{if } \ell \text{ is odd and } \phi^2(t)(t-t_1)^{-\ell} < 0 \text{ in } [0,1], \\ IV, & \text{if } \ell \text{ is odd and } \phi^2(t)(t-t_1)^{-\ell} > 0 \text{ in } [0,1]. \end{cases}$$

In [4], it is shown that for each fixed $t \in [0,1]$, there exists a FSS of (2.2),

$$\{w_1^{T_1}(t,\rho), w_2^{T_1}(t,\rho)\},$$

which is described by the following formulas.

Case 2.1. Let $T = I$. Then,

$$w_1^I(t,\rho) = \begin{cases} |\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_{t_1}^t \phi(\zeta)d\zeta}[1], & 0 \le t < t_1, \\ \csc\pi\mu\ |\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_{t_1}^t \phi(\zeta)d\zeta}[1], & t_1 < t \le 1, \end{cases} \tag{2.4}$$

$$w_2^I(t,\rho) = \begin{cases} |\phi(t)|^{-\frac{1}{2}}\ e^{-\rho\int_{t_1}^t \phi(\zeta)d\zeta}[1], & 0 \le t < t_1, \\ \sin\pi\mu\ |\phi(t)|^{-\frac{1}{2}}\ e^{-\rho\int_{t_1}^t \phi(\zeta)d\zeta}[1], & t_1 < t \le 1, \end{cases} \tag{2.5}$$

also

$$w_1^I(t_1,\rho) = \frac{\sqrt{2\pi}}{2}(i\rho)^{\frac{1}{2}-\mu}\csc\pi\mu\ e^{i\pi(-\frac{1}{4}+\frac{\mu}{2})}\ \frac{2^\mu\ \psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.6}$$

$$w_2^I(t_1,\rho) = \frac{\sqrt{2\pi}}{2}(i\rho)^{\frac{1}{2}-\mu}\ e^{i\pi(-\frac{1}{4}+\frac{\mu}{2})}\ \frac{2^\mu\ \psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.7}$$

where

$$\psi(t_1) = \lim_{t\to t_1}\ \phi^{\frac{-1}{2}}(t)\{\int_{t_1}^t \phi(\zeta)d\zeta\}^{\frac{1}{2}-\mu}.$$

Thus, we have

$$W(\rho) \equiv W(w_1^I(t,\rho), w_2^I(t,\rho)) = -2\rho[1] \tag{2.8}$$

as $\rho \to \infty$.

Case 2.2. Let $T = II$. Then,

$$w_1^{II}(t,\rho) = \begin{cases} |\phi(t)|^{-\frac{1}{2}}\ e^{i\rho\int_{t_1}^t \phi(\zeta)d\zeta}[1], & 0 \le t < t_1, \\ \csc\pi\mu\ |\phi(t)|^{-\frac{1}{2}}\ \{\ e^{i\rho\int_{t_1}^t \phi(\zeta)d\zeta}[1] + i\cos\pi\mu\ e^{-i\rho\int_{t_1}^t \phi(\zeta)d\zeta}[1]\ \}, & t_1 < t \le 1, \end{cases} \tag{2.9}$$

$$w_2^{II}(t,\rho)=\begin{cases}|\phi(t)|^{-\frac{1}{2}}\{e^{-i\rho\int_{t_1}^{t}\phi(\zeta)d\zeta}[1]+i\cos\pi\mu\ e^{i\rho\int_{t_1}^{t}\phi(\zeta)d\zeta}[1]\}, & 0\le t<t_1,\\ \sin\pi\mu\ |\phi(t)|^{-\frac{1}{2}}\ e^{-i\rho\int_{t_1}^{t}\phi(\zeta)d\zeta}[1], & t_1<t\le 1,\end{cases} \tag{2.10}$$

also

$$w_1^{II}(t_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu}\csc\pi\mu\ e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\ \frac{2^{\mu}\ \psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.11}$$

$$w_2^{II}(t_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu}\ e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\ \frac{2^{\mu}\ \psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.12}$$

$$W(\rho)\equiv W(w_1^{II}(t,\rho),w_2^{II}(t,\rho))=-2i\rho[1]. \tag{2.13}$$

Case 2.3. Let $T=III$. Then,

$$w_1^{III}(t,\rho)=\begin{cases}|\phi(t)|^{-\frac{1}{2}}\ e^{i\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma}[1], & 0\le t<t_1,\\ \frac{1}{2}\ \csc\frac{\pi\mu}{2}|\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma+i\frac{\pi}{4}}[1], & t_1<t\le 1,\end{cases} \tag{2.14}$$

$$w_2^{III}(t,\rho)=\begin{cases}|\phi(t)|^{-\frac{1}{2}}\ \{e^{-i\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma}[1]+ie^{i\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma}[1]\}, & 0\le t<t_1,\\ 2\ \sin\frac{\pi\mu}{2}|\phi(t)|^{-\frac{1}{2}}\ e^{-\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma+i\frac{\pi}{4}}[1], & t_1<t\le 1,\end{cases} \tag{2.15}$$

$$w_1^{III}(t_1,\rho)=\frac{\sqrt{2\pi}}{2}(i\rho)^{\frac{1}{2}-\mu}\csc\pi\mu\frac{2^{\mu}\ \psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.16}$$

$$w_2^{III}(t_1,\rho)=\frac{\sqrt{2\pi}}{2}(i\rho)^{\frac{1}{2}-\mu}e^{i\frac{\pi\mu}{2}}\ \sec(\frac{\pi\mu}{2})\ \frac{2^{\mu}\ \psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.17}$$

$$W(\rho)\equiv W(w_1^{III}(t,\rho),w_2^{III}(t,\rho))=-2i\rho[1]. \tag{2.18}$$

Case 2.4. Let $T=IV$. Then,

$$w_1^{IV}(t,\rho)=\begin{cases}|\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma}[1], & 0\le t<t_1,\\ \frac{1}{2}\ \csc\frac{\pi\mu}{2}|\phi(t)|^{-\frac{1}{2}}\{e^{i\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma-i\frac{\pi}{4}}[1]+e^{-i\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma+i\frac{\pi}{4}}[1]\}, & t_1<t\le 1,\end{cases} \tag{2.19}$$

$$w_2^{IV}(t,\rho)=\begin{cases}|\phi(t)|^{-\frac{1}{2}}\ e^{-\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma}[1], & 0\le t<t_1,\\ 2\ \sin\frac{\pi\mu}{2}|\phi(t)|^{-\frac{1}{2}}\ e^{-i\rho\int_{t_1}^{t}|\phi(\varsigma)|d\varsigma-i\frac{\pi}{4}}[1], & t_1<t\le 1,\end{cases} \tag{2.20}$$

and

$$w_1^{IV}(t_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu}\csc\pi\mu\frac{2^{\mu}\ \psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.21}$$

$$w_2^{IV}(t_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu}e^{-i\frac{\pi\mu}{2}}\ \sec(\frac{\pi\mu}{2})\ \frac{2^{\mu}\ \psi(t_1)}{\Gamma(1-\mu)}[1]. \tag{2.22}$$

Moreover,

$$W(\rho)\equiv W(w_1^{IV}(t,\rho),w_2^{IV}(t,\rho))=-2\rho[1]. \tag{2.23}$$

3. Asymptotic form of the solution

We consider the differential equation (2.2) with the initial conditions

$$C(0,\lambda)=1,\quad C'(0,\lambda)=0. \tag{3.1}$$

Since

$$C(t,\rho)=c_1(\rho)w_1^T(t,\rho)+c_2(\rho)w_2^T(t,\rho),$$

using of Cramer's rule, this leads to the equation

$$C(t,\rho)=\frac{1}{W(\rho)}(w_2'^T(0,\rho)w_1^T(t,\rho)-w_1'^T(0,\rho)w_2^T(t,\rho)), \tag{3.2}$$

where $W(\rho)=W(w_1^T,w_2^T)$.

Let $T=I$.

In this case, taking (2.4), (2.5) and (2.8) in to account we derive

$$C(t,\rho) \tag{3.3}$$
$$=\begin{cases} |\phi(0)|^{\frac12}|\phi(t)|^{-\frac12}\ \{\cosh(\rho\int_0^t|\phi(\zeta)|d\zeta)+O(\frac{1}{\rho^{\sigma_0}})\}, & 0\le t<t_1,\\ \frac12|\phi(0)|^{\frac12}|\phi(t)|^{-\frac12}\ \{\csc\pi\mu\ e^{\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]+\sin\pi\mu\ e^{-\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]\}, & t_1<t\le 1.\end{cases}$$

Thus, the following estimates are valid:

$$C(t,\rho)=\begin{cases}\frac12|\phi(0)|^{\frac12}|\phi(t)|^{-\frac12}\ e^{\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & 0\le t<t_1,\\ \frac12\csc\pi\mu\ |\phi(0)|^{\frac12}|\phi(t)|^{-\frac12}\ e^{\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & t_1<t\le 1,\end{cases}$$

where $E_k(t,\rho)=\sum_{n=1}^{\nu(t)}e^{\rho\alpha_k\beta_{kn}(t)}b_{kn}(t)$, and $\alpha_{-2}=\alpha_1=-1$, $\alpha_0=-\alpha_{-1}=i$, $\beta_{k\nu}(t)\neq 0$, $0<\delta\le\beta_{k1}(t)<\beta_{k2}(t)<\cdots\le\beta_{k\nu(t)}(t)\le 2\max\{R_+(1),R_-(1)\}$, where the integer-valued functions ν and b_{kn} are constant in every interval $[0,t_1-\varepsilon]$ and $[t_1+\varepsilon,1]$ for sufficiently small $\varepsilon>0$ and

$$R_+(t)=\int_0^t\sqrt{\max\{0,\phi^2(\zeta)\}}\ d\zeta,\quad R_-(t)=\int_0^t\sqrt{\max\{0,-\phi^2(\zeta)\}}\ d\zeta\cdot \tag{3.4}$$

Similarly, using (2.6)-(2.8) and (3.2), we obtain

$$C(t_1,\rho)=\frac{\sqrt{2\pi}|\phi(0)|^{\frac12}\rho^{\frac12-\mu}\ e^{i\pi(\frac14-\frac{\mu}{2})}\ 2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\ e^{i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta}\ E_k(t_1,\rho). \tag{3.5}$$

Let $T=II$.

In this case, from (2.9)-(2.13) and (3.2) we derive

$$C(t,\rho) \tag{3.6}$$
$$=\begin{cases}|\phi(0)|^{\frac12}|\phi(t)|^{-\frac12}\ \{\cosh(i\rho\int_0^t|\phi(\zeta)|d\zeta)+O(\frac{1}{\rho^{\sigma_0}})\}, & 0\le t<t_1,\\ \frac12|\phi(0)|^{\frac12}|\phi(t)|^{-\frac12}\ \{M_1(\rho)\ e^{i\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]+M_2(\rho)\ e^{-i\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]\ \}, & t_1<t\le 1,\end{cases}$$

where

$$\begin{cases}M_1(\rho)=\csc\pi\mu\ e^{i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta}-i\cot\pi\mu\ e^{-i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta},\\ M_2(\rho)=i\cot\pi\mu\ e^{i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta}+\csc\pi\mu\ e^{-i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta}.\end{cases} \tag{3.7}$$

By virtue (3.6) and (3.7), the following estimates are valid:

$$C(t,\rho)=\begin{cases}\frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{i\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & 0\le t<t_1,\\[2ex] \frac{1}{2}\csc\pi\mu\ |\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{i\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & t_1<t\le 1.\end{cases}$$

Similarly, from (2.11)-(2.13) and (3.2) we get

$$C(t_1,\rho)=\frac{\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}\rho^{\frac{1}{2}-\mu}\ e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\ 2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\ e^{i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta}\ E_k(t_1,\rho).$$

<u>Let $T=III$.</u>
In this case, it follows from (2.14), (2.15), (2.18) and (3.2) that

$$C(t,\rho)\tag{3.8}$$
$$=\begin{cases}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ \{\cosh(i\rho\int_0^t|\phi(\zeta)|d\zeta)+O(\frac{1}{\rho^{\sigma_0}})\}, & 0\le t<t_1,\\[2ex] \frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ \{N_1(\rho)\ e^{\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]+N_2(\rho)\ e^{-\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]\ \}, & t_1<t\le 1,\end{cases}$$

where

$$\begin{cases}N_1(\rho)=\frac{1}{2}\csc\frac{\pi\mu}{2}\ e^{i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta+i\frac{\pi}{4}}-\frac{1}{2}i\cot\frac{\pi\mu}{2}\ e^{-i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta+i\frac{\pi}{4}},\\[2ex] N_2(\rho)=2\sin\frac{\pi\mu}{2}\ e^{-i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta+i\frac{\pi}{4}},\end{cases}\tag{3.9}$$

Similarly, by virtue (3.8) and (3.9), we find

$$C(t,\rho)=\begin{cases}\frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{i\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & 0\le t<t_1,\\[2ex] \frac{1}{4}\csc\frac{\pi\mu}{2}\ |\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta+\rho\int_{t_1}^t|\phi(\zeta)|d\zeta+i\frac{\pi}{4}}E_k(t,\rho), & t_1<t\le 1.\end{cases}$$

Furthermore, using (2.16)-(2.18) and (3.2), we have

$$C(t_1,\rho)=\frac{\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}(i\rho)^{\frac{1}{2}-\mu}\ 2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\ e^{i\rho\int_0^{t_1}|\phi(\zeta)|d\zeta}\ E_k(t_1,\rho).$$

<u>Let $T=IV$.</u>
In this case, from (2.19), (2.20), (2.23) and (3.2) we obtain

$$C(t,\rho)\tag{3.10}$$
$$=\begin{cases}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ \{\cosh(\rho\int_0^t|\phi(\zeta)|d\zeta)+O(\frac{1}{\rho^{\sigma_0}})\}, & 0\le t<t_1,\\[2ex] \frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ \{F_1(\rho)\ e^{i\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]+F_2(\rho)\ e^{-i\rho\int_{t_1}^t|\phi(\zeta)|d\zeta}[1]\ \}, & t_1<t\le 1,\end{cases}$$

where

$$\begin{cases}F_1(\rho)=\frac{1}{2}\csc\frac{\pi\mu}{2}\ e^{\rho\int_0^{t_1}|\phi(\zeta)|d\zeta-i\frac{\pi}{4}},\\[2ex] F_2(\rho)=\frac{1}{2}\csc\frac{\pi\mu}{2}\ e^{\rho\int_0^{t_1}|\phi(\zeta)|d\zeta+i\frac{\pi}{4}}+2\sin\frac{\pi\mu}{2}\ e^{-\rho\int_0^{t_1}|\phi(\zeta)|d\zeta-i\frac{\pi}{4}}.\end{cases}\tag{3.11}$$

By virtue of (3.10) and (3.11), the following estimates are valid:

$$C(t,\rho)=\begin{cases}\frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & 0\le t<t_1,\\[2ex] \frac{1}{4}\csc\frac{\pi\mu}{2}\ |\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_0^{t_1}|\phi(\zeta)|d\zeta+i\rho\int_{t_1}^t|\phi(\zeta)|d\zeta-i\frac{\pi}{4}}E_k(t,\rho), & t_1<t\le 1.\end{cases}$$

Moreover, using (2.21)-(2.23) and (3.2), we obtain

$$C(t_1,\rho)=\frac{\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}(i\rho)^{\frac{1}{2}-\mu}\ 2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\ e^{\rho\int_0^{t_1}|\phi(\zeta)|d\zeta}\ E_k(t_1,\rho).$$

In addition, we deduce the following theorems.

Theorem 3.1. *Let $T=I$ and $C(t,\rho)$ be the solution of (2.2) under the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$. Then, the following estimate holds:*

$$C(t,\rho)=\frac{1}{2}\{\csc\pi\mu\}^{\nu}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{i\rho\int_0^t\phi(\zeta)d\zeta}E_k(t,\rho),\quad t\in D_\nu,\ \nu=0,1,$$

where $D_0=[0,t_1)$ and $D_1=(t_1,1]$. Moreover,

$$C(t_1,\rho)=\frac{\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}\rho^{\frac{1}{2}-\mu}\ e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\ 2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\ e^{i\rho\int_0^{t_1}\phi(\zeta)d\zeta}\ E_k(t_1,\rho).$$

Theorem 3.2. *Let $T=II$ and $C(t,\rho)$ be the solution of (2.2) under the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$. Then, the following estimates hold:*

$$C(t,\rho)=\begin{cases}\frac{1}{2}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & 0\le t<t_1,\\[2ex] \frac{1}{4}\csc\frac{\pi\mu}{2}\ |\phi(0)|^{\frac{1}{2}}|\phi(t)|^{-\frac{1}{2}}\ e^{\rho\int_0^{t_1}|\phi(\zeta)|d\zeta+i\rho\int_{t_1}^t|\phi(\zeta)|d\zeta-\frac{i\pi}{4}}E_k(t,\rho), & t_1<t\le 1,\end{cases}$$

$$C(t_1,\rho)=\frac{\sqrt{2\pi}|\phi(0)|^{\frac{1}{2}}\rho^{\frac{1}{2}-\mu}\ 2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\ e^{\rho\int_0^{t_1}\phi(\zeta)d\zeta}\ E_k(t_1,\rho).$$

4. Dual equations

For $s\in(0,1)$, consider the boundary value problem $L_1=L_1(\phi^2(t),s)$, $t\in[0,s]$, for Equation (2.2) with the conditions

$$y(0,\lambda)=1,\quad y'(0,\lambda)=0,\quad y(s,\lambda)=0.$$

Let $\ell=2\ell_0$. Then, for $s\in(0,1)\backslash\{t_1\}$, the problem L_1 has a countable set of positive eigenvalues $\{\lambda_n(s)\}_{n\ge1}$. From (3.3), we have the following asymptotic distribution for each $\lambda_n(s)$:

$$\sqrt{\lambda_n(s)}=\frac{n\pi-\frac{\pi}{2}}{\int_0^s\phi(\zeta)d\zeta}+O(\frac{1}{n}).\tag{4.1}$$

Further, according to (3.5), we have

$$\sqrt{\lambda_n(t_1)}=\frac{n\pi+(\frac{\pi\mu}{2}-\frac{3\pi}{4})}{\int_0^{t_1}\phi(\zeta)d\zeta}+O(\frac{1}{n}).$$

Now let $\ell=2\ell_0+1$. Then it follows from Theorem 3.2 that for $s\in(0,t_1)$, the problem L_1 has a countable set of negative eigenvalues $\{\lambda_n^-(s)\}_{n\ge1}$ as follow

$$\sqrt{-\lambda_n^-(s)}=\frac{n\pi}{\int_0^s|\phi(\zeta)|d\zeta}+O(\frac{1}{n}).\tag{4.2}$$

Also, for $t=t_1$,

$$\sqrt{-\lambda_n(t_1)}=\frac{n\pi+(\frac{\pi\mu}{2}-\frac{\pi}{4})}{\int_0^{t_1}|\phi(\zeta)|d\zeta}+O(\frac{1}{n}).$$

Moreover, for $t_1<s<1$, the problem L_1 has two sequences of negative and positive eigenvalues $\{\lambda_n(s)\}=\{\lambda_n^-(s)\}\cup\{\lambda_n^+(s)\}$, $n\in N$, as follows

$$\sqrt{-\lambda_n^-(s)}=-\frac{n\pi-\frac{\pi}{4}}{\int_0^{t_1}|\phi(\zeta)|d\zeta}+O(\frac{1}{n}),\quad \sqrt{\lambda_n^+(s)}=\frac{n\pi-\frac{\pi}{4}}{\int_{t_1}^s|\phi(t)|dt}+O(\frac{1}{n}).\tag{4.3}$$

Since the solution $C(t,\rho)$ defined by a fixed set of initial conditions is an entire function of ρ for each fixed $t\in[0,1]$, it follows from the classical Hadamard's factorization theorem [15] that $C(t,\rho)$ is expressible as an infinite product.

(a) Let $\ell=2\ell_0$ and $C(t,\lambda)$ be the solution of (2.2) satisfying the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$. For fixed $t\in(0,1)\backslash\{t_1\}$, by Halvorsen's result [9], $C(t,\lambda)$ is an entire function of order $\frac{1}{2}$. Hence, we can write $C(t,\lambda)$ as

$$C(t,\lambda)=h(t)\prod_{n\geq 1}(1-\frac{\lambda}{\lambda_n(t)}). \tag{4.4}$$

We know from [22] that for $t\in B_i$, $i=0,1$,

$$C(t,\lambda)=\frac{1}{2}\{\csc\pi\mu\}^i\ \phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\ \prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2}, \tag{4.5}$$

where $B_0=(0,t_1)$, $B_1=(t_1,1)$ and ζ_n, $n\geq 1$, is the sequence of positive zeros of $J'_{\frac{1}{2}}(z)$. Moreover,

$$C(t_1,\lambda)=\frac{1}{2}\phi^{\frac{1}{2}}(0)R_+^{\mu-\frac{1}{2}}(t_1)\psi(t_1)\ \prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2},$$

where j_n, $n=1,2,...$, is the sequence of positive zeros of $J'_\mu(z)$ and

$$\psi(t_1)=\lim_{t\to t_1}\ \phi^{\frac{-1}{2}}(t)\{\int_{t_1}^{t}\phi(s)ds\}^{\frac{1}{2}-\mu}.$$

(b) Let $\ell=2\ell_0+1$ and $C(t,\lambda)$ be the solution of (2.2) with satisfying the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$. From [17], for $0<t<t_1$,

$$C(t,\lambda)=|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{-1}{2}}\ R_-(t)\ \prod_{n\geq 1}\frac{\lambda-\lambda_n^-(t)}{z_n^2},$$

where $z_n=\frac{n\pi}{R_-(t)}$. Further,

$$C(t_1,\lambda)=\frac{|\phi(0)|^{\frac{1}{2}}\psi(t_1)}{2\mu}R_-^{\frac{1}{2}+\mu}(t_1)\ \prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t_1))R_-^2(t_1)}{\widetilde{r}_n^2},$$

where $\widetilde{r}_n$, $n\geq 1$, is the sequence of positive zeros of the Bessel function of order μ. Also, for $t\in(t_1,1)$,

$$\begin{aligned}C(t,\lambda)=&\frac{\pi}{8}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{-1}{2}}\ (R_-(t)R_+(t))^{\frac{1}{2}}\ \csc\frac{\pi\mu}{2}\\&\times\prod_{n\geq 1}\frac{(\lambda-\lambda_n^-(t))R_-^2(t_1)}{r_n^2}\ \prod_{n\geq 1}\frac{(\lambda_n^+(t)-\lambda)R_+^2(t)}{r_n^2},\end{aligned}$$

where r_n, $n\geq 1$, is the sequence of positive zeros of $J'_1(z)$.

Now, by the infinite product representation of the solutions, we derive the dual equations corresponding to Equation (2.2).
By the implicit function theorem, $\lambda_n(t)$ is twice continuously differentiable functions.

(i) Let $\ell=2\ell_0$. Then, for $t\in(0,1)\backslash\{t_1\}$, the condition $C(t,\lambda_n(t))=0$ gives, as usual,

$$\frac{\partial C}{\partial t}+\frac{\partial C}{\partial\lambda}.\lambda_n'=0,$$

and by differentiating again,

$$\frac{\partial^2 C}{\partial t^2} + 2\frac{\partial^2 C}{\partial t \partial \lambda}.\lambda_n' + \frac{\partial^2 C}{\partial \lambda^2}.(\lambda_n')^2 + \frac{\partial C}{\partial \lambda}.\lambda_n'' = 0. \tag{4.6}$$

By virtue of (2.2), the first term in (4.6) is zero at $(t, \lambda_n(t))$. Thus

$$2\frac{\partial^2 C}{\partial t \partial \lambda}.\lambda_n' + \frac{\partial^2 C}{\partial \lambda^2}.(\lambda_n')^2 + \frac{\partial C}{\partial \lambda}.\lambda_n'' = 0. \tag{4.7}$$

If we make use of the infinite product form of $C(t, \lambda)$, substituting the infinite product form of $C(t, \lambda)$ into (4.7), in the case when $t \in (0, 1)\backslash\{t_1\}$ we obtain the *dual* of the equation (2.2).

According to (4.4), we can write

$$C(t, \lambda) = h(t) \prod_{k \geq 1} (1 - \frac{\lambda}{\lambda_k(t)}), \tag{4.8}$$

where h is a function independent of λ. From (4.5) and $h_1(t) := h(t) \prod_{n \geq 1} \frac{\zeta_n^2}{R_+^2(t)\lambda_n(t)}$, we have

$$h_1 = \frac{1}{2}\{\csc \pi\mu\}^\nu \ \phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t) = h \ \prod_{k \geq 1} \frac{\zeta_k^2}{R_+^2(t)\lambda_k(t)}.$$

Therefore

$$h(t) = \frac{1}{2}\{\csc \pi\mu\}^\nu \ \phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t) \ \prod_{k \geq 1} \frac{R_+^2(t)\lambda_k(t)}{\zeta_k^2}. \tag{4.9}$$

Now, using (4.8), we calculate $\frac{\partial C}{\partial \lambda}$, $\frac{\partial^2 C}{\partial \lambda^2}$ and $\frac{\partial^2 C}{\partial t \partial \lambda}$ at $(t, \lambda_n(t))$. In determining of $\frac{\partial^2 C}{\partial \lambda \partial t}$, the interchange of summation and differentiation in $\frac{d}{dt} \sum\limits_{k \geq 1} \log(1 - \frac{\lambda}{\lambda_k(t)})$ is valid from [10] and the differentiated series

$$\sum_{k \neq n} \frac{-\lambda_n \lambda_k'(t)}{(\lambda_k(t) - \lambda_n)\lambda_k(t)}$$

is uniformly convergent. We define F_n by

$$F_n = F_n(t, \lambda_n(t)) = \prod_{k \neq n, k \geq 1} (1 - \frac{\lambda_n(t)}{\lambda_k(t)}).$$

Since

$$\frac{\partial C}{\partial \lambda} = h \sum_{i=1}^{\infty} \frac{-1}{\lambda_i(t)} \prod_{k \neq i, k \geq 1} (1 - \frac{\lambda}{\lambda_k(x)}),$$

we have

$$\frac{\partial C}{\partial \lambda}(t, \lambda_n(t)) = \frac{-hF_n}{\lambda_n(t)},$$

$$\frac{\partial^2 C}{\partial \lambda^2}(x, \lambda_n(t)) = \frac{2hF_n}{\lambda_n(t)} \sum_{i \neq n, i \geq 1}^{\infty} \frac{1}{\lambda_i}(1 - \frac{\lambda_n(t)}{\lambda_i(t)})^{-1},$$

$$\frac{\partial^2 C}{\partial \lambda \partial t}(t, \lambda_n(t)) = -\frac{h'(t)F_n}{\lambda_n(t)} + \frac{h(t)F_n\lambda_n'}{\lambda_n^2} - h(t)F_n \sum_{i \neq n, i \geq 1}^{\infty} \frac{\lambda_i'}{\lambda_i^2}(1 - \frac{\lambda_n(t)}{\lambda_i(t)})^{-1}$$
$$-\frac{h(t)F_n\lambda_n'}{\lambda_n} \sum_{i \neq n, i \geq 1}^{\infty} \frac{1}{\lambda_i}(1 - \frac{\lambda_n(t)}{\lambda_i(t)})^{-1}.$$

Substituting these terms into (4.7), we get

$$\lambda_n''(t) + \frac{2h'(t)\lambda_n'(t)}{h(t)} + 2\lambda_n(t)\lambda_n'(t) \sum_{i \neq n, i \geq 1} \frac{\lambda_i'(t)}{\lambda_i^2(t)}(1 - \frac{\lambda_n(t)}{\lambda_i(t)})^{-1} - 2\frac{(\lambda_n'(t))^2}{\lambda_n(t)} = 0, \tag{4.10}$$

where $h(t)$ is determined in (4.9).

Equation (4.10) is dual to the original equation (2.2) in the case when $\ell = 2\ell_0$ and involves only the function $\lambda_n(t)$.

(ii) Let $\ell = 2\ell_0 + 1$. Then, similarly, for $t \in (0, t_1)$ we obtain

$$\lambda_n''(t) + \frac{2f'(t)\lambda_n'(t)}{f(t)} + 2\lambda_n(t)\lambda_n'(t) \sum_{i\neq n, i\geq 1} \frac{\lambda_i'(t)}{\lambda_i^2(t)}(1 - \frac{\lambda_n(t)}{\lambda_i(t)})^{-1} - 2\frac{(\lambda_n'(t))^2}{\lambda_n(t)} = 0. \quad (4.11)$$

Moreover, for $t \in (t_1, 1)$ we get

$$\eta_{n,j}''(t) + 2Q'(t)(Q(t))^{-1}\eta_{n,j}'(t) - 2\left(\eta_{n,j}'(t)\right)^2(\eta_{n,j}(t))^{-1}$$

$$+2\eta_{n,j}(t)\eta_{n,j}'(t)\left\{\sum_{k\neq n, k\geq 1}\frac{\eta_{k,j}'(t)}{\eta_{k,j}^2(t) - \eta_{k,j}(t)\eta_{n,j}(t)} + \sum_{k\geq 1}\frac{\theta_{k,j}'(t)(\theta_{k,j}(t))^{-1}}{\eta_{k,j}(t) - \eta_{n,j}(t)}\right\} = 0, \quad (4.12)$$

$$\theta_{n,j}''(t) + 2Q'(t)(Q(t))^{-1}\theta_{n,j}'(t) - 2\left(\theta_{n,j}'(t)\right)^2(\theta_{n,j}(t))^{-1}$$

$$+2\theta_{n,j}(t)\eta_{n,j}'(t)\left\{\sum_{k\neq n, k\geq 1}\frac{\theta_{k,j}'(t)}{\theta_{k,j}^2(t) - \theta_{k,j}(t)\theta_{n,j}(t)} + \sum_{k\geq 1}\frac{\eta_{k,j}'(t)(\eta_{k,j}(t))^{-1}}{\theta_{k,j}(t) - \theta_{n,j}(t)}\right\} = 0, \quad (4.13)$$

where

$$f(t) = |\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{-1}{2}}\ R_-(t)\prod_{k\geq 1}\frac{-\lambda_k(t)}{z_k^2},\ 0 < t < t_1,$$

$$Q(t) = \frac{\pi}{8}|\phi(0)|^{\frac{1}{2}}|\phi(t)|^{\frac{-1}{2}}\ (R_-(t)R_+(t))^{\frac{1}{2}}\csc\frac{\pi\mu}{2}\prod_{k\geq 1}\frac{\theta_k(t)R_-^2(t_1)}{r_k^2}\prod_{k\geq 1}\frac{\eta_k(t)R_+^2(t)}{r_k^2},\ t_1 < t < 1,$$

and $\eta_n(t) = \lambda_n^+(t)$ and $\theta_n(t) = \lambda_n^-(t)$, $n \geq 1$. The system of equations (4.11)-(4.13) are dual to the original equation (15) in the case when $\ell = 2\ell_0 + 1$ and involves only the functions $\lambda_n(t)$, $\eta_n(t)$ and $\theta_n(t)$.

5. Canonical solution in singular case

According to (2.1), the problem $L_1 = L_1(\phi^2(t), s)$ defined by Equation (2.2) with the conditions $y(0,\lambda) = 1$, $y'(0,\lambda) = 0$, $y(s,\lambda) = 0$, can be transformed to the problem $L_2 = L_2(q(x), b)$ with the conditions

$$u(0,\lambda) = r_1,\quad u'(0,\lambda) = r_2,\quad u(b,\lambda) = 0, \quad (5.1)$$

where $b = \int_0^s |\phi(\varsigma)|d\varsigma$, $s \in (0,1)\backslash\{t_1\}$, $r_1 = |\phi(0)|^{\frac{1}{2}}$, $r_2 = \frac{1}{2}|\phi(0)|^{\frac{-3}{2}}\phi'(0)$. Let $\ell = 2\ell_0$, according to (2.1) and (4.1) for $b \in (0,, L)\backslash\{x_1\}$, the problem L_2 has a countable set of positive eigenvalues $\{\lambda_{1n}\}_{n\geq 1}$ as

$$\sqrt{\lambda_{1n}(b)} = \frac{n\pi - \frac{\pi}{2}}{b} + O(\frac{1}{n}). \quad (5.2)$$

Now, let $\ell = 2\ell_0 + 1$. Then, according to (2.1) and (4.2), for $b \in (0, x_1)$, the boundary value problem L_2 has a countable set of negative eigenvalues $\{\lambda_{2n}^-(b)\}_{n\geq 1}$ as

$$\sqrt{-\lambda_{2n}^-(b)} = \frac{n\pi}{b} + O(\frac{1}{n}). \quad (5.3)$$

Furthermore, it follows from (2.1) and (4.3) that the eigenvalues $\{\lambda_{2n}\}$ of boundary value problem L_2 for $x_1 < b < L$, consists of two sequences of negative and positive eigenvalues $\{\lambda_{2n}(b)\} = \{\lambda_{2n}^-(b)\} \cup \{\lambda_{2n}^+(b)\}$, $n \in N$, such that

$$\sqrt{-\lambda_{2n}^-(b)} = -\frac{n\pi - \frac{\pi}{4}}{x_1} + O(\frac{1}{n}), \qquad \sqrt{\lambda_{2n}^+(b)} = \frac{n\pi - \frac{\pi}{4}}{b - x_1} + O(\frac{1}{n}). \quad (5.4)$$

Since the solution $u(x,\rho)$ of Equation (1.6) defined by initial conditions (5.1) is an entire function of ρ for each fixed $x\in[0,L]$, it follows from the Hadamard's theorem ([15]) that $u(x,\rho)$ is expressible as an infinite product.

To complete our study, we prove the following theorems which are main results of this section.

Theorem 5.1. *Let $u(x,\lambda)$ be the solution of (1.6) satisfying the initial conditions $u(0,\lambda)=r_1$, $u'(0,\lambda)=r_2$. Then, in the case when $\ell=2\ell_0$, for $x\in w_i$, $i=0,1$,*

$$u(x,\lambda)=\frac{1}{2}r_1\,\{\csc\pi\mu\}^i\prod_{n\geq 1}\frac{(\lambda_{1n}(x)-\lambda)x^2}{\zeta_n^2}, \tag{5.5}$$

where $w_0=(0,x_1)$, $w_1=(x_1,L)$, $\lambda_{1n}(x)$, $n\geq 1$, the sequence represents the sequence of positive eigenvalues of the problem L_1 on $[0,x]$, and ζ_n, $n=1,2,...$, is the sequence of positive zeros of $J'_{\frac{1}{2}}(z)$.

Proof. It follows from (2.1) and (3.4) that $R_+(t)=x$. Hence, from (4.5) and $r_1=\phi^{\frac{1}{2}}(0)$, we arrive at (5.5). □

Similarly, for $\ell=2\ell_0+1$, we can obtain the following theorem.

Theorem 5.2. *Let $\ell=2\ell_0+1$. Then, the solution of (1.6) under the initial conditions (5.1) has the form*

$$u(x,\lambda)=\begin{cases} r_1x\ \prod_{n\geq 1}\frac{\lambda-\lambda_{2n}^-(x)}{\tilde{\zeta}_n^2}, & 0<x<x_1,\\ \frac{\pi}{8}r_1x\ \csc\frac{\pi\mu}{2}\ \prod_{n\geq 1}\frac{(\lambda-\lambda_{2n}^-(x))x_1^2}{r_n^2}\ \prod_{n\geq 1}\frac{(\lambda_{2n}^+(x)-\lambda)x^2}{r_n^2}, & x_1<x<L,\end{cases}$$

where $\tilde{\zeta}_n=\frac{n\pi}{x}$ and r_n, $n\geq 1$, is the sequence of positive zeros of $J'_1(z)$.

This completes the representation of the solution of (1.6) with the initial conditions (5.1) as an infinite product.

6. Solution of the inverse problem

In this section, using results obtained above, we provide a procedure for constructing the solution of the inverse problem corresponding to the problem $L_2=L_2(q(x),b)$ with one singular point $x_1\in(0,L)$. Here, the infinite product representation of the solution plays an important role for constructing the potential $q(x)$.

Theorem 6.1. *Let $\ell=2\ell_0$ and $\{\lambda_{1n}\}_{n=1}^{\infty}$ be the sequence of the eigenvalues of the boundary value problem $L_2=L_2(q(x),b)$ on $[0,x]$, for $x\in(0,L)\backslash\{x_1\}$. Then,*

$$q(x)=(\frac{1-\sigma_1''(x)}{2\sigma_1'(x)})'+(\frac{1-\sigma_1''(x)}{2\sigma_1'(x)})^2$$

where

$$\sigma_1(x)=\sum_{n=1}^{\infty}\frac{1}{\lambda_{1n}(x)}=\int_0^x\frac{1}{g^2(t)}\int_0^t g^2(v)dvdt, \tag{6.1}$$

$$g(x)=\frac{1}{2}r_1\{\csc\pi\mu\}^i\prod_{n\geq 1}\frac{x^2\lambda_{1n}(x)}{\zeta_n^2},\qquad x\in w_i,\ i=0,1,$$

where $w_0=(0,x_1)$, $w_1=(x_1,L)$ and ζ_n, $n\geq 1$, is the sequence of positive zeros of $J'_{\frac{1}{2}}(z)$.

Proof. From (4.4) we have

$$u(x,\lambda) = h(x)\prod_{n\geq 1}(1-\frac{\lambda}{\lambda_{1n}(x)}).$$

Therefore, for fixed $x \in [0, x_1]$,

$$-\frac{\partial}{\partial\lambda}\log u(x,\lambda) = -\frac{\partial}{\partial\lambda}(\log\prod_{n\geq 1}(1-\frac{\lambda}{\lambda_{1n}(x)}))$$

$$= \sum_{n\geq 1}\frac{1}{\lambda_{1n}(x)-\lambda} = \sum_{n\geq 1}\sum_{m\geq 0}\frac{\lambda^m}{\lambda_{1n}^{m+1}(x)}. \tag{6.2}$$

On the other hand, according to (4.1) for $|\lambda| < |\lambda_1|$,

$$\sum\frac{1}{|\lambda_{1n}-\lambda|} \leq \sum\frac{1}{|\lambda_{1n}|-|\lambda|} = \sum O(\frac{1}{n^2}) < \infty.$$

Thus, for $t \in (0, t_1)$, the series $\sum\limits_{n=1}^{\infty}\frac{1}{\lambda_{1n}(x)-\lambda}$ is absolutely convergent. So, in (6.2), the interchange of summations in $\sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{\infty}\frac{\lambda^m}{\lambda_{1n}^{m+1}}$ is valid. Hence, let $\sum\limits_{n=1}^{\infty}\frac{1}{\lambda_{1n}^{m+1}} =: \sigma_{m+1}$, we have from (6.2) that

$$-\frac{\partial}{\partial\lambda}\log u(x,\lambda) == \sum_{m=0}^{\infty}\sigma_{m+1}\lambda^m.$$

This gives

$$-\frac{\partial}{\partial\lambda}u(x,\lambda) = u(x,\lambda)\sum_{m=0}^{\infty}\sigma_{m+1}\lambda^m. \tag{6.3}$$

Now, by substituting

$$u(x,\lambda) = a_0(x) + a_1(x)\lambda + a_2(x)\lambda^2 + \cdots \tag{6.4}$$

into (6.3) and (1.6), the following systems are obtained:

$$\begin{cases} a_0(x)\sigma_1 + a_1(x) = 0, \\ \\ a_0(x)\sigma_n + a_1(x)\sigma_{n-1} + \cdots + a_{n-1}(x)\sigma_1 + na_n(x) = 0, \quad n \geq 2, \end{cases} \tag{6.5}$$

$$\begin{cases} a_0''(x) - a_0(x)q(x) = 0, \\ \\ a_n''(x) + a_{n-1}(x) - q(x)a_n(x) = 0, \quad n \geq 2. \end{cases} \tag{6.6}$$

Moreover, from (4.4) and (6.4) yield $u(x,0) = h(x) = a_0(x)$. Substituting (6.5) into (6.6), and $a_0''(x) = q(t)a_0(x)$ give us

$$a_0^2(x)\sigma_1''(x) + 2a_0(x)a_0'(x)\sigma_1'(x) - a_0^2(t) = 0. \tag{6.7}$$

Consequently,

$$\frac{d}{dx}(a_0^2(x)\sigma_1'(x)) = a_0^2(x).$$

From this, and $a_0(x) = h(x)$, $a_0(0) = 1$, $a_k(0) = 0$, $k \geq 1$, $\sigma_1(0) = 0$ we arrive at (6.1). Further, by virtue of (6.7), we have

$$\frac{a_0'(x)}{a_0(x)} = \frac{1-\sigma_1''(x)}{2\sigma_1'(x)}.$$

So, since $\frac{a_0''(x)}{a_0(x)} = (\frac{a_0'(x)}{a_0(x)})^2 + (\frac{a_0'(x)}{a_0(x)})'$, we conclude from (6.6) that

$$q(x) = \frac{a_0''(x)}{a_0(x)} = (\frac{1-\sigma_1''(x)}{2\sigma_1'(x)})^2 + (\frac{1-\sigma_1''(x)}{2\sigma_1'(x)})'.$$

This completes the proof of Theorem 6.1. □

Now let

$$\Omega_0(x)=\sum_{n=1}^{\infty}\frac{1}{\lambda_{2n}^{-}(x)},\quad x<x_1, \tag{6.8}$$

$$\Omega_1(x)=\sum_{n=1}^{\infty}(\frac{1}{\lambda_{2n}^{-}(x)}+\frac{1}{\lambda_{2n}^{+}(x)}),\quad x>x_1.$$

By a similar method, in the case when $\ell=2\ell_0+1$, for $x<x_1$ we obtain the potential $q(x)$ as

$$q(x)=(\frac{1-\Omega_0''(x)}{2\Omega_0'(x)})'+(\frac{1-\Omega_0''(x)}{2\Omega_0'(x)})^2.$$

Moreover,

$$\frac{d}{dx}(Q^2(x)\Omega_1'(x))=Q^2(x). \tag{6.9}$$

Integrating (6.9) from x_1 up to x, and from $\Omega_2'(x_1)=0$, we have

$$Q^2(x)\Omega_1'(x)=\int_{x_1}^{x}Q^2(v)dv,$$

and so,

$$\Omega_1(x)=\Omega_1(L)-\int_{x}^{x_1}\frac{1}{Q^2(x)}(\int_{x_1}^{x}Q^2(v)dv)dx. \tag{6.10}$$

Thus, we obtain the following theorem.

Theorem 6.2. *Let $\ell=2\ell_0+1$. Then for $j=0,1$,*

$$q(x)=(\frac{1-\Omega_j''(x)}{2\Omega_j'(x)})'+(\frac{1-\Omega_j''(x)}{2\Omega_j'(x)})^2,\qquad x\in w_j,$$

where $w_0=(0,x_1)$, $w_1=(x_1,L)$, and $\Omega_0(x),\Omega_1(x)$ are defined as in (6.8) *and* (6.10)*, respectively.*

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