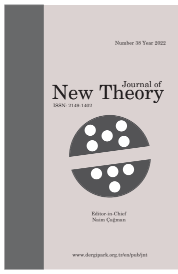
Journal of
New Theory
ISSN: 2149-1402

38 (2022) 14-24
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Timelike V-Bertrand Curves in Minkowski 3-Space E_1^3

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Article History
Received: 30 Dec 2021
Accepted: 22 Mar 2022
Published: 31 Mar 2022
10.53570/jnt.1051013
Research Article

Abstract — In this paper, the timelike V-Bertrand curve, a new type Bertrand curve in Minkowski 3-space E_1^3, is characterized. Based on the timelike V-Bertrand curve, the properties of the timelike T, N, and B Bertrand curves are obtained. From the timelike V-Bertrand curve, f-Bertrand curves and Bertrand surfaces are defined. We support the existence of these new curves and surfaces with examples. Finally, we discuss the results for further research.

Keywords — *Bertrand curves, V-Bertrand curves, timelike V-Bertrand curves, Minkowski 3-space E_1^3*

Mathematics Subject Classification (2020) — 53A04, 53A05

1. Introduction

The theory of curves has been a popular topic and many studies have been done on them. The Euclidean case (or more generally the Riemann case) of regular curves, a special type of curve, has been explored by many mathematicians. Characterization of a regular curve is one of the important problems in Euclidean space. Also, determining the Serret-Frenet vectors and the curvatures of regular curves is a common way to characterize a space curve in 3-dimensional space.

Minkowski space is one of the mathematical structures in which Einstein's relativity theory is best expressed. Since the inner product in Minkowski 3-space has an index, a vector has three different casual character. Therefore, while there exists only one Serret-Frenet formula in Euclidean 3-space, there exist five different Serret-Frenet formulas in Minkowski 3-space.

Bertrand curves are one of the most studied topics in the theory of curves. These curves have been firstly defined by Bertrand [1]. In this study, he has given an answer to the Saint Venant's open problem in which whether a curve exists on the surface produced by its principal normal vector and whether there exists another curve linearly dependent with principal normal vector of this curve [2]. The necessary and sufficient condition for existence of such a second curve is it satisfies the equation $a\kappa + b\tau = 1$ such that $a, b \in \mathbb{R}$, $a \neq 0$, and κ and τ are curvatures [3]. Moreover, Izumiya and Takeuchi have shown that all Bertrand curves can be obtained from a sphere, and they have given a method in [4] to obtain a Bertrand curve from a sphere. Recently, Camcı et al. [5] have studied Bertrand curves with a novel approach. İlarslan et al. have defined null Cartan and pseudo null Bertrand curves in Minkowski 3-space E_1^3 [6]. Further, (1,3)-Bertrand curves in a timelike (1,3)-normal plane in Minkowski space-time E_1^4 have been examined [7]. Also, Matsuda and Yorozu have shown that there is no Bertrand curve in Euclidean n-space E^n such that $n \geq 4$ and have defined (1,3)-Bertrand curves in Euclidean 4-space E^4 [8]. Lucas and Ortega-Yagües have characterized helices in $\mathbb{S}^3$ as the only

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twisted curves in $\mathbb{S}^3$ having infinite Bertrand conjugate curves [9]. Dede et al. have defined directional Bertrand curves [10]. Additionally, a new type Bertrand curve, called V-Bertrand curve, has been firstly defined and investigated by Camcı in [11].

In Section 2, we present some of definitions and properties to be used in the next sections. In Section 3, we describe timelike V-Bertrand curves in Minkowski 3-space E_1^3 and give a characterization of a timelike V-Bertrand curve. In Section 4, we define f-Bertrand curves using timelike curves. In Section 5, we give a method to obtain another Bertrand curve from a Bertrand curve. In Section 6, we define Bertrand surfaces by timelike curves. Finally, we discuss the need for further research. This study is a part of the first author's master's thesis [12].

2. Preliminaries

We start with recalling the definitions and theorems given by Camcı in [11]. Let $\gamma : I \to \mathbb{R}^3$ be a unit-speed curve with arc-length parameter "s". If Serret-Frenet apparatus are denoted with $\{T, N, B, \kappa, \tau\}$, then we can define a curve $\beta : I \to \mathbb{R}^3$ as

$$\beta(s) = \int V(s)ds + \lambda(s)N(s) \tag{1}$$

where $\lambda : I \to \mathbb{R}^3$ is a differentiable function and V is a unit vector field with

$$V : I \to T(\mathbb{R}^3), V(s) = u(s)T(s) + v(s)N(s) + \omega(s)B(s), \quad u, v, \omega \in C^\infty(I, \mathbb{R})$$

Definition 2.1. [11] Let $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$ be Serret-Frenet apparatus of the curve β defined in (1). If $\{N, \bar{N}\}$ is linearly dependent (e.g. $N = \varepsilon\bar{N}$, $\varepsilon = \pm1$), then (γ, β) is V-Bertrand curve mate and γ is called V-Bertrand curve. If $V = T$, then $(\gamma,\ \beta)$ is a classical Bertrand mate.

Theorem 2.2. [11] Let γ be a unit-speed curve with Serret-Frenet apparatus $\{T, N, B, \kappa, \tau\}$. The curve γ is a V-Bertrand curve if and only if the following equation holds:

$$\lambda(\kappa \tan\theta + \tau) = u \tan\theta - \omega \tag{2}$$

where

$$\lambda(s) = -\int v(s)ds$$

and θ is a constant angle between T and $\bar{T}$.

Definition 2.3. [11] Let γ be a unit-speed and non-planar curve ($\tau \neq 0$) with Serret-Frenet apparatus $\{T, N, B, \kappa, \tau\}$. If there exist $\lambda \neq 0$ and $\theta \in \mathbb{R}$ satisfying the equation

$$\lambda\kappa + \lambda \cot\theta\tau = 1 \tag{3}$$

then we say that the curve γ is a Bertrand curve (or T-Bertrand curve). In addition, if the equation

$$\lambda\kappa \tan\theta + \lambda\tau = -1 \tag{4}$$

holds, then we say that the curve γ is a B-Bertrand curve.

Remark 2.4. [11] If $u(s) = 1$ and $v(s) = \omega(s) = 0$, then the pair (γ, β) is a T-Bertrand curve mate. Also, if $\omega(s) = 1$ and $u(s) = v(s) = 0$, then the pair (γ, β) is a B-Bertrand curve mate. Furthermore, if $v(s) = 1$ and $u(s) = \omega(s) = 0$, then we say that the pair (γ, β) is an N-Bertrand curve mate.

Next, recall that Minkowski 3-space E_1^3 is Euclidean 3-space E^3 equipped with the metric

$$g := -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3 [13]. In this space, a vector can has one of three casual characters according to this metric. If $g(u, u) > 0$ or $u = 0$, then u is a spacelike vector, if $g(u,u) < 0$, then u is a timelike vector, and if $g(u,u) = 0$ and $u \neq 0$, then u is a null (lightlike) vector. Moreover, an arbitrary curve $\alpha = \alpha(s)$ in Minkowski 3-space E_1^3 can be called according to its the velocity vector $\alpha^{'}(s)$. A curve α is called spacelike, timelike, or null, if $\alpha^{'}(s)$ is spacelike, timelike, or null, respectively. For a timelike curve with Serret-Frenet apparatus $\{T, N, B, \kappa, \tau\}$, the following formulas hold:

$$T' = \kappa N, \ N' = \kappa T + \tau B, \text{ and } B' = -\tau N \tag{5}$$

where

$$g(T,T) = -1, \quad g(N,N) = 1, \quad g(B,B) = 1 \tag{6}$$

$$g(N,B) = 0, \quad g(T,N) = 0, \quad g(T,B) = 0 \tag{7}$$

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N \tag{8}$$

3. Timelike V-Bertrand Curves in Minkowski 3-Space E_1^3

In this section, we define timelike V-Bertrand curves in Minkowski 3-space E_1^3 and investigate some of their basic properties. In addition, we give a characterization for this type curves.

Definition 3.1. Let $\gamma : I \to E_1^3$, $\gamma = \gamma(s)$ be a unit-speed timelike curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\beta : I \to E_1^3$, $\beta = \beta(s)$ be a regular curve with Frenet apparatus $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$. We can define a curve β by

$$\beta(s) = \int V(s)ds + \lambda(s)N(s) \tag{9}$$

where $\lambda : I \to \mathbb{R}^3$ is a differentiable function and V is a unit vector field with

$$V : I \to T(\mathbb{R}^3), V(s) = u(s)T(s) + v(s)N(s) + \omega(s)B(s), \quad u, v, \omega \in C^\infty(I, \mathbb{R}).$$

If $\{N, \bar{N}\}$ is linearly dependent (e.g. $N = \varepsilon\bar{N}$, $\varepsilon = \pm 1$), then the pair (γ, β) is called a timelike V-Bertrand curve mate and γ is called a timelike V-Bertrand curve. Moreover, especially, if $V = T$ (N or B), then (γ, β) is a timelike T (N or B)-Bertrand curve mate.

Theorem 3.2. Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of this curve. The curve γ is a timelike V-Bertrand curve if and only if it satisfies the following condition:

$$\lambda(\tau - \kappa \tanh\theta) = u \tanh\theta - \omega \tag{10}$$

such that

$$\lambda = -\int v(s)ds \tag{11}$$

and θ is a constant angle between T and $\overline{T}$.

Proof. Let $\gamma : I \to E_1^3$, $\gamma = \gamma(s)$ be a unit-speed timelike V-Bertrand curve and $\beta : I \to E_1^3$, $\beta = \beta(\bar{s})$ be V-Bertrand curve mate of γ. Also, let Frenet apparatus of these curves be $\{T, N, B, \kappa, \tau\}$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$, respectively.

($\Rightarrow$) Derivating β with respect to s, we have the following equation

$$\begin{aligned} \frac{d\bar{s}}{ds}\bar{T} &= uT + vN + \omega B + \lambda' N + \lambda N' \\ &= (u + \lambda\kappa)T + (\lambda' + v)N + (\omega + \lambda\tau)B \end{aligned} \tag{12}$$

Since $\{N, \bar{N}\}$ is linearly dependent, we have

$$\lambda = -\int v(s)ds \tag{13}$$

After, it follows that equation (12), we have

$$\bar{T} = \frac{ds}{d\bar{s}}(u + \lambda\kappa)T + \frac{ds}{d\bar{s}}(\omega + \lambda\tau)B \tag{14}$$

From the equation (14), we get

$$\cosh\theta = \frac{ds}{d\bar{s}}(u + \lambda\kappa) \tag{15}$$

$$\sinh\theta = \frac{ds}{d\bar{s}}(\omega + \lambda\tau) \tag{16}$$

From the equations (15) and (16), we get

$$\lambda(\tau - \kappa\tanh\theta) = u\tanh\theta - \omega$$

Thus, the equation (14) is rewritten as

$$\bar{T} = \cosh\theta T + \sinh\theta B \tag{17}$$

Also, if the derivative of equation (17) according to the arc-parameter s is taken, then we get

$$\frac{d\bar{s}}{ds}\overline{\kappa}\overline{N} = \theta' \sinh\theta T + (\kappa\cosh\theta - \tau\sinh\theta)N + \theta' \cosh\theta B \tag{18}$$

As $\{N, \bar{N}\}$ is linearly dependent, the angle θ is a constant.

($\Leftarrow$) Let the equation (10) be valid for the constant θ. Derivating the equation (9), we have the equation (12). From the equations (11) and (12), we get

$$\bar{T} = \frac{ds}{d\bar{s}}(u + \lambda\kappa)T + \frac{ds}{d\bar{s}}(\omega + \lambda\tau)B = \cosh(w(s))T + \sinh(w(s))B \tag{19}$$

From the equations (10) and (19), we obtain

$$\tanh(w(s)) = \frac{u + \lambda\kappa}{\omega + \lambda\tau} = \tanh\theta \tag{20}$$

From the equation (20), $w(s) = \theta$. Since θ is a constant, if the derivative of the equation (19) is taken, then it is seen that $\{N, \bar{N}\}$ is linearly dependent. Therefore, the curve γ is a V-Bertrand curve.

□

Corollary 3.3. Let γ be a unit-speed and non-planar timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of the curves in Minkowski 3-space E_1^3. If $\overline{\lambda} = \lambda\tanh\theta$ and $\overline{\mu} = -\lambda$ such that λ and θ are non-zero constant numbers, then

1. γ is a timelike T-Bertrand curve if and only if $\overline{\lambda}\kappa + \overline{\mu}\tau = -\tanh\theta$. Further, if $u(s) = 1$ and $v(s) = \omega(s) = 0$ in the equation $V(s) = u(s)T(s) + v(s)N(s) + \omega(s)B(s)$, then (γ, β) is a timelike T-Bertrand curve mate. From the equation (9), we have

$$\beta(s) = \int T(s)ds + \lambda(s)N(s)$$

If the integral constant is assumed as zero in this equation, then (γ, β) is a classical timelike Bertrand curve mate.

2. γ is a timelike N-Bertrand curve if and only if $\frac{\tau}{\kappa} = \tanh\theta$. Also, if $u(s) = w(s) = 0$ and $v(s) = 1$ in the equation $V(s) = u(s)T(s) + v(s)N(s) + \omega(s)B(s)$, then (γ, β) is a timelike N-Bertrand curve mate. From Theorem 3.2, $\lambda = -s + c$ and the timelike N-Bertrand curve γ is a general helix such that θ is a constant.

3. γ is a timelike B-Bertrand curve if and only if $\overline{\lambda}\kappa + \overline{\mu}\tau = 1$. Morever, let γ be a timelike anti-Salkowski curve, i.e., τ is a constant. If $\lambda = \frac{1}{\tau}$, then

$$(\lambda\tanh\theta)\kappa - \lambda\kappa = 1$$

In this case, any timelike anti-Salkowski curve is a timelike B-Bertrand curve.

Example 3.4. Let us consider the curve $\gamma(s) = (\sqrt{2}\sinh s, \sqrt{2}\cosh s, s)$ in Minkowski 3-space E_1^3 provided in [14]. It is clear that γ is a timelike curve. The Frenet vectors and curvatures of γ are as follows:

$$\begin{aligned} T &= (\sqrt{2}\cosh s, \sqrt{2}\sinh s, 1) \\ N &= (\sinh s, \cosh s, 0) \\ B &= \left(\cosh s, \sinh s, \sqrt{2}\right) \\ \kappa &= \sqrt{2} \\ \tau &= -1 \end{aligned} \tag{21}$$

If $V = B$ ($u = v = 0$ and $w = 1$) is taken, then (γ, β) timelike B-Bertrand curve mate is obtained in Definition 3.1. To find the curve β, if timelike B-Bertrand curve characterization is used, then we have

$$\lambda = \frac{\sqrt{2}}{\sqrt{2} + 2\tanh\theta}$$

If the vectors N and B in the equation (21) and λ are written in the Definition 3.1, then we obtain

$$\beta(s) = \left((1+\lambda)\sinh s, (1+\lambda)\cosh s, \sqrt{2}s\right)$$

The tangent vector of the curve β is as follows:

$$\overline{T} = \frac{1}{\sqrt{2 - (1+\lambda)^2}}\left((1+\lambda)\cosh s, (1+\lambda)\sinh s, \sqrt{2}s\right)$$

If $1 + \lambda = \frac{1}{\sqrt{2}}$, then the curve β is obtained as

$$\beta(s) = \left(\frac{1}{\sqrt{2}}\sinh s, \frac{1}{\sqrt{2}}\cosh s, \sqrt{2}s\right)$$

Hence, the graph of the timelike B-Bertrand curve mate (γ, β) is as follows:

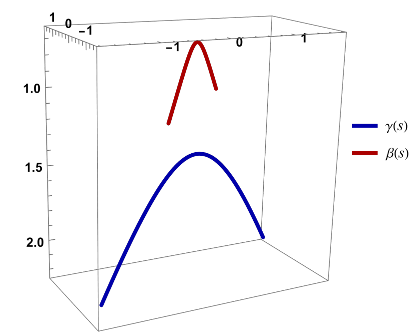


Fig. 1. The timelike B-Bertrand curve mate (γ, β)

4. f-Bertrand Curves Obtained from Timelike Curves

In this section, we propose f-Bertrand curves by using timelike curves. Morever, we provide three examples for f-Bertrand curves.

Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of the curve in Minkowski 3-space E_1^3. Let V be a timelike unit vector field defined in the Definition 3.1. If $v = 0$, then $-u^2+w^2=-1$. For $\epsilon=\pm 1$, then $w=\epsilon\sqrt{u^2-1}$. Applying transformation in the equation (10), we have

$$u\tanh\theta-\epsilon\sqrt{u^2-1}=f \tag{22}$$

If this quadratic equation is solved according to the variable u, then we have

$$u^{\pm}=\frac{f\tanh\theta\pm\sqrt{f^2+1-(\tanh\theta)^2}}{(\tanh\theta)^2-1} \tag{23}$$

From (23), $w_{1,2}^{\pm}=\epsilon\sqrt{(u^{\pm})^2-1}$. Therefore, there are four different situations for timelike unit vector field:

$$V_1^{\pm}=u^+T+w_1^{\pm}B \quad V_2^{\pm}=u^-T+w_2^{\pm}B$$

Thus, $\beta_1^{\pm}$ and $\beta_2^{\pm}$ can be defined as

$$\begin{aligned}\beta_1^{\pm}(s)&=\int V_1^{\pm}ds+\lambda N\\ \beta_2^{\pm}(s)&=\int V_2^{\pm}ds+\lambda N\end{aligned} \tag{24}$$

Then, the curve γ is a timelike V_1^+, V_1^-,V_2^+, and V_2^--curve. Thus, the following definition can be given.

Definition 4.1. Each of the curves $\beta_1^+(s)$, $\beta_1^-(s)$, $\beta_2^+(s)$, and $\beta_2^-(s)$ defined in (23) is called an f-Bertrand curve mate of a timelike curve γ and the timelike curve γ is called an f-Bertrand curve.

Example 4.2. Let us consider the timelike curve γ provided in Example 3.4. To find $\tanh\theta-$Bertrand mates of the timelike curve γ, we suppose that $f=\tanh\theta$ in the equation (22). From the equations (10) and (22),

$$\tanh\theta=-\frac{\lambda}{1+\lambda\sqrt{2}}$$

Morever, $u^+=1-2(\cosh\theta)^2$ and $u^-=-1$ from the equation (23). Therefore, we have $w_1^+=\sinh 2\theta$, $w_1^-=-\sinh 2\theta$, and $w_2^{\pm}=0$. Hence, the f-Bertrand curve mates of the timelike curve γ are as follows:

$$\beta_1^{\pm}(s)=\begin{pmatrix}\left(\left(1-2(\cosh\theta)^2\right)\sqrt{2}\pm(\sinh 2\theta+\lambda)\right)\sinh s,\\ \left(\left(1-2(\cosh\theta)^2\right)\sqrt{2}\pm(\sinh 2\theta+\lambda)\right)\cosh s,\\ \left(\left(1-2(\cosh\theta)^2\right)\pm\left(\sqrt{2}\sinh 2\theta\right)\right)s\end{pmatrix}$$

$$\beta_2^{\pm}(s)=\beta_2(s)=\left(\left(\sqrt{2}+\lambda\right)\sinh s,\left(\sqrt{2}+\lambda\right)\cosh s,s\right)$$

For $\lambda=\sqrt{2}$, the curve pairs (γ,β_1^+), (γ,β_1^-), and (γ,β_2) are presented in the Fig. 2.

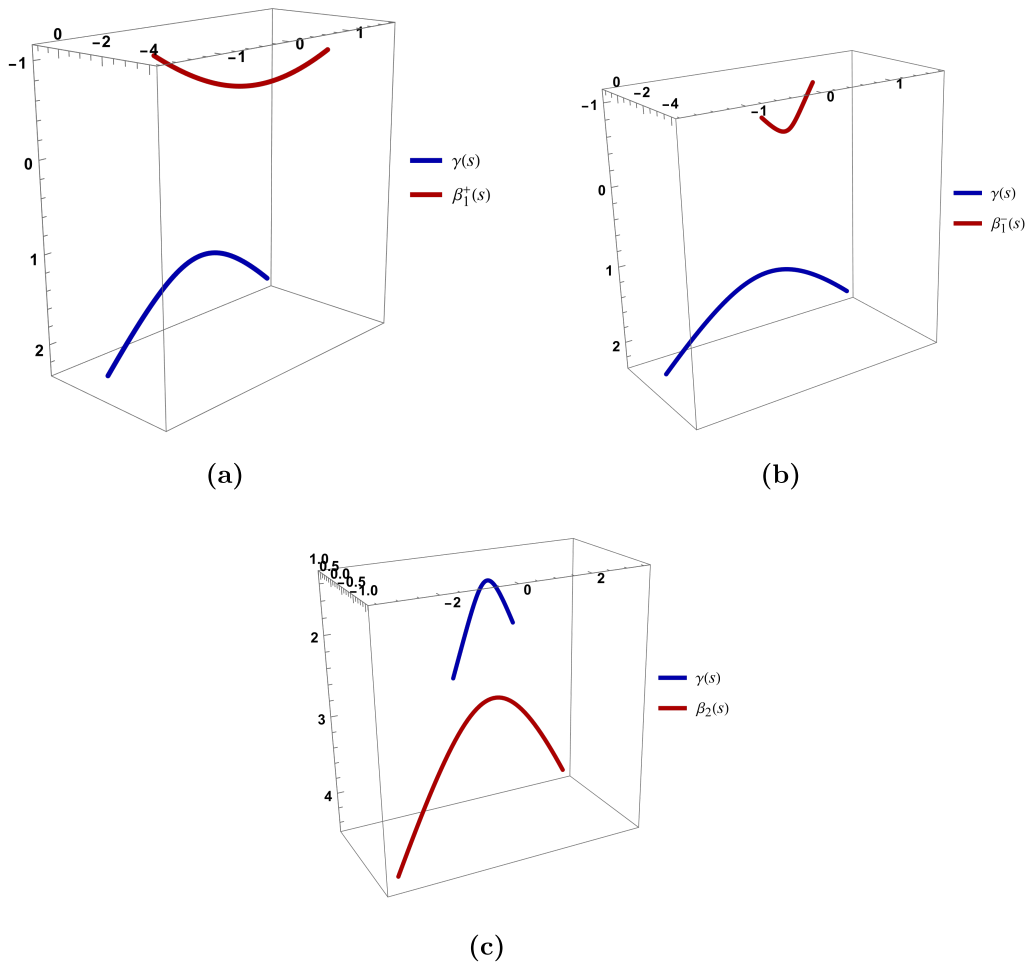


Fig. 2. (a) The curve pair (γ, β_1^+) for $\lambda = \sqrt{2}$ (b) The curve pair (γ, β_1^-) for $\lambda = \sqrt{2}$, and (c) The curve pair (γ, β_2) for $\lambda = \sqrt{2}$

5. Timelike and Spacelike Bertrand Curve Obtained From Timelike Bertrand Curve

In this section, we obtain new timelike and spacelike Bertrand curves using a timelike curve.

Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of the curve in Minkowski 3-space E_1^3. Considering u and w are constants and $v = 0$ in the unit vector field V in Definition 3.1, V can be rewritten as $V(s) = uT(s) + wB(s)$. Let $\gamma_V = \int V(s)ds$ and its Frenet vectors and curvatures is $\{T_V, N_V, B_V, \kappa_V, \tau_V\}$. In this section, the conditions for a curve γ_V to be a Bertrand curve are investigated.

Lemma 5.1. Let V be a timelike unit vector field. In this case, curvatures of γ are written as follows by curvatures of γ_V:

$$\begin{aligned} \kappa &= w\kappa_V + u\tau_V \\ \tau &= u\kappa_V + w\tau_V \end{aligned}$$

Proof. If V is a timelike unit vector field, we have $-u^2 + w^2 = -1$. Since the tangent vector of curve γ_V is the vector V, the curve γ_V is a timelike curve. Therefore,

$$T_V = uT + wB \tag{25}$$

If the derivative of this equation is taken and $N_V = N$, then

$$\kappa_V = u\kappa - w\tau \tag{26}$$

Applying the cross product to the equation (25) by N_V from the right, we get

$$B_V = uB + wT$$

If we derivative this equation, we have

$$\tau_V = -w\kappa + u\tau \tag{27}$$

From equations (26) and (27), the curvatures of the curve γ are obtained as follows:

$$\begin{aligned} \kappa &= w\kappa_V + u\tau_V \\ \tau &= u\kappa_V + w\tau_V \end{aligned} \tag{28}$$

□

The following theorem is given from the Lemma 5.1.

Theorem 5.2. Let V be a timelike unit vector field. γ is a timelike Bertrand curve if and only if γ_V is a timelike Bertrand curve.

Lemma 5.3. Let V be a spacelike unit vector field. In this case, curvatures of γ are written as follows by curvatures of γ_V:

$$\begin{aligned} \kappa &= -u\kappa_V + w\tau_V \\ \tau &= -w\kappa_V + u\tau_V \end{aligned}$$

Proof. Let V be a spacelike unit vector field. Thus, $-u^2 + w^2 = 1$. Because the tangent vector of curve γ_V is the vector V, the curve γ_V is a spacelike curve. Hereby,

$$T_V = uT + wB \tag{29}$$

If the equation (29) is differentiated and $N_V = N$, thereby

$$\kappa_V = u\kappa - w\tau \tag{30}$$

Applying the cross product to the equation (29) by N_V from the right, the following equation is obtained:

$$B_V = uB + wT$$

If we derivative this equation, we have

$$\tau_V = w\kappa - u\tau \tag{31}$$

From equations (30) and (31), the curvatures of the curve γ are obtained as follows:

$$\begin{aligned} \kappa &= -u\kappa_V + w\tau_V \\ \tau &= -w\kappa_V + u\tau_V \end{aligned} \tag{32}$$

□

The following theorem is given from the Lemma 5.3.

Theorem 5.4. Let V be a spacelike unit vector field. γ is a timelike Bertrand curve if and only if γ_V is a spacelike Bertrand curve whose binormal is a timelike curve.

6. Bertrand Surface Obtained From Timelike Bertrand Curve

In this section, we suggest the concept of Bertrand surfaces and provide an example for Bertrand surfaces.

Let γ be a unit-speed timelike curve and $\{T, N, B, \kappa, \tau\}$ be Frenet apparatus of the curves in Minkowski 3-space E_1^3. Because of timelike Bertrand (timelike T-Bertrand) characterization, we have the equation

$$\lambda \tanh\theta\kappa - \lambda\tau = -\tanh\theta$$

If both sides of this equation are multiplied by a real number t, the following equation is obtained

$$\lambda t \tanh\theta\kappa - \lambda t\tau = -t\tanh\theta$$

Putting $-t\tanh\theta$ instead of f in the equation (23), we find

$$u^{\pm}(t) = \frac{-t\left(\tanh\theta\right)^2 \pm \sqrt{t^2\left(\tanh\theta\right)^2 + 1 - \left(\tanh\theta\right)^2}}{\left(\tanh\theta\right)^2 - 1} \tag{33}$$

Also,

$$w_1^{\pm} = \epsilon\sqrt{\left(u^{+}(t)\right)^2 - 1} \text{ and } w_2^{\pm} = \epsilon\sqrt{\left(u^{-}(t)\right)^2 - 1} \tag{34}$$

Thus, the following definition can be given.

Definition 6.1. Let γ be a timelike Bertrand curve. Each of the following surfaces ψ_1^+, ψ_1^-, ψ_2^+, and ψ_2^- is called a Bertrand surface of γ.

$$\begin{aligned} \psi_1^{\pm}(t,s) &= \int V_1^{\pm} ds + \lambda N \\ \psi_2^{\pm}(t,s) &= \int V_2^{\pm} ds + \lambda N \end{aligned} \tag{35}$$

such that $V_1^{\pm}(t,s) = u^{+}(t)T(s) + w_1^{\pm}(t)B(s)$ and $V_2^{\pm}(t,s) = u^{-}(t)T(s) + w_2^{\pm}(t)B(s)$ by $u^{\pm}$, $w_1^{\pm}$, and $w_2^{\pm}$ in the equations (33) and (34).

Example 6.2. Let γ be a timelike curve provided in Example 3.4. To find a Bertrand surface of the curve γ, if the curvatures of the curve γ are written by using timelike T-Bertrand characterization, we get

$$\lambda = -\frac{\tanh\theta}{1 + \sqrt{2}\tanh\theta}$$

If $\tanh\theta = -\frac{\sqrt{2}}{3}$, then

$$\begin{aligned} u^{+}(t) &= \frac{2}{7}t - \frac{3}{7}\sqrt{2t^2 + 7} \\ w_1^{+}(t) &= \sqrt{\left(\frac{2}{7}t - \frac{3}{7}\sqrt{2t^2 + 7}\right)^2 - 1} \end{aligned} \tag{36}$$

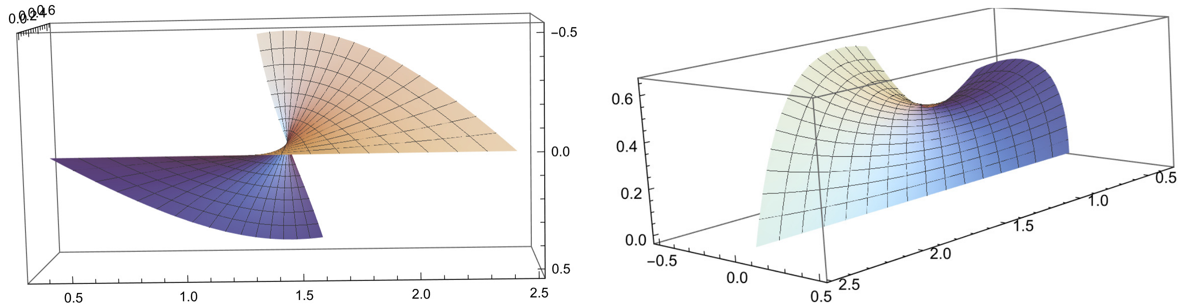
The surface ψ_1^+ in the equation (35) is as follows:

$$\psi_1^{+}(t,s) = u^{+}(t)\int T(s)ds + w_1^{+}(t)\int B(s)ds + \lambda N(s) \tag{37}$$

From the equation (36), the equation (37) is rearranged as follows:

$$\psi_1^{+}(t,s) = \begin{pmatrix} \left(\frac{2}{7}\sqrt{2}t - \frac{3}{7}\sqrt{2}\sqrt{2t^2+7} + \frac{1}{7}\sqrt{22t^2 + 14 - 12t\sqrt{2t^2+7}} + \sqrt{2}\right)\sinh s, \\ \left(\frac{2}{7}\sqrt{2}t - \frac{3}{7}\sqrt{2}\sqrt{2t^2+7} + \frac{1}{7}\sqrt{22t^2 + 14 - 12t\sqrt{2t^2+7}} + \sqrt{2}\right)\cosh s, \\ \left(\frac{2}{7}st - \frac{3}{7}s\sqrt{2t^2+7} + \frac{1}{7}s\sqrt{22t^2 + 14 - 12t\sqrt{2t^2+7}}\right)\sqrt{2} + \sqrt{2} \end{pmatrix}$$

The graph of the surface ψ_1^+ is provided in Fig. 3.



(a) **(b)**

Fig. 3. The Bertrand surface ψ_1^+ of the curve γ

7. Conclusion

In this study, we characterized V-Bertrand curves in Minkowski 3-space by V-Bertrand curves in Euclidean 3-space, a new type of Bertrand curve defined by Camcı [11]. Firstly, the characterization of timelike V-Bertrand curves was given by a timelike curve. Afterwards, we defined T-Bertrand, N-Bertrand, and B-Bertrand curves by the timelike V-Bertrand curve and their characterization. Some of the obtained important results are the following: a timelike T-Bertrand curve is a timelike Bertrand curve and a timelike N-Bertrand curve is a timelike circular helix. Furthermore, in the timelike V-Bertrand curve characterization, four f-Bertrand curves were obtained from a timelike V-Bertrand curve and a mapping f. Additionaly, using these f-Bertrand curve characterizations, four Bertrand surfaces were defined by timelike Bertrand curves. Finally, a method was given to obtain a spacelike curve whose binormal vector is a timelike vector and another timelike Bertrand curve from a timelike Bertrand curve. Thus, timelike V-Bertrand curves in Minkowski 3-space, a new curve, has been brought to the literature. With the idea used in this study, the researchers can develop this study for other Frenet frames.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

Acknowledgement

This work was supported by the Office of Scientific Research Projects Coordination at Çanakkale Onsekiz Mart University, Grant Number: FYL-2019-2927.

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