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THE LINEAR ALGEBRA OF A GENERALIZED TRIBONACCI MATRIX

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Abstract. In this paper, we consider a generalization of a regular Tribonacci matrix for two variables and show that it can be factorized by some special matrices. We produce several new interesting identities and find an explicit formula for the inverse and $k-$th power. We also give a relation between the matrix and a matrix exponential of a special matrix.

1. Introduction

Integer sequences are widely used in many areas such as physics, engineering, arts and nature. There have been several studies in the literature that concern about the second order integer sequences and their generalizations such as Fibonacci, Lucas, Pell and Jacobsthal, see [8, 9, 11–13, 17]. Horadam interested in the generalized Fibonacci sequence $\{W_n(a,b;p,q)\}_{n\geq 0}$, where a,b are nonnegative integers and p,q are arbitrary integers, and studied some properties of the sequence, see [11, 12]. Another generalization of the Fibonacci sequence is called as the Tribonacci sequence. The Tribonacci sequence is the most familiar series of numbers obtained by generalizing Fibonacci sequence as orders.

For $n \geq 0$, we use the following definition of the sequence of Tribonacci numbers which is given by third order recurrence relation

$$t_{n+3} = t_{n+2} + t_{n+1} + t_n$$

with initial conditions

$$t_0 = t_1 = 1, \quad t_2 = 2.$$

The first few terms of the Tribonacci numbers are given in Table 1.

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n	0	1	2	3	4	5	6	7	8	9	10	11	12
t_n	1	1	2	4	7	13	24	44	81	149	274	504	927

TABLE 1. The first few terms of the Tribonacci sequence

The characteristic polynomial $x^3 - x^2 - x - 1 = 0$ of the third order Tribonacci recurrence has a unique real root of maximum modulus and this is

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} \approx 1.83929,$$

the Tribonacci constant, see [21]. Many researchers studied some properties of the Tribonacci sequence, see [4–6, 10, 15, 20, 22, 23, 25].

A matrix T_n of order $n+1$ with entries

$$t_{i,j} = \begin{cases} \frac{2t_j}{t_{i+2}+t_i-1}, & \text{if } 0 \leq j \leq i, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

is defined in [26] and the Tribonacci space sequences $\ell_p(T)$ are introduced. For $n = 4$, the matrix T_4 will look as follows

$$T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{15} & \frac{1}{15} & \frac{2}{15} & \frac{4}{15} & \frac{7}{15} \end{bmatrix}.$$

Definition 1. *A square matrix R is regular if and only if R is a stochastic matrix and some power R^k, for $k \geq 1$, has all entries nonzero.*

Thus from the definition of the regular matrix, we obtain that the matrix defined in (1) is a regular matrix.

Inspiring by this study, we define a two variable generalization of the matrix given in (1) and obtain several interesting new properties. We are also interested in matrix factorization of the defined matrix which is a method of representing a matrix as a product of some matrices. There are various types of matrix factorizations such as singular value decomposition, LU factorization, Cholesky factorization, etc. This method is used to simplify calculations, especially in solving a problem that is difficult to solve in its original form. Several authors are interested in matrix factorizations of some special matrices, see [1, 2, 7, 18, 19, 27].

2. A Generalization of the Regular Tribonacci Matrix

In this section, we give a generalization of the matrix defined in (1). We define a matrix $T_n(x,y) = [t_{i,j}(x,y)]$ of order $n+1$ with entries

$$t_{i,j}(x,y) = \begin{cases} \frac{2t_j}{t_{i+2}+t_i-1} x^{i-j} y^j, & \text{if } 0 \le j \le i, \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $n = 4$, the matrix will look as follows

$$T_4(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 \\ \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 \\ \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 \\ \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 \end{bmatrix}.$$

We will denote the (i,j) entry of this matrix as $(T_n(x,y))_{i,j}$. It is easy to see that when x or y is zero, $t_{i,j}(x,y)$ will be trivial. Therefore we generally assume that x and y in $T_n(x,y)$ are non-zero real numbers. It is clear that for $x = y = 1$ we have

$$t_{i,j}(1,1) = t_{i,j}$$

and so in this case we obtain the regular Tribonacci matrix (1).

2.1. **Multiplication of two Tribonacci matrices.** The Tribonacci matrix $T_n(x,y)$ has some interesting properties and applications. Thus we give some of these properties now. For $n, j \in \mathbb{N}$, we define

$$(x \oplus y)_j^n = \sum_{k=0}^{n} t_{k+j,k+j} x^{n-k} y^k.$$

Theorem 1. *For any positive integer n and any real numbers x, y, z and w, we have*

$$\Big(T_n(x,y)T_n(w,z)\Big)_{i,j} = \Big(T_n\big((x \oplus yw)_j, yz\big)\Big)_{i,j}. \tag{2}$$

Proof. From the definition of the matrix $T_n(x,y)$ and the rules of the matrix multiplication, the (i,j) entry of $T_n(x,y)T_n(w,z)$ is 0 for $j > i$. For $j \le i$ it can be obtained as

$$\begin{aligned} \sum_{k=j}^{i} t_{i,k}(x,y)t_{k,j}(w,z) &= \sum_{k=j}^{i} \frac{2t_k}{t_{i+2}+t_i-1} x^{i-k} y^k \frac{2t_j}{t_{k+2}+t_k-1} w^{k-j} z^j \\ &= \frac{2t_j}{t_{i+2}+t_i-1} \sum_{k=j}^{i} \frac{2t_k}{t_{k+2}+t_k-1} x^{i-k} y^k w^{k-j} z^j \\ &= t_{i,j} \sum_{k=j}^{i} t_{k,k} x^{i-k} y^k w^{k-j} z^j \end{aligned}$$

$$
\begin{aligned}
&= t_{i,j}\sum_{k=0}^{i-j} t_{k+j,k+j}x^{i-j-k}y^{k+j}w^k z^j \\
&= t_{i,j}(yz)^j \sum_{k=0}^{i-j} t_{k+j,k+j}x^{i-j-k}(yw)^k \\
&= t_{i,j}(x\oplus yw)_j^{i-j}(yz)^j
\end{aligned}
$$

This is also the (i,j) entry of $T_n\big((x\oplus yw)_j, yz\big)$, so equation (2) holds. □

For $w=x$ and $z=y$ in (2), we

$$(T_n^2(x,y))_{i,j} = T_n\left(x(1\oplus y)_j, y^2\right)_{i,j}.$$

Using formula (2) again, multiplying $T_n^2(x,y)$ and $T_n(x,y)$, we get

$$(T_n^3(x,y))_{i,j} = T_n\left(x(1\oplus y\oplus y^2)_j, y^3\right)_{i,j}.$$

Then using the mathematical induction method, the following results can be obtained.

$$(T_n^k(x,y))_{i,j} = T_n\left(x(1\oplus y\oplus\cdots\oplus y^{k-1})_j, y^k\right)_{i,j}.$$

2.2. **The inverse of the matrix $T_n(x,y)$.** The inverse of the Tribonacci matrix $T_n(x,y)$ is given by the following theorem.

Theorem 2. *The $(i,j)-$entry of the inverse of the matrix $T_n(x,y)$ is*

$$(T_n(x,y)^{-1})_{i,j} = \begin{cases} \frac{t_{i+2}+t_i-1}{2t_iy^i}, & \text{if } i=j, \\ -\frac{(t_{i+2}+t_i-1-2t_i)x}{2t_iy^i}, & \text{if } i=j+1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By straightforward computation of matrix multiplication, we get the desired result. □

2.3. **The factorization of the Tribonacci matrix.** We define the matrices of order $n+1$ with the following entries

$$
\begin{aligned}
(S_n(x,y))_{i,j} &= \begin{cases} t_{i,j}(x,y)t^{-1}_{j-1,j-1}(x,y)+t_{i,j+1}(x,y)t^{-1}_{j,j-1}(x,y) & i\geq j, \\ 0 & i<j, \end{cases} \\
\overline{T}_{n-1}(x,y) &= \begin{bmatrix} 1 & 0 \\ 0 & T_{n-1}(x,y) \end{bmatrix}, \quad n\geq 1, \\
G_n &= S_n, \quad G_k(x,y) = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k(x,y) \end{bmatrix}, \quad 1\leq k\leq n-1.
\end{aligned}
$$

Let us consider the product of the matrices $T_n(x,y)$ and $\overline{T}^{-1}_{n-1}(x,y)$. Here we represent the (i,j) entry of the matrices $T_n^{-1}(x,y)$ and $\overline{T}^{-1}_{n-1}(x,y)$ as $t^{-1}_{i,j}(x,y)$ and

$\bar{t}_{i,j}^{-1}(x,y)$, respectively. From the definitions of the matrices, the (i,j) entry of $T_n(x,y)\overline{T}_{n-1}^{-1}(x,y)$ for $i<j$ equals 0 and for $i \geq j$, we have

$$\sum_{k=j}^{i} t_{i,k}(x,y)\bar{t}_{k,j}^{-1}(x,y) \quad = \quad \sum_{k=j}^{i} t_{i,k}(x,y)t_{k-1,j-1}^{-1}(x,y). \tag{3}$$

Then it can be seen that the term of the sum (3) is nonzero only for $k-1=j-1$ and $k-1=j$, that is, for $k=j$ and $k=j+1$. Thus

$$\sum_{k=j}^{i} t_{i,k}(x,y)t_{k-1,j-1}^{-1}(x,y) \quad = \quad t_{i,j}(x,y)t_{j-1,j-1}^{-1}(x,y)+t_{i,j+1}(x,y)t_{j,j-1}^{-1}(x,y).$$

Therefore we obtained the following result.

Lemma 1. *For any positive integer n and any real numbers x and y, we have*

$$T_n(x,y)=S_n(x,y)\overline{T}_{n-1}(x,y).$$

Example 1.

$$\begin{aligned}
&S_5(x,y)\overline{T}_4(x,y)\\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0 \\ \frac{1}{4}x^2 & -\frac{1}{4}xy & y & 0 & 0 & 0 \\ \frac{1}{8}x^3 & -\frac{1}{8}x^2y & 0 & y & 0 & 0 \\ \frac{1}{15}x^4 & -\frac{1}{15}x^3y & 0 & \frac{1}{15}xy & \frac{14}{15}y & 0 \\ \frac{1}{28}x^5 & -\frac{1}{28}x^4y & 0 & \frac{1}{28}x^2y & -\frac{3}{98}xy & \frac{195}{196}y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 \\ 0 & \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 \\ 0 & \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 \\ 0 & \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0 \\ \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 & 0 \\ \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 & 0 \\ \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 & 0 \\ \frac{1}{28}x^5 & \frac{1}{28}x^4y & \frac{1}{14}x^3y^2 & \frac{1}{7}x^2y^3 & \frac{1}{4}xy^4 & \frac{13}{28}y^5 \end{bmatrix}\\
&= T_5(x,y).
\end{aligned}$$

Using Lemma 1 and the definition of the matrices $G_k(x,y)$, we present the decomposition of $T_n(x,y)$ in the following.

Theorem 3. *The matrix $T_n(x,y)$ can be factorized as*

$$T_n(x,y)=G_n(x,y)G_{n-1}(x,y)\cdots G_1(x,y).$$

In particular,

$$T_n=G_nG_{n-1}\cdots G_1,$$

where $T_n:=T_n(1,1)$, $G_k:=G_k(1,1)$, $k=1,2,\ldots,n$.

For the inverse of the matrix $T_n(x,y)$, we get

$$T_n^{-1}(x,y)=G_1^{-1}(x,y)G_2^{-1}(x,y)\cdots G_n^{-1}(x,y).$$

Example 2. *Since*

$$T_5(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0 \\ \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 & 0 \\ \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 & 0 \\ \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 & 0 \\ \frac{1}{28}x^5 & \frac{1}{28}x^4y & \frac{1}{14}x^3y^2 & \frac{1}{7}x^2y^3 & \frac{1}{4}xy^4 & \frac{13}{28}y^5 \end{bmatrix},$$

we can factorize this matrix as

$G_5(x,y)G_4(x,y)G_3(x,y)G_2(x,y)G_1(x,y) =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0 \\ \frac{1}{4}x^2 & -\frac{1}{4}xy & y & 0 & 0 & 0 \\ \frac{1}{8}x^3 & -\frac{1}{8}x^2y & 0 & y & 0 & 0 \\ \frac{1}{15}x^4 & -\frac{1}{15}x^3y & 0 & \frac{1}{15}xy & \frac{14}{15}y & 0 \\ \frac{1}{28}x^5 & -\frac{1}{28}x^4y & 0 & \frac{1}{28}x^2y & -\frac{3}{98}xy & \frac{195}{196}y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 \\ 0 & \frac{1}{4}x^2 & -\frac{1}{4}xy & y & 0 & 0 \\ 0 & \frac{1}{8}x^3 & -\frac{1}{8}x^2y & 0 & y & 0 \\ 0 & \frac{1}{15}x^4 & -\frac{1}{15}x^3y & 0 & \frac{1}{15}xy & \frac{14}{15}y \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}x & \frac{1}{2}y & 0 & 0 \\ 0 & 0 & \frac{1}{4}x^2 & -\frac{1}{4}xy & y & 0 \\ 0 & 0 & \frac{1}{8}x^3 & -\frac{1}{8}x^2y & 0 & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}x & \frac{1}{2}y & 0 \\ 0 & 0 & 0 & \frac{1}{4}x^2 & -\frac{1}{4}xy & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}x & \frac{1}{2}y \end{bmatrix}.$$

3. Some Applications of the Matrix $T_n(x,y)$

In this section, we give some applications of the defined matrix $T_n(x,y)$. Firstly, we present a relation between the matrices $T_n(x,ay)$ and $T_n(x,-y)$ for a nonzero real number a.

Theorem 4. *For a nonzero real number a, the matrices $T_n(x,ay)$ and $T_n(x,-y)$ satisfy the following*

$$T_n\Big(x,\frac{y}{a}\Big)^{-1} = T_n(x,-y)^{-1}T_n(x,ay)T_n(x,-y)^{-1}.$$

Proof. The proof can be done easily by definition of the matrices and matrix multiplication. □

We give another factorization of the matrices $T_n(x,y)$ and $T_n(-x,y)$ where the variables x and y are separated from these matrices.

Theorem 5. *Let $D_n(x) := \operatorname{diag}\{1,x,x^2,\dots,x^n\}$ be a diagonal matrix. For any positive integer k and any non-zero real numbers x and y, we have*

$$T_k(x,y) = D_k(x)T_k(1,1)D_k^{-1}(x/y),$$
$$T_k(-x,y) = D_k(x)T_k(-1,1)D_k^{-1}(x/y).$$

Remark 1. *The entries of the matrix $T_n(x,y)$ can be separated by the indices, that is for $i \geq j$*

$$\Big(T_n(x,y)\Big)_{i,j} = \frac{2t_j}{t_{i+2}+t_i-1}x^{i-j}y^j = \frac{x^i}{t_{i+2}+t_i-1}2t_j\left(\frac{y}{x}\right)^j = a_ib_j$$

where

$$a_i = \frac{x^i}{t_{i+2}+t_i-1} \text{ and } b_j = 2t_j\left(\frac{y}{x}\right)^j.$$

In [19], the authors give some properties of such matrices. The related results provide the alternative proofs for Theorem 2 and Theorem 5.

Theorem 6. *Let $K_n(x,y) = [k_{i,j}]$ be a matrix with entries $k_{i,j} = t_jx^{i-j}y^j$ and $D_n^{'}$ be a diagonal matrix with diagonal entries $\{1, \frac{1}{2}, \cdots, \frac{2}{t_{i+2}+t_i-1}, \cdots, \frac{2}{t_{n+2}+t_n-1}\}$. Then we have*

$$T_n(x,y) = D_n^{'}K_n(x,y).$$

Proof. Multiplying $T_n(x,y)$ from the left with the diagonal matrix with entries $\{1, 2, \cdots, \frac{t_{i+2}+t_i-1}{2}, \cdots, \frac{t_{n+2}+t_n-1}{2}\}$, we get clearly the matrix $K_n(x,y)$. Hence the result follows. $\square$

Example 3. *For $n = 5$, we have*

$$\begin{aligned}
T_5(x,y) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0 \\ \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 & 0 \\ \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 & 0 \\ \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 & 0 \\ \frac{1}{28}x^5 & \frac{1}{28}x^4y & \frac{1}{14}x^3y^2 & \frac{1}{7}x^2y^3 & \frac{1}{4}xy^4 & \frac{13}{28}y^5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{28} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 & 0 \\ x^2 & xy & 2y^2 & 0 & 0 & 0 \\ x^3 & x^2y & 2xy^2 & 4y^3 & 0 & 0 \\ x^4 & x^3y & 2x^2y^2 & 4xy^3 & 7y^4 & 0 \\ x^5 & x^4y & 2x^3y^2 & 4x^2y^3 & 7xy^4 & 13y^5 \end{bmatrix} \\
&= D_5^{'}K_5(x,y).
\end{aligned}$$

Now, we present a matrix whose Cholesky factorization includes the matrix $T_n(1,1)$. First, we need the following result.

Lemma 2 ([16])**.** *For $n \geq 0$, the Tribonacci numbers t_n satisfy*

$$\sum_{k=1}^{n} t_k^2 = \frac{4t_nt_{n+1} - (t_{n+1} - t_{n-1})^2 + 1}{4}. \tag{4}$$

Theorem 7. *A matrix $Q_n = [c_{i,j}]$ with entries*

$$c_{i,j} = \frac{4t_k t_{k+1} - (t_{k+1} - t_{k-1})^2 + 1}{(t_{i+2} + t_i - 1)(t_{j+2} + t_j - 1)}$$

where $k = \min\{i, j\}$, is a symmetric matrix and its Cholesky factorization is $T_n(1,1)T_n(1,1)^T$.

Proof. Since

$$c_{i,j} = \tfrac{4t_k t_{k+1} - (t_{k+1} - t_{k-1})^2 + 1}{(t_{i+2} + t_i - 1)(t_{j+2} + t_j - 1)} = c_{j,i},$$

Q_n is symmetric. Now, we will show that $Q_n = T_n(1,1)T_n(1,1)^T$. By matrix multiplication,

$$\begin{aligned} T_n(1,1)T_n(1,1)^T = \sum_{k=0}^{n} t_{i,k} t_{j,k} &= \sum_{k=0}^{n} \frac{2t_k}{t_{i+2} + t_i - 1} \frac{2t_k}{t_{j+2} + t_j - 1} \\ &= \frac{4}{(t_{i+2} + t_i - 1)(t_{j+2} + t_j - 1)} \sum_{k=0}^{n} t_k^2. \end{aligned}$$

The proof is completed by substituting (4) in the last equation. □

In the last part of this section, we will give a relation between the matrix $T_n(x, y)$ and the exponential of a special matrix. Matrix exponentials are defined by simply plugging matrices into the usual Maclaurin series for the exponential function. In other words, for any square matrix M, the exponential of M is defined to be the matrix

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots + \frac{M^k}{k!} + \cdots.$$

For any square matrix M, we have the following result:

Theorem 8 ([3,24])**.**

(i) *For any numbers r and s, we have $e^{(r+s)M} = e^{rM} e^{sM}$.*
(ii) $(e^M)^{-1} = e^{-M}$.
(iii) *By taking the derivative with respect to x of each entry of e^{Mx}, we get the matrix $\frac{d}{dx} e^{Mx} = M e^{Mx}$.*

Definition 2. *The matrix $M_n = [m_{i,j}]$ is defined by*

$$m_{i,j} = \begin{cases} \frac{t_j}{t_i}, & \text{if } i = j+1, \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

We want to obtain a relation between $T_n(x, y)$ and $e^{M_n x}$, so we prove the following auxiliary result.

Lemma 3. *For every nonnegative integer k, the entries of the matrix M_n^k are given by*

$$(M_n^k)_{i,j} = \begin{cases} \frac{t_j}{t_i}, & \text{if } i = j + k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof will be done by induction on k. The case $k = 0$ follows straightforward. Let us assume the inductive hypothesis on $M_n^{k+1} = M_n M_n^k$. It is not hard to see for $i \neq j + k + 1$, $(M_n^{k+1})_{i,j} = 0$. For $i = j + k + 1$, we have

$$(M_n^{k+1})_{i,j} = \frac{t_{i-1}}{t_i} \frac{t_j}{t_{j+k}} = \frac{t_{j+k}}{t_{j+k+1}} \frac{t_j}{t_{j+k}} = \frac{t_j}{t_{j+k+1}}.$$

□

Theorem 9. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have*

$$(T_n(0,1)^{-1} T_n(x,1))_{i,j} = (i-j)!(e^{M_n x})_{i,j}.$$

Proof. Suppose that there is a matrix L_n such that $(T_n(0,1)^{-1}T_n(x,1))_{i,j} = (i-j)!(e^{L_n x})_{i,j}$. Then we have

$$\frac{d}{dx}(T_n(0,1)^{-1}T_n(x,1))_{i,j} = L_n(i-j)!(e^{L_n x})_{i,j} = L_n(T_n(0,1)^{-1}T_n(x,1))_{i,j}$$

and so

$$\frac{d}{dx}(T_n(0,1)^{-1}T_n(x,1))_{i,j} \mid_{x=0} = L_n.$$

Thus there is at most one matrix L_n such that $(T_n(0,1)^{-1}T_n(x,1))_{i,j} = (i-j)!(e^{L_n x})_{i,j}$. It can be easily seen that $L = M_n$, where M_n is the matrix given in Definition 2, by calculating $\frac{d}{dx}(T_n(0,1)^{-1}T_n(x,1))_{i,j} \mid_{x=0}$. We conclude that $M_n^k = 0$ for $k \geq n+1$, thus

$$e^{M_n x} = \sum_{k=0}^{n} M_n^k \frac{x^k}{k!}.$$

For $i < j$, we see that $(e^{M_n x})_{i,j} = 0$ and we also have $(e^{M_n x})_{i,i} = 1$. Now, suppose that $i > j$ and let $i = j + k$.

$$(e^{M_n x})_{i,j} = (M_n^k)_{i,j} \frac{x^k}{k!} = \frac{t_j}{t_{j+k}} \frac{x^k}{k!} = \frac{1}{k!}(T_n(0,1)^{-1}T_n(x,1))_{i,j}.$$

Hence the proof is completed. □

Example 4. *We obtain the matrix* $\frac{d}{dx}T_5(0,1)^{-1}T_5(x,1)$ *by taking the derivative of each entry of the matrix* $T_5(0,1)^{-1}T_5(x,1)$ *with respect to* x*. Thus*

$$\frac{d}{dx}T_5(0,1)^{-1}T_5(x,1)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2}x & \frac{1}{4} & 0 & 0 & 0 & 0\\ \frac{3}{8}x^2 & \frac{1}{4}x & \frac{1}{4} & 0 & 0 & 0\\ \frac{4}{5}x^3 & \frac{1}{5}x^2 & \frac{4}{15}x & \frac{4}{15} & 0 & 0\\ \frac{5}{28}x^4 & \frac{1}{7}x^3 & \frac{3}{14}x^2 & \frac{2}{7}x & \frac{1}{4} & 0 \end{bmatrix}.$$

Hence we have

$$M_5=T_5(0,1)^{-1}\frac{d}{dx}T_5(x,1)\mid_{x=0}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{4}{7} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{7}{13} & 0 \end{bmatrix}$$

and

$$M_5^2=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\times\frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\times\frac{4}{7} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{4}{7}\times\frac{7}{13} & 0 & 0 \end{bmatrix},$$

$$M_5^3=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2}\times\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\times\frac{1}{2}\times\frac{4}{7} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\times\frac{4}{7}\times\frac{7}{13} & 0 & 0 & 0 \end{bmatrix},$$

$$M_5^4=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2}\times\frac{1}{2}\times\frac{4}{7} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\times\frac{1}{2}\times\frac{4}{7}\times\frac{7}{13} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_5^5=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2}\times\frac{1}{2}\times\frac{4}{7}\times\frac{7}{13} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let M_n be the matrix defined in (5) and $U_n(x) = e^{M_n x}$. At the end of this section, we will find the explicit inverse of the matrix $R_n(x) = [I_n - \lambda U_n(x)]^{-1}$ for real number λ such that $\mid \lambda \mid< 1$. To achieve this, we need the following result.

Lemma 4 ([14], Corollary 5.6.16)**.** *A matrix A of order n is nonsingular if there is a matrix norm $||\cdot||$ such that $||I - A|| < 1$. If this condition is satisfied,*

$$A^{-1} = \sum_{k=0}^{\infty}(I - A)^k.$$

Theorem 10. *The matrix $R_n(x)$ is defined for real number λ such that $\mid \lambda \mid< 1$. The entries of the matrix are*

$$(R_n(x))_{i,i} = \frac{1}{1-\lambda}$$

and

$$(R_n(x))_{i,j} = (U_n(x))_{i,j}\mathfrak{L}i_{j-i}(\lambda)$$

for $i > j$, where $\mathfrak{L}i_n(z)$ is the polylogarithm function

$$\mathfrak{L}i_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}.$$

Proof. The statement in Lemma 4 is equivalent to: If $||\cdot||$ is a matrix norm and if $||A|| < 1$ for a square matrix of order n, then $I - A$ is invertible and $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$. Then for $|\lambda| < 1$, we can write

$$(R_n(x))_{i,j} = \sum_{k=0}^{\infty}(U_n(x))^k\lambda^k = \sum_{k=0}^{\infty}(U_n(xk))_{i,j}\lambda^k = (U_n(x))_{i,j}\sum_{k=0}^{\infty}\lambda^k k^{i-j}.$$

We obtain the desired result by writing the sum for $i = j$ and $i > j$. $\square$

Example 5.

$$\begin{aligned} I_4 - \lambda U_4(x) &= I_4 - \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ x\lambda & \lambda & 0 & 0 & 0 \\ \frac{1}{4}\lambda x^2 & \frac{1}{2}\lambda x & \lambda & 0 & 0 \\ \frac{1}{24}\lambda x^3 & \frac{1}{8}\lambda x^2 & \frac{1}{2}\lambda x & \lambda & 0 \\ \frac{1}{168}\lambda x^4 & \frac{1}{42}\lambda x^3 & \frac{1}{7}\lambda x^2 & \frac{4}{7}\lambda x & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ -x\lambda & 1-\lambda & 0 & 0 & 0 \\ -\frac{1}{4}\lambda x^2 & -\frac{1}{2}\lambda x & 1-\lambda & 0 & 0 \\ -\frac{1}{24}\lambda x^3 & -\frac{1}{8}\lambda x^2 & -\frac{1}{2}\lambda x & 1-\lambda & 0 \\ -\frac{1}{168}\lambda x^4 & -\frac{1}{42}\lambda x^3 & -\frac{1}{7}\lambda x^2 & -\frac{4}{7}\lambda x & 1-\lambda \end{bmatrix}. \end{aligned}$$

The inverse of this matrix equals

$$\begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 \\ \frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 & 0 & 0 \\ \frac{1}{4}\frac{\lambda^2+\lambda}{(1-\lambda)^3}x^2 & \frac{1}{2}\frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 & 0 \\ \frac{1}{24}\frac{\lambda^3+4\lambda^2+\lambda}{(1-\lambda)^4}x^3 & \frac{1}{8}\frac{\lambda^2+\lambda}{(1-\lambda)^3}x^2 & \frac{1}{2}\frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 \\ \frac{1}{168}\frac{\lambda^4+11\lambda^3+11\lambda^2+\lambda}{(1-\lambda)^5}x^4 & \frac{1}{42}\frac{\lambda^3+4\lambda^2+\lambda}{(1-\lambda)^4}x^3 & \frac{1}{7}\frac{\lambda^2+\lambda}{(1-\lambda)^3}x^2 & \frac{4}{7}\frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} \end{bmatrix}.$$

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