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Research Article

Double-toroidal and triple-toroidal commuting graph

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Abstract

In this paper, all finite non-abelian groups whose commuting graphs can be embed on the double-torus or triple-torus are classified.

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1. Introduction

Let G be a non-abelian group and $Z(G)$ be its center. The *commuting graph* of G, denoted by $\Gamma_c(G)$, is a simple undirected graph in which the vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. This graph is precisely the complement of the non-commuting graph of a group considered in [1] and [11]. Commuting graphs of groups were first mentioned in the seminal paper of Brauer and Fowler [7] which was concerned with the classification of the finite simple groups. However, the ever-increasing popularity of the topic is often attributed to a question, posed in 1975 by Paul Erdös and answered affirmatively by Neumann [13], asking whether or not a non-commuting graph having no infinite complete subgraph possesses a finite bound on the cardinality of its complete subgraphs. In recent years, the commuting graphs of groups have become a topic of research for many mathematicians (see, for example, [2], [5], [8], [10]). In [8], Das and Nongsiang determine (up to isomorphism) all finite non-abelian groups whose commuting graphs are planar or toroidal. There is also a ring theoretic version of the commuting graph (see, for example, [3], [4], [12]).

In the present paper, we deal with a topological aspect, namely, the genus of the commuting graphs of finite non-abelian groups. The primary objective of this paper is, to determine (up to isomorphism) all finite non-abelian groups whose commuting graphs are double-toroidal or triple-toroidal.

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2. Some prerequisites

In this section, we recall certain graph theoretic terminologies (see, for example, [14] and [15]) and some well-known results which have been used extensively in the forthcoming sections. All graphs in this paper are undirected, with no loops or multiple edges.

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Let $x, y \in V(\Gamma)$. Then x and y are said to be *adjacent* if $x \neq y$ and there is an edge $x - y$ in $E(\Gamma)$ joining x and y.

Given a graph Γ, let U be a nonempty subset of $V(\Gamma)$. Then the *induced subgraph* of Γ on U is defined to be the graph $\Gamma[U]$ in which the vertex set is U and the edge set consists precisely of those edges in Γ whose endpoints lie in U. If $\{\Gamma_\alpha\}_{\alpha\in\Lambda}$ is a family of subgraphs of a graph Γ, then the union $\underset{\alpha\in\Lambda}{\cup}\Gamma_\alpha$ denotes the subgraph of Γ whose vertex set is $\underset{\alpha\in\Lambda}{\cup} V(\Gamma_\alpha)$ and the edge set is $\underset{\alpha\in\Lambda}{\cup} E(\Gamma_\alpha)$. The graph obtained by taking the union of graphs Γ_1 and Γ_2 with disjoint vertex sets is the disjoint union or sum, written $\Gamma_1 + \Gamma_2$. In general, $m\Gamma$ is the graph consisting of m pairwise disjoint copies of Γ.

The *genus* of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. Clearly, if $\tilde{\Gamma}$ is a subgraph of Γ, then $\gamma(\tilde{\Gamma}) \leq \gamma(\Gamma)$. The surface with one, two and three handles is the torus, double-torus and triple-torus, respectively. The graphs embeddable on the surfaces of genus 0, 1, 2, 3 are the *planar*, *toroidal*, *double-toroidal* and *triple-toroidal* graphs, respectively.

A *block* of a graph Γ is a connected subgraph B of Γ that is maximal with respect to the property that removal of a single vertex (and the incident edges) from B does not make it disconnected, that is, the graph $B \setminus \{v\}$ is connected for all $v \in V(B)$. Given a graph Γ, there is a unique finite collection $\mathfrak{B}$ of blocks of Γ, such that $\Gamma = \underset{B\in\mathfrak{B}}{\cup} B$. The collection $\mathfrak{B}$ is called the *block decomposition* of Γ. In [6, Corollary 1], it has been proved that the genus of a graph is the sum of the genera of its blocks.

We conclude the section with the following two useful results.

Lemma 2.1 ([15], Theorem 6-38). *If $n \geq 3$, then*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

where K_n is the complete graph of order n.

Lemma 2.2 ([15], Theorem 6-37). *If $m, n \geq 2$, then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

where $K_{m,n}$ is the complete bipartite graph with parts of size m and n.

3. Commuting graph

In this section, we shall determine all finite non-abelian groups whose commuting graphs are of genus at most 3. The following theorems give all planar and toroidal commuting graphs.

Theorem 3.1 ([8], Theorem 5.7). *Let G be a finite non-abelian group. Then $\Gamma_c(G)$ is planar if and only if G is isomorphic to one of the following groups:*

(1) $S_3, D_8, Q_8, D_{10}, A_4, D_{12}, Q_{12}, D_8 \times \mathbb{Z}_2, Q_8 \times \mathbb{Z}_2, Sz(2), S_4, SL(2,3), A_5$,
(2) $\langle a, b : a^4 = b^4 = 1, a^b = a^{-1}\rangle \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$,
(3) $\langle a, b : a^8 = b^2 = 1, a^b = a^5\rangle \cong M_{16}$,
(4) $\langle a, b \mid a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1}\rangle \cong SG(16, 3)$,
(5) $\langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2cb\rangle \cong D_8 \circ \mathbb{Z}_4$.

Theorem 3.2 ([8], Theorem 6.6)**.** *Let G be a finite non-abelian group. Then $\Gamma_c(G)$ is toroidal if and only if G is isomorphic to one of the following groups:*

(1) $D_{14}, D_{16}, Q_{16}, SD_{16}, S_3 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$,
(2) $\langle a, b : a^7 = b^3 = 1, a^b = a^2\rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

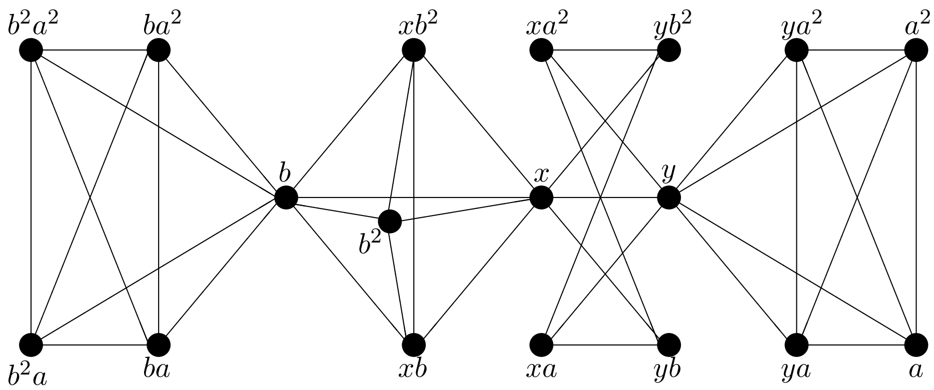

Figure 1. The subgraph $\bar{K} \cup \bar{L} \cup \bar{M} \cup \bar{N}$ of the commuting graph $\Gamma_c(S_3 \times S_3)$.

Remark 3.3. Consider the group $G = S_3 \times S_3 \cong \langle\ (1,2,3),\ (1,2),\ (4,5,6), (4,5)\rangle$. Let $a = (1,2,3), b = (4,5,6), x = (1,2), y = (4,5)$. Let $K = \{b, ba, ba^2, b^2a, b^2a^2\}$, $L = \{b, xb, b^2, xb^2, x\}$, $M = \{x, xa, xa^2\} \cup \{y, yb, yb^2\}$ and $N = \{a, a^2, y, ya, ya^2\}$. Suppose $\bar{K} = \Gamma_c(G)[K]$, $\bar{L} = \Gamma_c(G)[L]$, $\bar{M} = \Gamma_c(G)[M]$ and $\bar{N} = \Gamma_c(G)[N]$. Then $\bar{K} \cong \bar{L} \cong \bar{N} \cong K_5$ and $\bar{M} \cong K_{3,3}$. Note that $K \cap L = \{b\}$, $L \cap M = \{x\}$ and $M \cap N = \{y\}$. Thus the graphs $\bar{K} \cup \bar{L} \cup \bar{M} \cup \bar{N}$ is as shown in Figure 1. Clearly, from Figure 1, $\bar{K} \cup \bar{L} \cup \bar{M} \cup \bar{N}$ has four blocks $\bar{K}$, $\bar{L}$, $\bar{M}$ and $\bar{N}$. Thus $\gamma(\bar{K} \cup \bar{L} \cup \bar{M} \cup \bar{N}) = \gamma(\bar{K}) + \gamma(\bar{L}) + \gamma(\bar{M}) + \gamma(\bar{N}) = 4$. Thus $\Gamma_c(S_3 \times S_3) \geq \gamma(\bar{K} \cup \bar{L} \cup \bar{M} \cup \bar{N}) = 4$.

The following two lemmas will be useful in the sequel.

Lemma 3.4. *Let G be a p-group of order p^n, where $n > 1$.*

(1) *Then G has an abelian subgroup of order p^2.*
(2) *If $p = 3$, $n \geq 3$, then G has an abelian subgroup of order 27 or $G \setminus Z(G)$ has four commuting disjoint subsets of size 6.*
(3) *If $p = 2$, $n \geq 4$, then G has an abelian subgroup of order 8 and if $n \geq 5$ and $|Z(G)| \geq 4$, then G has an abelian subgroup of order 16.*
(4) *If $p = 2$, $n \geq 5$, and $|Z(G)| = 2$, then G has an abelian subgroup of order 16 or $G \setminus Z(G)$ has four commuting disjoint subsets of size 5.*

Proof. (a) Since G is a $p-$group, we have $|Z(G)| > 1$. Let x be a non-identity element of $Z(G)$ and consider the subgroup $\langle x, y\rangle$, for any $y \in G \setminus \langle x\rangle$.

(b) Consider a subgroup H of G of order 27. Suppose that H is non-abelian. Then, $|Z(H)| = 3$ and the centralizers of the non-central elements of H are of order 9. Since any two distinct centralizers of the non-central elements of H intersect at $Z(H)$, it follows that the number of such centralizers is 4. Thus it follows that $G \setminus Z(G)$ has four commuting disjoint subsets of size 6.

(c) This is Lemma 5.1 of [8].

(d) Consider a subgroup of G of order 32. Using GAP[9] or otherwise, one can see that G has an abelian subgroup of order 16 or $G \setminus Z(G)$ has four commuting disjoint subsets of size 5. □

Lemma 3.5. *Let G be a finite group such that $7 \mid |G|$. If the sylow 7-subgroup is normal in G, then either G has an abelian subgroup of order greater than or equal to 14 or $|G| \leq 42$.*

Proof. Suppose G has no abelian subgroup of order greater than or equal to 14. Let P be the sylow 7-subgroup of G. In view of Lemma 3.4, Part (a), $|P| = 7$. Let $x \in G$, such that $\circ(x) = 7$. Then, $|C_G(x)| = 7$; otherwise $\langle x, y\rangle$ for $y \in C_G(x) \setminus \langle x\rangle$ is an abelian subgroup of G of order atleast 14. Given that the sylow 7-subgroup is normal in G and thus it follows that $|Cl_G(x)| \leq 6$. Since $|G| = |C_G(x)||Cl_G(x)|$, we have $|G| \leq 42$. □

Theorem 3.6. *Let G be a finite non-abelian group. Then, the commuting graph of G is double-toroidal if and only if G is isomorphic to one of the following groups:*

(1) $D_{18}, D_{20}, Q_{20}, S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2, S_3 \times \mathbb{Z}_4$,
(2) $\langle x, y, z : x^3 = y^3 = z^2 = [x, y] = 1, x^z = x^{-1}, y^z = y^{-1}\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$,
(3) $\langle x, y : x^8 = y^3 = 1, y^x = y^{-1}\rangle \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_8$,
(4) $\langle x, y, z : x^4 = y^3 = z^2 = 1, y^x = y^{-1}, [x, z] = [y, z] = 1\rangle \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$,
(5) $\langle x, y : x^4 = y^3 = (yx^2)^2 = [x^{-1}yx, y] = 1\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$,
(6) $\langle x, y, z : x^4 = y^4 = z^3 = 1, y^x = y^{-1}, z^{y^2} = z^{-1}, z^{x^2} = z^{-1}$,
$x^{-1}zx^{-1} = (zy)^2\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$.

Proof. Let G be a finite non-abelian group whose commuting graph is double-toroidal. Then $\Gamma_c(G)$ has no subgraphs isomorphic to K_9, $K_8 + K_5$ or $3K_5$.

(1) Suppose $|Z(G)| \geq 8$. Since G is non-abelian, we have $|G/Z(G)| \geq 4$. Let $xZ(G)$ and $yZ(G)$ be two distinct non-identity elements of $G/Z(G)$. Then the induced subgraph of $\Gamma_c(G)$ by the set $xZ(G) \cup yZ(G)$ has a subgraph isomorphic to $2K_8$, which is a contradiction. Thus $|Z(G)| \leq 7$.

(2) Suppose $|Z(G)| = 7$. If p is a prime and $p = 3, 5$ or $p > 7$, then $p \nmid |G|$; otherwise, for an element x of G of order p, $\langle x, Z(G)\rangle$ is an abelian group of order $7p$. Thus $|G| = 2^i 7^j$. If $i \geq 2$, then G has an abelian subgroup of order 4 and hence an abelian subgroup of order 28, which is a contradiction. By Lemma 3.4, we have $j = 1$ and so $|G| = 14$, which is a contradiction. Thus $|Z(G)| \leq 6$.

(3) Suppose $|Z(G)| = 6$. If p is a prime and $p \geq 5$, then $p \nmid |G|$; otherwise, for an element x of G of order p, $\langle x, Z(G)\rangle$ is an abelian group of order $6p$. Thus $|G| = 2^i 3^j$. By Lemma 3.4, we have $i \leq 4$ and $j \leq 2$. If $i = 4$, then by Lemma 3.4, G has an abelian subgroup of order 8 and hence a subgroup of order 24, a contradiction. So $i \leq 3$. Similarly if $j = 2$, then G has an abelian group of order 18, a contradiction. It follows that $|G| = 24$ and so $G \cong D_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_3$. The commuting graphs of both these groups are isomorphic to $3K_6$. Hence the commuting graphs of $D_8 \times \mathbb{Z}_3$ and $Q_8 \times \mathbb{Z}_3$ are not double-toroidal.

(4) Suppose $|Z(G)| = 5$. If p is a prime and $p = 3$ or $p \geq 7$, then clearly $p \nmid |G|$. Thus, we have $|G| = 2^i 5^j$. If $i \geq 2$, then G has an abelian subgroup of order 4 and hence an abelian subgroup of order 20, which is a contradiction. By Lemma 3.4, we have $j = 1$ and so $|G| = 10$, which is a contradiction. Thus $|Z(G)| \neq 5$.

(5) Suppose $|Z(G)| = 4$. If p is a prime and $p \geq 5$, then clearly $p \nmid |G|$. Thus $|G| = 2^i 3^j$. By Lemma 3.4 and since $|Z(G)| = 4$, we have $i \leq 4$ and $j \leq 1$ and so $|G| = 16, 24$ or 48. Groups of order 16 with $|Z(G)| = 4$ are planar, see [8, Lemma 5.5]. Groups of order 24 with $|Z(G)| = 4$, are

- $\langle x, y : x^8 = y^3 = 1, y^x = y^{-1}\rangle \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_8$,
- $S_3 \times \mathbb{Z}_4$,
- $\langle x, y, z : x^4 = y^3 = z^2 = 1, y^x = y^{-1}, [x, z] = [y, z] = 1\rangle \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$,
- $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

All these groups are AC-groups, with each of them having 3 centralizers of size 8 and one centralizer of size 12. Thus by [8, Proposition 3.4], the commuting graphs of these groups are double-toroidal.

If $|G| = 48$, then $|G/Z(G)| = 12$. If $\bar{x}$ is an element of $G/Z(G)$ of order 6, then $\langle x, Z(G)\rangle$ is an abelian group of order 24, which is a contradiction. Thus $G/Z(G)$ has no element of order 6 and so $G/Z(G) \cong A_4$. Thus $G/Z(G)$ has two elements $\bar{x}, \bar{y}$ of order 3, such that $\bar{x} \notin$

$\langle \bar{y} \rangle$. Therefore, the induced subgraph of $\Gamma_c(G)$ by the set $xZ(G) \cup x^2Z(G) \cup yZ(G) \cup y^2Z(G)$ is isomorphic to $2K_8$, which is a contradiction.

(6) Suppose $|Z(G)| = 3$. If p is a prime and $p \geq 5$, then clearly $p \nmid |G|$. Thus, we have $|G| = 2^i 3^j$. By Lemma 3.4, we have $i \leq 4$ and $j \leq 2$. Suppose $i \geq 2$. Then, by Lemma 3.4, a sylow 2-subgroup of G contains an abelian subgroup of order 4 and hence G contains an abelian subgroup of order 12, which is a contradiction. Hence $i = 1$. Therefore $|G| = 18$. There is only one group of order 18 with $|Z(G)| = 3$, namely $\mathbb{Z}_3 \times S_3$ and its commuting graph is toroidal. Thus $|Z(G)| \neq 3$.

(7) Suppose $|Z(G)| = 2$. If p is a prime and $p \geq 7$, then clearly $p \nmid |G|$. Thus $|G| = 2^i 3^j 5^k$. By Lemma 3.4, we have $i \leq 4$, $j \leq 2$ and $k \leq 1$. Suppose $j = 2$. By Lemma 3.4, a sylow 3-subgroup S of G is an abelian subgroup of order 9 and so $\langle S, Z(G) \rangle$ is an abelian subgroup of order 18, which is a contradiction. Therefore $j \leq 1$ and thus $|G| \mid 2^4.3.5$.

By Theorem 3.1, groups of order 6, 8, 10 and 12 are planar and by [8, Lemma 6.2], groups of order 16 with $|Z(G)| = 2$ are toroidal. Groups of order 30 has an abelian subgroup of order 15. Thus $|G| \in \{20, 24, 40, 48, 60, 80, 120, 240\}$.

Group of order 20 with $|Z(G)| = 2$ are D_{20} and Q_{20}. These two groups are AC-groups, with each of them has one centralizer of size 10 and 5 centralizers of size 4. Thus by [8, Proposition 3.4], their commuting graphs are double-toroidal.

Groups of order 24 with $|Z(G)| = 2$ are

- $SL(2, 3)$,
- $\mathbb{Z}_2 \times A_4$,
- Q_{24},
- D_{24},
- $\langle x, y, z : x^2 = y^2 = z^3 = (xz)^2 = (yx)^4 = 1, y^z = y^{-1} \rangle \cong (\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

The group $SL(2, 3)$ is planar. The group $\mathbb{Z}_2 \times A_4$ is toroidal. If G is one of the groups Q_{24}, D_{24} or $(\mathbb{Z}_6 \times \mathbb{Z}_2) : \mathbb{Z}_2$, then G has an abelian subgroup of order 12. Thus K_{10} is a subgraph of $\Gamma_c(G)$, which is a contradiction.

Note that if G has an abelian subgroup of order greater than of equal to 12, then its commuting graph is not double-toroidal. Groups of order 40 with $|Z(G)| = 2$ and has no abelian subgroup of order greater than or equal to 12 are

- $\langle x, y : y^5 = x^8 = 1, x^y = xy \rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_8$,
- $\langle x, y, z : y^2 = x^4 = z^5 = 1, y^x = y^{-1}, y^z = y^{-1}, x^z = xz \rangle \cong \mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$.

Each of the groups $\mathbb{Z}_5 \rtimes \mathbb{Z}_8$, and $\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$, has 5 abelian subgroups of order 8, namely the sylow 2-subgroups and intersection of any two is the center. Thus the commuting graphs of these groups are not double-toroidal.

Groups of order 48 with $|Z(G)| = 2$ and has no abelian subgroup of order greater than or equal to 12 are

- $\langle x, y, z : y^3 = z^4 = 1, x^2 = z^2, y^x = y^{-1}, y^{-1}zy^{-1}z^{-1}y^{-1}z = xz^{-1}xy^{-1}zy = 1 \rangle \cong SL(2, 3) \circ \mathbb{Z}_2$,
- $GL(2, 3)$,
- $\langle x, y, z : x^2 = y^3 = z^4 = (xz^2)^2 = 1, y^z = y^{-1}, (xx^y)^z = x^{-1} \rangle \cong A_4 \rtimes \mathbb{Z}_4$,
- $\mathbb{Z}_2 \times S_4$.

The groups $SL(2, 3) \circ \mathbb{Z}_2$ and $GL(2, 3)$ are AC-groups. Each of these groups has 3 centralizers of size 8 and the rest are of size less than or equal to 6. Thus by [8, Proposition 3.4], $\gamma(\Gamma_c(GL(2, 3))) = \gamma(\Gamma_c(SL(2, 3) \circ \mathbb{Z}_2)) = 3$, that is, $\Gamma_c(GL(2, 3))$ and $\Gamma_c(SL(2, 3) \circ \mathbb{Z}_2)$ are not double-toroidal.

The group $A_4 \rtimes \mathbb{Z}_4$ has four abelian subgroups of order 8, say A, B, C, D, such that $A \cap B = A \cap C = A \cap D = B \cap C = B \cap D = Z(G)$ and $|C \cap D| = 4$. Suppose $(C \cap D) \setminus Z(G) = \{u, v\}$. Then $\gamma(\Gamma_c(G)) \geq \gamma(\Gamma_c(G)[A \setminus Z(G)]) + \gamma(\Gamma_c(G)[B \setminus Z(G)]) +$

$\gamma(\Gamma_c(G)[(C \setminus (Z(G)) \cup \{u\}]) + \gamma(\Gamma_c(G)[(D \setminus (Z(G)) \cup \{v\})]) = 1+1+1+1 = 4$. Thus the the commuting graph of $A_4 \rtimes \mathbb{Z}_4$ is not double-toroidal.

For the group $G = \mathbb{Z}_2 \times S_4$, let $A = \mathbb{Z}_2 \times \langle (1,4,2,3) \rangle$, $B = \mathbb{Z}_2 \times \langle (1,3,4,2) \rangle$, $C = \mathbb{Z}_2 \times \langle (1,3),(2,4) \rangle$ and $D = \mathbb{Z}_2 \times \langle (1,2,3,4) \rangle$. Then $A \cap B = A \cap C = A \cap D = B \cap C = B \cap D = Z(G) = \mathbb{Z}_2 \times \{()\}$ and $C \cap D = \mathbb{Z}_2 \times \langle (1,3)(2,4) \rangle$. Let $H = \mathbb{Z}_2 \times \langle (1,3)(2,4) \rangle$. Thus $\gamma(\Gamma_c(G)) \geq \gamma(\Gamma_c(G)[A \setminus Z(G)]) + \gamma(\Gamma_c(G)[B \setminus Z(G)]) + \gamma(\Gamma_c(G)[(C \setminus H) \cup \{(1,(1,2)(3,4))\}]) + \gamma(\Gamma_c(G)[(D \setminus H) \cup \{(x,(1,2)(3,4))\}]) = 1+1+1+1 = 4$, where $\mathbb{Z}_2 = \langle x \rangle$. Thus the the commuting graph of $A_4 \rtimes \mathbb{Z}_4$ is not double-toroidal.

Let G be a group of order 80. Let P_1 and P_2 be two sylow 5-subgroups of G. Then $2K_8$ is a subgraph of $\Gamma_c(G)[\langle P_1, P_2, Z(G) \rangle \setminus Z(G)]$ and so $2K_8$ is a subgraph of $\Gamma_c(G)$, which is a contradiction. Thus the sylow 5-subgroup of G is normal in G. Let $P = \langle x \rangle$ be the sylow 5-subgroup of G. Thus $|Cl_G(x)| = 4$. Now since $|C_G(x)||Cl_G(x)| = 80$, we have $|C_G(x)| = 20$. Note that $Z(G) \subset C_G(x)$. Thus $|Z(C_G(x))| \geq 10$. But $|Z(C_G(x))| = 10$ is not possible; otherwise $C_G(x)/Z(C_G(x))$ is cyclic and hence $C_G(x)$ is abelian. Therefore $|Z(C_G(x))| = 20$, that is $C_G(x)$ is abelian and so G has an abelian subgroup of order 20. Thus $\Gamma_c(G)$ is not double-toroidal.

Solvable groups of order 60 and 120 has a Hall subgroup of order 15, which is abelian. There is no non-solvable group of order 60 with $|Z(G)| = 2$. Non-solvable groups of order 120 with $|Z(G)| = 2$ are $SL(2,5)$ and $\mathbb{Z}_2 \times A_5$. Each of these groups has 6 abelian subgroups of order 10 and the intersection of any two of these subgroups is the center. Thus the commuting graphs of $SL(2,5)$ and $\mathbb{Z}_2 \times A_5$ are not double-toroidal.

Solvable groups of order 240 has a Hall subgroup of order 15, which is abelian. There are 8 non-solvable groups of order 240, but all these groups has an abelian subgroups of order 12. Therefore, there are no commuting graphs of groups of order 240 which are double-toroidal.

(8) Suppose $|Z(G)| = 1$. By Lemma 3.4, we have $|G| = 2^i 3^j 5^k 7^l$, where $i \leq 4$, $j \leq 2$, $k \leq 1$ and $l \leq 1$. Thus $|G| \mid 2^4.3^2.5.7$.

If $7 \mid |G|$, then by Lemma 3.5, we have $|G| \leq 42$. Thus $|G| = 14, 21, 28, 42$. Up to isomorphism, groups of order 14 and 21 are D_{14} and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, respectively. Both the commuting graphs of these groups are toroidal. Thus it follows that $|G| = 28, 42$. There are no group of order 28 with trivial center. Group of order 42 with trivial center are $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \langle x^2 = y^3 = z^7 = 1, (xz)^2 = 1, xyx = y, zy = yz^2 \rangle$ and D_{42}. The group $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ has 7 abelian subgroups of size 6 and one abelian subgroup of size 7 and the intersection of these subgroups is the trivial subgroup. Thus the commuting graph of $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ is not double-toroidal. The dihedral group D_{42} has an abelian subgroup of order 21. Thus its commuting graph is not double-toroidal.

Suppose $9 \mid |G|$. Then $7 \nmid |G|$. Let n_3 be the number of sylow 3-subgroup of G. Then, $n_3 \equiv 1 \bmod 3$ and $n_3 \mid 2^4.5$. Thus $n_3 = 1$ or $n_3 \geq 4$. Suppose $n_3 \geq 4$. Let P_1, P_2 be sylow 3-subgroups of G. Let $Q_1 = P_1 \setminus \{e\}$, then $\gamma(\Gamma_c(G)[Q_1]) = 2$. Note that $|P_1 \cap P_2| \leq 3$. Let $Q_2 = P_2 \setminus P_1$. Then $|Q_2| \geq 6$. Therefore $\gamma(\Gamma_c(G)) \geq \gamma(\Gamma_c(G)[Q_1]) + \gamma(\Gamma_c(G)[Q_2]) = 3$, a contradiction. Hence, the sylow 3-subgroup of G is normal in G. Let P be the sylow 3-subgroup of G. Clearly P is solvable. Thus $|G/P| = 2^i 5^j$, and so, by Burnside's theorem, G/P is solvable. Thus if $5 \mid |G|$, then G has a Hall subgroup of order 45, and groups of order 45 are abelian, which is a contradiction. Therefore $5 \nmid |G|$ and so $|G| \in \{18, 36, 72, 144\}$.

There are two groups of order 18 with trivial center, namely, D_{18} and

$$\langle x, y, z : x^3 = y^3 = z^2 = [x,y] = 1, x^z = x^{-1}, y^z = y^{-1} \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2.$$

Both these groups are AC-groups. The centralizers of the non-central elements of any of these groups are of size 9 and 2. There is exactly one centralizer of size 9 of any of these groups. Thus by [8, Proposition 3.4], their commuting graphs are double-toroidal.

There are two groups of order 36 with trivial center, up to isomorphism, namely $S_3 \times S_3$ and

$$\langle x, y : x^4 = y^3 = (yx^2)^2 = [x^{-1}yx, y] = 1\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4.$$

By Remark 3.3, genus of the commuting graph of $S_3 \times S_3$ is greater than or equal to 4. The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ is an AC-group, with centralizers of non-central elements are of size 4 and 9. There is exactly one centralizer of size 9. By [8, Proposition 3.4], $\gamma(\Gamma_c((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4)) = 2$. Thus $\Gamma_c((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4)$ is double-toroidal.

There are 6 non-abelian groups with trivial center of order 72, up to isomorphism. They are

- $\langle x, y, z : x^2 = y^2 = z^9 = (xz)^2 = (z^{-1}yx)^2 = 1, y^z = (yx)^2, y^{z^3} = y^{-1}\rangle \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_2$,
- $\langle x, y : x^3 = y^8 = (y^{-1}x)^2y^2x^{-1} = (y^4x)^2 = 1\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$,
- $\langle x^2 = y^2 = z^3 = (xz)^2 = (yx)^4 = (yz^{-1})^2(yz)^2 = (z^{-1}(yx)^2)^2 = 1\rangle \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2$,
- $\langle x, y, z : x^3 = y^4 = z^4 = (x^{-1}y^2)^2 = (z^2x)^2 = yxyzx^{-1}zx = 1, y^z = y^{-1}\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$.
- $\langle x, y, z, u : x^2 = y^2 = z^3 = u^3 = (xu)^2 = (xz)^2 = (yz)^3 = (xyz)^2 = 1, uz = zu, yu = uy\rangle \cong (\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$,
- $A_4 \times S_3$.

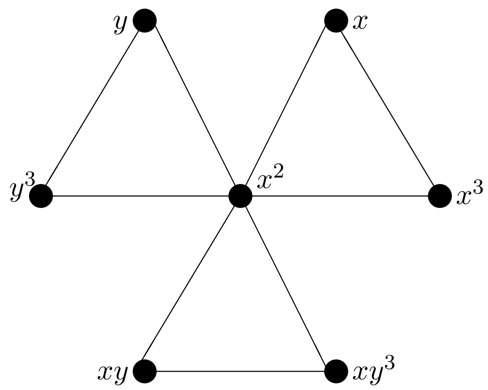

Figure 2. Commuting graph of the Quaternion group $Q_8 \cong \langle x, y : x^4 = 1, x^2 = y^2, xyx^{-1} = y^{-1}\rangle$, taken all the non-identity elements as vertices.

Let $\bar{G} = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$. The group $\bar{G}$ consist of one sylow 3-subgroup of order 9 and 9 sylow 2-subgroups of order 8. The sylow 2-subgroups of $\bar{G}$ are isomorphic to Q_8 and the sylow 3-subgroup is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. The intersection of any two of these subgroups is trivial. Thus $\bar{G}$ is exactly the union of these subgroups. Let L be any of these subgroups and $x \in L, x \neq 1$. Then $C_{\bar{G}}(x) \subseteq L$. Thus the commuting graph of $\bar{G}$ consist of 10 components. One of the component is $\Gamma_c\bar{G}[H]$, where $H \cup \{1\}$ is the sylow 3-subgroup of $\bar{G}$. The other 9 components are $\Gamma_c\bar{G}[K_i]$, where $K_i \cup \{1\}, i = 1, 2, \ldots, 9$, are the sylow 2-subgroups of $\bar{G}$. Now, $\Gamma_c\bar{G}[H] \cong K_8$ and from Figure 2, $\Gamma_c\bar{G}[K_i]$, for $i = 1, 2, \ldots, 9$, are planar. Thus $\Gamma_c\bar{G}$ is double-toroidal.

The groups $((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_2, (\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ and $A_4 \times S_3$ has an abelian subgroup of order 12. Thus the commuting graphs of these groups are not double-toroidal. The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$ has 9 abelian subgroups of order 8 and one of order 9. The intersection of any two of these subgroups is trivial. Thus $K_8 + K_5$ is a subgraph of $\Gamma_c((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8)$, showing that the commuting graph of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$ is not double-toroidal. The group

$(S_3\times S_3)\rtimes\mathbb{Z}_2$ has $S_3\times S_3$ as a subgroup. By Remark 3.3, the genus of the commuting graph of $S_3 \times S_3$ is at least 4. Note that $\Gamma_c(S_3 \times S_3)$ is a subgraph of $\Gamma_c((S_3 \times S_3) \rtimes \mathbb{Z}_2)$. Thus the genus of the commuting graph of $(S_3 \times S_3) \rtimes \mathbb{Z}_2$ is atleast 4. Hence $\Gamma_c((S_3 \times S_3) \rtimes \mathbb{Z}_2)$ is not double-toroidal.

There are 3 nonabelian groups of order 144 with trivial center. These are:-

- $\langle x, y, z : y^2 = z^3 = (yz)^2 = 1, x^y = x^3, xzxz^{-1}x^{-2}z = xz^{-1}xyxzxy = 1\rangle \cong ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_2$
- $S_3 \times S_4$
- $A_4 \times A_4$

The group $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_2$ has 9 abelian subgroups of order 8 and one of order 9. The intersection of any two of these subgroups is trivial. The groups $S_3 \times S_4$ and $A_4 \times A_4$ has an abelian subgroup of order 12. Thus, the commuting graphs of these groups are not double-toroidal.

Suppose $|G| = 2^i.3.5$. Then $|G| \in \{30, 60, 120, 240\}$. Group of order 30 are solvable, and hence has a Hall subgroup of order 15, which is abelian. Solvable group of order 60 has a Hall subgroup of order 15, which is abelian. Non-solvable group of order 60 is A_5. But the commuting graph of A_5 is planar. Solvable group of order 120 has a Hall subgroup of order 15 which is abelian. There is only one non-solvable group with trivial center of order 120, namely S_5. It has 10 abelian subgroups of order 6 and the intersection of any two of these subgroups is trivial. Thus the commuting graph of S_5 is not double-toroidal.

Suppose $|G| = 2^i.5$, that is $|G| \in \{10, 20, 40, 80\}$. There is only one non-abelian group of order 10 upto isomorphism, namely D_{10} and its commuting graph is planar. There is only one non-abelian group with trivial center of order 20, namely, $Sz(2)$ and its commuting graph is planar. There is no non-abelian group of order 40 with trivial center. There is only one non-abelian group of order 80 with trivial center, namely,

$$\langle x, y : x^2 = y^5 = (xy^{-1}xy)^2 = (xy^{-1})^5 = (xy^{-2}xy^2)^2 = 1\rangle \cong \\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_5.$$

This group has an abelian subgroup of order 16. Therefore, its commuting graph is not double-toroidal. This completes the proof. □

The proof of Theorem 3.7 below is as nearly the same as the proof of Theorem 3.6. But we have put it separately for the sake of completeness of Theorem 3.7.

Theorem 3.7. *Let G be a finite non-abelian group. Then, the commuting graph of G is triple-toroidal if and only if G is isomorphic to one of the following groups:*

(1) $GL(2, 3), D_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_3$,
(2) $\langle x, y, z : y^3 = z^4 = 1, x^2 = z^2, y^x = y^{-1}, y^{-1}zy^{-1}z^{-1}y^{-1}z = xz^{-1}xy^{-1}zy = 1\rangle \cong SL(2, 3) \circ \mathbb{Z}_2$,

Proof. Let G be a finite non-abelian group whose commuting graph is triple-toroidal. Then $\Gamma_c(G)$ has no subgraphs isomorphic to K_{10}, $K_9 + K_5$, $2K_8$, $K_8 + 2K_5$ or $4K_5$.

(1) Suppose $|Z(G)| \geq 8$. Since G is non-abelian, we have $|G/Z(G)| \geq 4$. Let $xZ(G)$ and $yZ(G)$ be two distinct non-identity elements of $G/Z(G)$. Then the induced subgraph of $\Gamma_c(G)$ by the set $xZ(G) \cup yZ(G)$ has a subgraph isomorphic to $2K_8$, which is a contradiction. Thus $|Z(G)| \leq 7$.

(2) Suppose $|Z(G)| = 7$. If p is a prime and $p = 3, 5$ or $p > 7$, then $p \nmid |G|$; otherwise, for an element x of G of order p, $\langle x, Z(G)\rangle$ is an abelian group of order $7p$. Thus $|G| = 2^i7^j$. If $i \geq 2$, then G has an abelian subgroup of order 4 and hence an abelian subgroup of order 28, which is a contradiction. By Lemma 3.4, we have $j = 1$ and so $|G| = 14$, which is a contradiction. Thus $|Z(G)| \leq 6$.

(3) Suppose $|Z(G)| = 6$. If p is a prime and $p \geq 5$, then $p \nmid |G|$; otherwise, for an element x of G of order p, $\langle x, Z(G)\rangle$ is an abelian group of order $6p$. Thus $|G| = 2^i3^j$. By

Lemma 3.4, we have $i \leq 4$ and $j \leq 2$. If $i = 4$, then by Lemma 3.4, G has an abelian subgroup of order 8 and hence a subgroup of order 24, a contradiction. So $i \leq 3$. Similarly if $j = 2$, then G has an abelian group of order 18, a contradiction. It follows that $|G| = 24$ and so $G \cong D_8 \times \mathbb{Z}_3, Q_8 \times \mathbb{Z}_3$. The commuting graphs of both these groups are isomorphic to $3K_6$. Hence the commuting graphs of $D_8 \times \mathbb{Z}_3$ and $Q_8 \times \mathbb{Z}_3$ are triple-toroidal.

(4) Suppose $|Z(G)| = 5$. If p is a prime and $p = 3$ or $p \geq 7$, then clearly $p \nmid |G|$. Thus, we have $|G| = 2^i 5^j$. If $i \geq 2$, then G has an abelian subgroup of order 4 and hence an abelian subgroup of order 20, which is a contradiction. By Lemma 3.4, we have $j = 1$ and so $|G| = 10$, which is a contradiction. Thus $|Z(G)| \neq 5$.

(5) Suppose $|Z(G)| = 4$. If p is a prime and $p \geq 5$, then clearly $p \nmid |G|$. Thus $|G| = 2^i 3^j$. By Lemma 3.4 and since $|Z(G)| = 4$, we have $i \leq 4$ and $j \leq 1$ and so $|G| = 16, 24$ or 48. Groups of order 16 with $|Z(G)| = 4$ are planar, see [8, Lemma 5.5]. Groups of order 24 with $|Z(G)| = 4$, are

- $\langle x, y : x^8 = y^3 = 1, y^x = y^{-1} \rangle \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_8$,
- $S_3 \times \mathbb{Z}_4$,
- $\langle x, y, z : x^4 = y^3 = z^2 = 1, y^x = y^{-1}, [x, z] = [y, z] = 1 \rangle \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$,
- $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

By Theorem 3.6, the commuting graphs of these groups are double-toroidal.

If $|G| = 48$, then $|G/Z(G)| = 12$. If $\bar{x}$ is an element of $G/Z(G)$ of order 6, then $\langle x, Z(G) \rangle$ is an abelian group of order 24, which is a contradiction. Thus $G/Z(G)$ has no element of order 6 and so $G/Z(G) \cong A_4$. Thus $G/Z(G)$ has two elements $\bar{x}, \bar{y}$ of order 3, such that $\bar{x} \notin \langle \bar{y} \rangle$. Therefore, the induced subgraph of $\Gamma_c(G)$ by the set $xZ(G) \cup x^2 Z(G) \cup yZ(G) \cup y^2 Z(G)$ is isomorphic to $2K_8$, which is a contradiction.

(6) Suppose $|Z(G)| = 3$. If p is a prime and $p \geq 5$, then clearly $p \nmid |G|$. Thus, we have $|G| = 2^i 3^j$. By Lemma 3.4, we have $i \leq 4$ and $j \leq 2$. Suppose $i = 4$. Then, by Lemma 3.4, a sylow 2-subgroup of G contains an abelian subgroup of order 8 and hence G contains an abelian subgroup of order 24, which is a contradiction. Suppose $i \geq 2$ and $j = 2$. Then a sylow 2-sugroup of G has an abelian subgroup M of order 4 and hence $H = \langle M, Z(G) \rangle$ is an abelian subgroup of G of order 12. Let K be a sylow 3-subgroup of G. Then $H \cap K = Z(G)$. Thus $K_9 + K_5$ is a subgraph of $\Gamma_c(G)[(H \cup K) \setminus Z(G)]$, and hence $K_9 + K_5$ is a subgraph of $\Gamma_c(G)$, which is a contradiction. Note that there is no group of order 24 with $|Z(G)| = 3$. Therefore $|G| = 18$. There is only one group of order 18 with $|Z(G)| = 3$, namely $\mathbb{Z}_3 \times S_3$ and its commuting graph is toroidal. Thus $|Z(G)| \neq 3$.

(7) Suppose $|Z(G)| = 2$. If p is a prime and $p \geq 7$, then clearly $p \nmid |G|$. Thus $|G| = 2^i 3^j 5^k$. By Lemma 3.4, we have $i \leq 4$, $j \leq 2$ and $k \leq 1$. Suppose $j = 2$. By Lemma 3.4, a sylow 3-subgroup S of G is an abelian subgroup of order 9 and so $\langle S, Z(G) \rangle$ is an abelian subgroup of order 18, which is a contradiction. Therefore $j \leq 1$ and thus $|G| \mid 2^4.3.5$.

By Theorem 3.1, groups of order 6, 8, 10 and 12 are planar and by [8, Lemma 6.2], groups of order 16 with $|Z(G)| = 2$ are toroidal. Groups of order 30 has an abelian subgroup of order 15. Thus $|G| \in \{20, 24, 40, 48, 60, 80, 120, 240\}$.

Group of order 20 with $|Z(G)| = 2$ are D_{20} and Q_{20}. By Theorem 3.6, the commuting graphs of these groups are double-toroidal.

Groups of order 24 with $|Z(G)| = 2$ are

- $SL(2, 3)$,
- $\mathbb{Z}_2 \times A_4$,
- Q_{24},
- D_{24},
- $\langle x, y, z : x^2 = y^2 = z^3 = (xz)^2 = (yx)^4 = 1, y^z = y^{-1} \rangle \cong (\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

The group $SL(2,3)$ is planar. The group $\mathbb{Z}_2 \times A_4$ is toroidal. If G is one of the groups Q_{24}, D_{24} or $(\mathbb{Z}_6 \times \mathbb{Z}_2) : \mathbb{Z}_2$, then G has an abelian subgroup of order 12. Thus K_{10} is a subgraph of $\Gamma_c(G)$, which is a contradiction.

Note that if G has an abelian subgroup of order greater than of equal to 12, then its commuting graph is not triple-toroidal. Groups of order 40 with $|Z(G)| = 2$ and has no abelian subgroup of order greater than or equal to 12 are

- $\langle x, y : y^5 = x^8 = 1, x^y = xy\rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_8$,
- $\langle x, y, z : y^2 = x^4 = z^5 = 1, y^x = y^{-1}, y^z = y^{-1}, x^z = xz\rangle \cong \mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$.

Each of the groups $\mathbb{Z}_5 \rtimes \mathbb{Z}_8$, and $\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$, has 5 abelian subgroups of order 8, namely the sylow 2-subgroups and intersection of any two is the center. Thus the commuting graphs of these groups are not triple-toroidal.

Groups of order 48 with $|Z(G)| = 2$ and has no abelian subgroup of order greater than or equal to 12 are

- $\langle x, y, z : y^3 = z^4 = 1, x^2 = z^2, y^x = y^{-1}, y^{-1}zy^{-1}z^{-1}y^{-1}z = xz^{-1}xy^{-1}zy = 1\rangle \cong SL(2,3) \circ \mathbb{Z}_2$,
- $GL(2,3)$,
- $\langle x, y, z : x^2 = y^3 = z^4 = (xz^2)^2 = 1, y^z = y^{-1}, (xx^y)^z = x^{-1}\rangle \cong A_4 \rtimes \mathbb{Z}_4$,
- $\mathbb{Z}_2 \times S_4$.

The groups $SL(2,3) \circ \mathbb{Z}_2$ and $GL(2,3)$ are AC-groups. Each of these groups has 3 centralizers of size 8 and the rest are of size less than or equal to 6. Thus by [8, Proposition 3.4], $\gamma(\Gamma_c(GL(2,3))) = \gamma(\Gamma_c(SL(2,3)\circ\mathbb{Z}_2)) = 3$, that is, $\Gamma_c(GL(2,3))$ and $\Gamma_c(SL(2,3)\circ\mathbb{Z}_2)$ are triple-toroidal.

The group $A_4 \rtimes \mathbb{Z}_4$ has four abelian subgroups of order 8, say A, B, C, D, such that $A \cap B = A \cap C = A \cap D = B \cap C = B \cap D = Z(G)$ and $|C \cap D| = 4$. Suppose $(C \cap D) \setminus Z(G) = \{u, v\}$. Then $\gamma(\Gamma_c(G)) \geq \gamma(\Gamma_c(G)[A \setminus Z(G)]) + \gamma(\Gamma_c(G)[B \setminus Z(G)]) + \gamma(\Gamma_c(G)[(C \setminus (Z(G)) \cup \{u\}]) + \gamma(\Gamma_c(G)[(D \setminus (Z(G)) \cup \{v\})]) = 1+1+1+1 = 4$. Thus the the commuting graph of $A_4 \rtimes \mathbb{Z}_4$ is not triple-toroidal.

For the group $G = \mathbb{Z}_2 \times S_4$, let $A = \mathbb{Z}_2 \times \langle(1,4,2,3)\rangle$, $B = \mathbb{Z}_2 \times \langle(1,3,4,2)\rangle$, $C = \mathbb{Z}_2 \times \langle(1,3),(2,4)\rangle$ and $D = \mathbb{Z}_2 \times \langle(1,2,3,4)\rangle$. Then $A \cap B = A \cap C = A \cap D = B \cap C = B \cap D = Z(G) = \mathbb{Z}_2 \times \{()\}$ and $C \cap D = \mathbb{Z}_2 \times \langle(1,3)(2,4)\rangle$. Let $H = \mathbb{Z}_2 \times \langle(1,3)(2,4)\rangle$. Thus $\gamma(\Gamma_c(G)) \geq \gamma(\Gamma_c(G)[A \setminus Z(G)]) + \gamma(\Gamma_c(G)[B \setminus Z(G)]) + \gamma(\Gamma_c(G)[(C \setminus H) \cup \{(1,(1,2)(3,4))\}]) + \gamma(\Gamma_c(G)[(D \setminus H) \cup \{(x,(1,2)(3,4))\}]) = 1+1+1+1 = 4$, where $\mathbb{Z}_2 = \langle x\rangle$. Thus the the commuting graph of $A_4 \rtimes \mathbb{Z}_4$ is not triple-toroidal.

Let G be a group of order 80. Let P_1 and P_2 be two sylow 5-subgroups of G. Then $2K_8$ is a subgraph of $\Gamma_c(G)[\langle P_1, P_2, Z(G)\rangle \setminus Z(G)]$ and so $2K_8$ is a subgraph of $\Gamma_c(G)$, which is a contradiction. Thus the sylow 5-subgroup of G is normal in G. Let $P = \langle x\rangle$ be the sylow 5-subgroup of G. Thus $|Cl_G(x)| = 4$. Now since $|C_G(x)||Cl_G(x)| = 80$, we have $|C_G(x)| = 20$. Note that $Z(G) \subset C_G(x)$. Thus $|Z(C_G(x))| \geq 10$. But $|Z(C_G(x))| = 10$ is not possible; otherwise $C_G(x)/Z(C_G(x))$ is cyclic and hence $C_G(x)$ is abelian. Therefore $|Z(C_G(x))| = 20$, that is $C_G(x)$ is abelian and so G has an abelian subgroup of order 20. Thus $\Gamma_c(G)$ is not triple-toroidal.

Solvable groups of order 60 and 120 has a Hall subgroup of order 15, which is abelian. There is no non-solvable group of order 60 with $|Z(G)| = 2$. Non-solvable groups of order 120 with $|Z(G)| = 2$ are $SL(2,5)$ and $\mathbb{Z}_2 \times A_5$. Each of these groups has 6 abelian subgroups of order 10 and the intersection of any two of these subgroups is the center. Thus the commuting graphs of $SL(2,5)$ and $\mathbb{Z}_2 \times A_5$ are not triple-toroidal.

Solvable groups of order 240 has a Hall subgroup of order 15, which is abelian. There are 8 non-solvable groups of order 240, but all these groups has an abelian subgroups of order 12. Therefore, there are no commuting graphs of groups of order 240 which are triple-toroidal.

(8) Suppose $|Z(G)| = 1$. By Lemma 3.4, we have $|G| = 2^i 3^j 5^k 7^l$, where $i \leq 4$, $j \leq 2$, $k \leq 1$ and $l \leq 1$. Thus $|G| \mid 2^4.3^2.5.7$.

If $7 \mid |G|$, then by Lemma 3.5, we have $|G| \leq 42$. Thus $|G| = 14, 21, 28, 42$. Up to isomorphism, groups of order 14 and 21 are D_{14} and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, respectively. Both the commuting graphs of these groups are toroidal. Thus it follows that $|G| = 28, 42$. There are no group of order 28 with trivial center. Group of order 42 with trivial center are $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \langle x^2 = y^3 = z^7 = 1, (xz)^2 = 1, xyx = y, zy = yz^2 \rangle$ and D_{42}. The group $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ has 7 abelian subgroups of size 6 and one abelian subgroup of size 7 and the intersection of these subgroups is the trivial subgroup. Thus the commuting graph of $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ is not triple-toroidal. The dihedral group D_{42} has an abelian subgroup of order 21. Thus its commuting graph is not triple-toroidal.

Suppose $9 \mid |G|$. Then $7 \nmid |G|$. Let n_3 be the number of sylow 3-subgroup of G. Then, $n_3 \equiv 1 \bmod 3$ and $n_3 \mid 2^4.5$. Thus $n_3 = 1$ or $n_3 \geq 4$. Suppose $n_3 \geq 4$. Let P_1, P_2, P_3 be sylow 3-subgroups of G. Let $Q_1 = P_1 \setminus \{e\}$, then $\gamma(\Gamma_c(G)[Q_1]) = 2$. Note that $|P_1 \cap P_i| \leq 3$, for $i = 2, 3$. Let $Q_i = P_i \setminus P_1$, for $i = 2, 3$. Then $|Q_2|, |Q_3| \geq 6$. Also $|P_2 \cap P_3| \leq 3$ and so, since $1 \in P_2 \cap P_3$ and $1 \notin Q_2, Q_3$, we have $|Q_2 \cap Q_3| \leq 2$. If $Q_2 \cap Q_3 = \emptyset$, then $\Gamma_c(G)[Q_i] \cong K_6, i = 2, 3$ and so $\gamma(\Gamma_c(G)) \geq \gamma(\Gamma_c(G)[Q_1]) + \gamma(\Gamma_c(G)[Q_2]) + \gamma(\Gamma_c(G)[Q_3]) = 4$, a contradiction. So, suppose $|Q_2 \cap Q_3| \geq 1$. Let $y \in Q_2 \cap Q_3$. Let $\bar{Q_2} = Q_2 \setminus \{y\}$ and $\bar{Q_3} = (Q_3 \setminus (Q_2 \cap Q_3)) \cup \{y\}$. Then $\Gamma_c(G)[\bar{(Q_i)}] \cong K_5, i = 2, 3$. Therefore $\gamma(\Gamma_c(G)) \geq \gamma(\Gamma_c(G)[Q_1]) + \gamma(\Gamma_c(G)[\bar{Q_2}]) + \gamma(\Gamma_c(G)[\bar{Q_3}]) = 4$, a contradiction. Hence, the sylow 3-subgroup of G is normal in G. Let P be the sylow 3-subgroup of G. Clearly P is solvable. Thus $|G/P| = 2^i 5^j$, and so, by Burnside's theorem, G/P is solvable. Thus if $5 \mid |G|$, then G has a Hall subgroup of order 45, and groups of order 45 are abelian, which is a contradiction. Therefore $5 \nmid |G|$ and so $|G| \in \{18, 36, 72, 144\}$.

There are two groups of order 18 with trivial center, namely, D_{18} and

$$\langle x, y, z : x^3 = y^3 = z^2 = [x, y] = 1, x^z = x^{-1}, y^z = y^{-1} \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2.$$

By Theorem 3.6, the commuting graphs of these groups are double-toroidal.

There are two groups of order 36 with trivial center, up to isomorphism, namely $S_3 \times S_3$ and

$$\langle x, y : x^4 = y^3 = (yx^2)^2 = [x^{-1}yx, y] = 1 \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4.$$

By Remark 3.3, genus of the commuting graph of $S_3 \times S_3$ is greater than or equal to 4. By Theorem 3.6, the commuting graph of the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ is double-toroidal.

There are 6 non-abelian groups with trivial center of order 72, up to isomorphism. They are

- $\langle x, y, z : x^2 = y^2 = z^9 = (xz)^2 = (z^{-1}yx)^2 = 1, y^z = (yx)^2, y^{z^3} = y^{-1} \rangle \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_2$,
- $\langle x, y : x^3 = y^8 = (y^{-1}x)^2 y^2 x^{-1} = (y^4 x)^2 = 1 \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$,
- $\langle x^2 = y^2 = z^3 = (xz)^2 = (yx)^4 = (yz^{-1})^2 (yz)^2 = (z^{-1}(yx)^2)^2 = 1 \rangle \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2$,
- $\langle x, y, z : x^3 = y^4 = z^4 = (x^{-1}y^2)^2 = (z^2 x)^2 = yxyzx^{-1}zx = 1, y^z = y^{-1} \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$,
- $\langle x, y, z, u : x^2 = y^2 = z^3 = u^3 = (xu)^2 = (xz)^2 = (yz)^3 = (xyz)^2 = 1, uz = zu, yu = uy \rangle \cong (\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$,
- $A_4 \times S_3$.

By Theorem 3.6,, the commuting graph of the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$ is double-toroidal.

The groups $((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_2, (\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ and $A_4 \times S_3$ has an abelian subgroup of order 12. Thus the commuting graph of these groups are not triple-toroidal. The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$ has 9 abelian subgroups of order 8 and one of order 9. The intersection of any two of these subgroups is trivial. Thus $K_8 + 2K_5$ is a subgraph of $\Gamma_c((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8)$, showing that the commuting graph of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$ is not triple-toroidal. The group $(S_3 \times S_3) \rtimes \mathbb{Z}_2$ has $S_3 \times S_3$ as a subgroup. By Remark 3.3, the genus of the commuting graph of $S_3 \times S_3$ is at least 4. Note that $\Gamma_c(S_3 \times S_3)$ is a subgraph of $\Gamma_c((S_3 \times S_3) \rtimes \mathbb{Z}_2)$. Thus the genus of the commuting graph of $(S_3 \times S_3) \rtimes \mathbb{Z}_2$ is atleast 4. Hence $\Gamma_c((S_3 \times S_3) \rtimes \mathbb{Z}_2)$ is not triple-toroidal.

There are 3 nonabelian groups of order 144 with trivial center. These are:-

- $\langle x, y, z : y^2 = z^3 = (yz)^2 = 1, x^y = x^3, xzxz^{-1}x^{-2}z = xz^{-1}xyxzxy = 1\rangle \cong ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_2$,
- $S_3 \times S_4$,
- $A_4 \times A_4$.

The group $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_2$ has 9 abelian subgroups of order 8 and one of order 9. The intersection of any two of these subgroups is trivial. The groups $S_3 \times S_4$ and $A_4 \times A_4$ has an abelian subgroup of order 12. Thus, the commuting graph of these groups are not triple-toroidal.

Suppose $|G| = 2^i.3.5$. Then $|G| \in \{30, 60, 120, 240\}$. Group of order 30 are solvable, and hence has a Hall subgroup of order 15, which is abelian. Solvable group of order 60 has a Hall subgroup of order 15, which is abelian. Non-solvable group of order 60 is A_5. But the commuting graph of A_5 is planar. Solvable group of order 120 has a Hall subgroup of order 15 which is abelian. There is only one non-solvable group with trivial center of order 120, namely S_5. It has 10 abelian subgroups of order 6 and the intersection of any two of these subgroups is trivial. Thus the commuting graph of S_5 is not triple-toroidal.

Suppose $|G| = 2^i.5$, that is $|G| \in \{10, 20, 40, 80\}$. There is only one non-abelian group of order 10 upto isomorphism, namely D_{10} and its commuting graph is planar. There is only one non-abelian group with trivial center of order 20, namely, $Sz(2)$ and its commuting graph is planar. There is no non-abelian group of order 40 with trivial center. There is only one non-abelian group of order 80 with trivial center, namely,

$$\langle x, y : x^2 = y^5 = (xy^{-1}xy)^2 = (xy^{-1})^5 = (xy^{-2}xy^2)^2 = 1\rangle$$
$$\cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_5.$$

This group has an abelian subgroup of order 16. Therefore, its commuting graph is not triple-toroidal. This completes the proof. □

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