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Preservice Middle School Mathematics Teachers' Development of Flexibility and Strategy Use by Geometric Thinking in Dynamic Geometry Environments

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The current study, aimed to examine the development of flexibility and strategy use through instructional sequence encouraging the preservice middle school mathematics teachers' geometric thinking and problem-solving performance on dynamic geometry environment. Mixed-method research design was used in the study. The study was conducted with 46 preservice middle school mathematics teachers selected by criterion sampling strategy. The data were collected through the tests including open-ended and multiple-choice questions. In the quantitative part of the study, it was attempted to identify whether the tasks affect problem solving skills and geometric thinking. The said individuals participated in six-week instructional sequence designed by the tasks of geometric constructions performed by GeoGebra. For the qualitative part, document analysis technique was used in order to illustrate the ways of this effect in detail and illustrate the development of flexibility and strategy use. It was observed that tasks by DGE improved the preservice middle school mathematics teachers' scores of problem-solving and geometric thinking. Also, they solved the problems by representing the properties of higher thinking levels than the levels they had before participating in the instructional sequence enacted by GeoGebra. It is believed that this could encourage the improvement of their flexibility and strategy use.

Introduction

The learners can improve their problem-solving skills as one of the most critical cognitive activity used in real life and professional experiences (Jonassen, 2000). Problem solvers are expected to make modifications their thoughts and behaviors in different contexts. Hence, they should have the ability of working in a flexible way since the more flexible they are, the more successfully they perform their problem-solving skills and the more creative they are in solving process (Demetriou, 2004; Elia, et al., 2009). Problem solving skills can be improved through instructions providing opportunities of comparing and making connection between different representations and mathematical ideas; their previous knowledge and new

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concept through their thinking and examining others' reasoning (Stein & Lane, 1996) in a flexible way based on using strategies flexibly in changing situations. Therefore, it is important to provide instructional sequence encouraging the use of tools for solving problems having multiple solutions or multiple ways to solution (Leikin & Levav-Waynberg, 2009; Sullivan et al., 2006; Zaslavsky & Sullivan, 2011). To line with this view, this kind of instructional sequence can be provided using dynamic geometry software (DGS). DGS proposes opportunities such as to eliminate drawing errors through geometric constructions that can cause misconceptions and to transform static figures of Euclidean geometry into dynamic figures by dragging the segments or points in case of preserving the critical properties defining the figures (Uygun, 2020; Scher, 2000). In this respect, DGS makes "mathematics into a laboratory science rather than [a] game of mental gymnastics, dominated by computation and symbolic manipulation'' so that "mathematics becomes an investigation of interesting phenomena, and the role of the mathematics student becomes that of the scientist" (Olive, 2002, p. 17 as cited in Gawlick, 2004, p. 1). Hence, with the help of DGS, opportunities for learners "to explore, propose, make conjectures and try to demonstrate" (Engström, 2004, p. 4) can be provided. To line with this view, DGS can help learners propose new solution strategies or ways for familiar problems, analyze and explore the properties of figures in detail and hierarchical relationship by dragging, and construct the figures in different ways and explore their justification effectively (Ruthven et al., 2008). In this respect, it can be stated that DGS can enhance posing different strategies to solve problems and using them in a flexible way and encourage the learners to adapt on changing or new situations. The research in the literature shows that use of problem solving strategies encourage their success and learning (Cai, 2003). In this respect, they do not only need to know and use problem solving strategies but also to be flexible to use these strategies in new situations and problems that they face with in mathematics lessons and real life (Baroody, 2003).

Given these explanations show that making further research about the effects of using DGS to develop preservice teachers' flexibility in problem solving and strategy use is necessitated. Also, this necessity exists in geometry in particular because students cannot improve their understanding, thinking and problem-solving skills adequately (Elchuck, 1992; Idris, 1999/2009). Moreover, they are expected to make connection between figures and their properties by geometric thinking. Hence, they illustrate inadequate performance in geometry lessons. Previous research explains the reasons of this case by ineffective instruction, geometric thinking level and inabilities about geometric content (Cangelosi, 1996; Idris, 2009). So, the effects of tasks supported by DGS used in instructional sequence encouraging geometric thinking and problem-solving performance are value to be examined by the process illustrating the preservice middle school mathematics teachers' (PMSMT) development of flexibility and strategy use. Also, kinds of problem solution strategies emerged in the process of engagement in these tasks supported by DGS and how the flexibility and strategy use in problem solving emerged in this instructional sequence was provided were examined in order to report the effects in detail. In other words, this development process was examined in connection with the improvement on geometric thinking and problem-solving performance in order to report the effects in detail effectively. Therefore, the purpose of the present study was to explore the answers of the following research questions:

(1) Is there significant difference between pretest and posttest scores of the PMSMT's van Hiele geometric thinking?
(2) Is there significant difference between pretest and posttest scores of the PMSMT's problem solving performance?

(3) How the PMSMT apply the strategies to solve geometry problems benefiting from flexibility before and after instructional sequence?

Theoretical Framework

DGE proposes the opportunities of making drawings that can be moved and forming geometrical solutions easily (Strasser, 2002). There have been research examining the effects of DGE on the development of geometric thinking with the help of these opportunities (De Villiers, 2004). Moreover, its contribution on improvement of problem-solving skills and formation of strategies has been examined in the literature (Fey et al., 2010; Olive & Makar, 2010; Wilson et al., 1993). With the help of the functions of the DGS, the problems can be analyzed effectively, the solution plan can be explored, and solution strategies can be formed (Hölzl, 2001; Wilson et al., 1993). To line with this view, the individuals can produce different strategies by exploring and reasoning about the context and use them in alternative situations. Hence, it can be stated that DGE enhances strategy use and flexibility.

The development of problem-solving skills have become one of the major goals of the mathematics education. Stanic and Kilparick (1989) emphasize this goal by stating "the term problem solving has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is, and why we should teach mathematics in general and problem solving in particular" (p. 1). In this respect, there have been many research examining the effects of ways and strategies for the development of problem-solving skills focusing on heuristic processes (Polya, 1957), inductive and deductive thinking styles (Lakatos, 1976), decision-making strategies (Schoenfeld, 2010), modelling activities (Lesh & Zawojewski, 2007), and mathematical practices (Uygun & Akyuz, 2019). The strategies used in solving problems can be used to make inferences about the individuals' mathematical thinking, understanding about the concept and performance on problem solving (Schoenfeld, 1992). The individuals are expected to use heuristics strategies such as working backwards, substituting numbers, subdividing problem, looking for a counter example in solving problems by making decisions based on their knowledge about concept and problem-solving strategies (Carlson & Bloom, 2005). However, even though many previous studies show that individuals can perform problem solving strategies [e.g. guess-check-revise, make a table, make an organized list, look for a pattern, use logical reasoning], they cannot successfully use heuristic strategies in solving unfamiliar problems (De Bock et al., , 1998; Schoenfeld, 1992).
Problem solving as an inquiry-based learning process including using and adopting existing knowledge into new unfamiliar cases in order to acquire and construct new knowledge (Hammouri, 2003) is useful in learning geometrical concepts and geometric thinking (Uygun & Akyuz, 2019). Moreover, it can be stated that the problem-solving ability is closely related to the quality of learning and understanding concepts (Sudarsono et al., 2021) through learning process in which the relationships between teachers, students, curriculum, learning demands, materials, tools and facilities emerge systemically and synergistically (MEB, 2018). At this point, Van Hiele theory (Hiele, 1958) explaining the learners' geometric thinking processes through five hierarchical levels can propose a beneficial way in order to overview of the preservice mathematics teachers' thought patterns in problem solving processes as well as in learning and geometrical concepts. The first level of van Hiele geometric thinking level is Level 0, namely visualization, refers to learners' examination of geometrical shapes and problems based on their visualizations. Level I, namely analysis, refers to the ability of analysing geometric concepts, rules, and their properties. Level II, namely informal deduction, refers to the ability of making and discovering connections among geometrical concepts, their definitions and properties benefiting from informal explanations and arguments. Level III,

namely deduction, refers to the ability of using and understanding postulates, axioms, and theorems and producing formal arguments and using mathematical symbols in deductive proof. Level IV, namely rigor, refers to the ability of studying in various geometry systems by analysing and comparing them (Mason, 1998). Moreover, the previous research (Hendroanto et al., 2019; Rizki et al., 2018; Yudianto et al., 2018) have examined and emphasized the connection among Van Hiele geometric thinking levels and problem-solving ability including using and producing problem solving strategies. The research also insist on the necessity of further research examining this connection in the context of preservice mathematics teachers' development of problem solving skills and geometric thinking levels in different contexts. In the light of these explanations, in the current study, it was focused on the development of PMSMT's development of problem solving and geometric thinking. Moreover, this study paid attention to these issues in the contexts of flexibility and strategy use in problem solving processes considering their geometric thinking levels by differentiating from the previous research.

When the learners face with an unfamiliar problem, they have cognitive conflict since the situation does not fit their existing knowledge. Then, they start working on it. Through this studying process, they try to make some modifications on their existing knowledge and thinking by learning additional ones. "…they confirm or redefine their conceptual knowledge, relearn mathematics content and become more open to alternative ways of learning mathematics" (Steele & Widman, 1997, 190) since problem solving is not remembering the memorized facts or using and following well-learned operations or procedures. At that point, flexibility can be useful since it is an important aspect of individuals' mathematical proficiency (National Research Council, 2001) describing to switch between alternative strategies and methods and to select the most appropriate strategies among them. Also, the efficiency of the selected strategy and ease of switching among the strategies and to adapt the strategy to the context can be paid attention on (Star, 2005; Star & Rittle-Johnson, 2008; Verschaffel et al., 2007). In mathematics education, the students are expected to solve mathematics problems accurately, quickly, and adaptively (Heinze et al., 2009). At that point, adaptively solving problems emphasizes to solve problems and perform tasks flexibly by generating meaningful ways and strategies considering the characteristics of the concept, tasks, and context (Baroody & Dowker, 2003; Verschaffel et al., 2007). To line with this view, Verschaffel, Luwel, Torbeyns and Van Dooren (2009) explained the flexibility by connecting with selecting strategies adaptively as "the conscious or unconscious selection and use of the most appropriate solution strategy on a given mathematical tem or problem, for a given individual, in a given context" (p.343).

Blöte and colleagues (2001) state that flexibility occurs through a process beginning by acquiring knowledge about multiple strategies and ending by choosing appropriate strategy based on particular preferences based on context. In this respect, it can be claimed that flexibility can be improved with the help of instructional interventions in relation to development of conceptual knowledge and understanding of the domain (Blöte et al., 2001; Rittle-Johnson & Star, 2007). Hence, it can be stated that mathematics teachers and preservice mathematics teachers as experts expected to have deep and significant knowledge in mathematics classrooms should acquire flexibility in using strategy for mathematical tasks and problems. With this motivation, it was aimed to examine the development of the PMSMT's flexibility and strategy use on solving problems through the instructional intervention supported by DGS in the context of triangles as a geometrical concept. In previous research, the development of flexibility for different grade level of students has been examined in the contexts such as computation, algebra, general problem solving (Elia et al., 2009) but this study by focusing on geometry and examining flexibility considering the development of geometrical thinking through instructional sequence supported by DGS differentiates from the previous research. The instructional sequence was performed with the help of GeoGebra because of its properties such as dragging and providing flexible motions and actions. In the previous studies, GeoGebra is used as a beneficial tool by constructing geometric shapes and making modifications and transformations on the shapes and relating them. With this motivation, in the current study, GeoGebra was used in order to construct different types of triangles, their elements and roles and properties of these elements and the examination of the possibility of the construction of the triangles based on some known and unknown elements. In that respect, through engagement in the tasks through the instructional sequence, the learners can examine the solution of the problems and form their own strategies and use them in different contexts adaptively by improving their problem-solving skills, flexibility and geometric thinking. Moreover, it was observed that most of the studies in this field have predominantly focused on the identification of the connection among problem solving skills and geometric thinking levels. Differently, this identification of this connection was examined based on their development of learning process encouraged by hypothetical learning trajectory designed by GeoGebra in the context of flexibility and strategy use rather than determining problem solving achievement and geometric thinking levels.

Method

The study was designed by mixed-method research design including qualitative and quantitative research methods. Sequential explanatory strategy as a type of mixed-method research design was used. In this strategy, the study can start with collection and analysis of quantitative data and then, qualitative data can be used in order to obtain detailed and deep information about research problems (Creswell, 2003). In the current study, the quantitative data were analyzed initially. Then, the findings of this analysis process were detailed by qualitative data. In the present study, pretest and posttest were conducted to the participants before and after the six-week instructional sequence performed by GeoGebra. Quantitative data were used in order to determine if there was a statistically significant difference between the PMSMT's van Hiele geometric thinking levels and the scores of the tests related to their problem-solving process of triangles. In other words, quantitative data were used in order to examine the effects of the instructional sequence on the PMSMT's van Hiele geometric thinking and geometry problem solving skills and strategies. Then, document analysis was performed in the qualitative dimension of the study. Qualitative data were used in order to picturize the effects of instructional sequence supported by the tasks on them clearly and interpret and understand the PMSMT's improvement on flexibility and strategy use considering their geometric thinking effectively by reporting their problem-solving process.

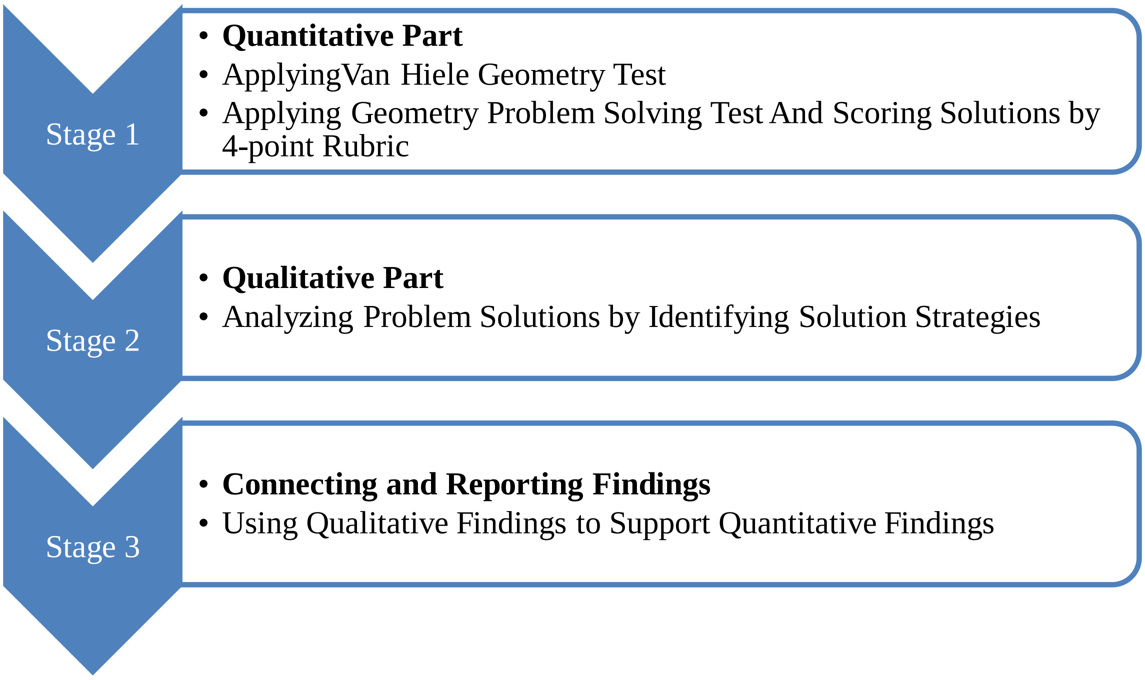


Figure 1. Mixed method research process

Participants

The participants of the current study were composed of 46 preservice middle school mathematics teachers (PMSMT) enrolled in the department of elementary mathematics education at a public university in Turkey. Of these junior participants, 28 were female and 18 were male students. They were selected by criterion sampling strategy. Because individuals' prior knowledge affects the flexibility (Rittle-Johnson et al., 2012), the criteria of being familiar with the knowledge about main geometrical figures especially triangles, their properties, and theorems about them and using GeoGebra, and taken the courses *Using Technology in Mathematics Education*, *Geometry* and *Analytic Geometry* in previous semesters. They had learned how to study on GeoGebra, to use the tools and segments on it and to know its facilities in this course. Another set of criteria for selection was the course of *Problem Solving in Mathematics Education* since they learned the problem solving strategies such as working backwards, making organized list.

Instruments

In the present study, two types of instruments as pretest and posttest were used in order to test the effect of instructional sequence on the PMSMT's geometric thinking, performance of problem solving, improvement on flexibility and strategy use. The first type of instrument used in the study was the Van Hiele Geometry Test (VHGT). VHGT as a standard multiple-choice test was developed and used in order to determine the PMSMT's geometric thinking level by Usiskin (1982). This test includes 25 multiple-choice geometry questions to be administered in 35 minutes. These questions are separated into five groups representing different geometric thinking levels. Each group of five questions represent a level from 0 to 4 respectively. The levels have on their own characteristics. In Level 0, learners focus on physical attributes of geometrical shapes as a whole. Level 1, they examine specific properties of shapes without relational understanding. Level 2, they attain relational understanding and make inferences. Level 3, they can construct and analyze proofs and statements about geometric shapes. Level 4, they can analyze formal and logical statements about mathematical systems based on abstract deductions (Usiskin, 1982). The examination of the PMSMT's geometric thinking and its reflections on flexibility and strategy use was examined by these characteristics.

The second type of instrument included the test composed of eight open-ended questions about triangles. This test was used in order to identify the problem-solving performance, and preferred and formed problem solution strategies. It was prepared based on the learning objectives of six-week instructional sequence designed and the tasks used (see Table 1). Also, the validity and reliability studies were performed by Uygun (2016). Two academics having a PhD degree in mathematics education analyzed the questions. Then, this test was conducted to three preservice mathematics teachers which were not the participants of the current study. Afterwards, necessary revisions were made on the problems based on the suggestions of experts and preservice mathematics teachers. The problems of the test were formed considering the hypothetical learning trajectory (HLT) designed based on literature of mathematics education about triangles, van Hiele geometric thinking levels, the objectives of Turkish middle school mathematics curriculum and textbooks.

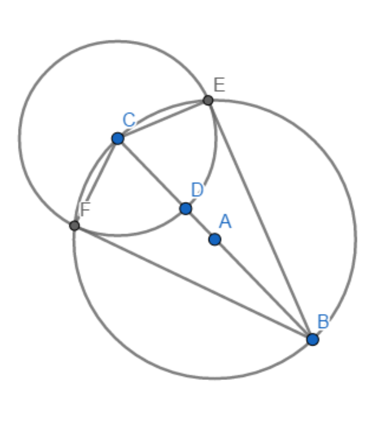
Instructional Sequence and Tasks

The instructions were enacted by the researcher based on the Hypothetical Learning Trajectory (HLT) designed about triangles by Uygun (2016). This HLT was used since this previous research has showed that it encourages the development of the PMSMT's conceptual knowledge and social learning environment. On the other hand, differentiating from the research of Uygun (2016), the tasks of the HLT were performed using GeoGebra in a computer-lab instead of compass and straight edge.

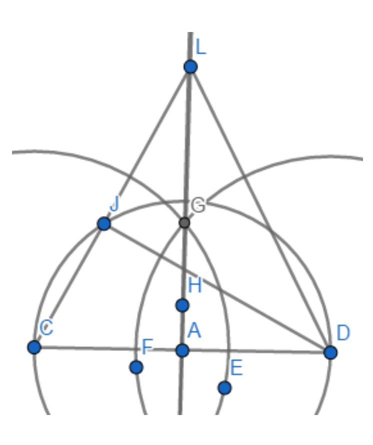
Table 1. Hypothetical Learning Trajectory with Geometric Construction Tasks for Triangles

Learning Goal	Concepts	Supporting Tasks	Tools and Imagery
Evaluating the formation of triangles	Definitions of triangles Formation of triangles	Classification of triangles Basic drawings of triangles	Diagrams GeoGebra
Reasoning on auxiliary elements of triangles	Auxiliary elements of triangles	Definitions, constructions, and properties of auxiliary elements	GeoGebra Drawings
Reasoning on congruence and similarity	Transformation geometry Congruence & Similarity	Formation of images Comparing triangles and their images	GeoGebra Dot paper

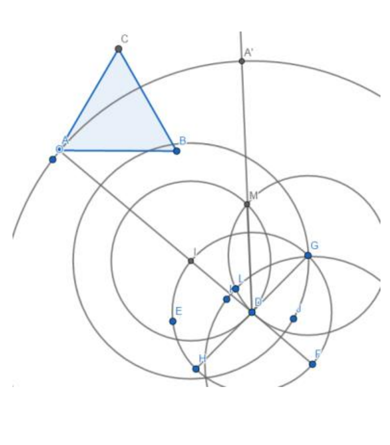
The instructional sequence performed by this HLT lasted six weeks. The lessons for each objective lasted two weeks and three hours in each week. This HLT was designed by examining literature of mathematics education about triangles, properties of van Hiele geometric thinking levels, the objectives of Turkish middle school mathematics curriculum, geometric constructions of different geometric shapes and textbooks. Through instructional sequence, the PMSMT engaged in the activity sheets and the tasks on them with their peers and then, they made discussions about their ideas on the tasks in whole class discussions under the guidance of the instructor. Hypothetical learning process can be explained through behaviors, actions, examinations, and discussions of the PMSMT while engaging in supporting tasks to reach learning goals. For example, in the first stage, the PMSMT can comprehend the definition and formation of triangles based on their elements by examining the classification and drawings of triangles. In the first objective, the PMSMT discussed about the critical attributes of different types of triangles and definitions of them. They constructed these triangles by the attributes and discussed their classification of them based on angle and edge. Then, they analyzed the possibility of construction of the triangles by knowing some of the elements. They engaged in the tasks representing different group of elements of triangles. For example, "When we know the measures of h_a and V_a and m(BAC) = 90^0 in the triangle of ABC, is it possible to draw/construct this triangle? How?" and "When we know the measures of h_a and a and m(BAC) = 90^0 in the triangle of ABC, is it possible to draw/construct this triangle? How?". They made geometric construction on GeoGebra as in Figure 2a. In the second objective of the HLT, the PMSMT discussed definitions, constructions and properties of auxiliary elements of triangles. They engaged in the tasks of construction of these elements, showing the concurrency of them by construction and their places on a plane based on a triangle. For example, "Construct a triangle and its altitude by compass and straight edge.". They made the geometric constructions on GeoGebra as in Figure 2b. In the third objective, the PMSMT engaged in the tasks of congruence and similarity of triangles using transformations by geometric constructions. They analyzed formation of images of triangles through transformations and making comparisons between triangles and their images. For example, "Construct the image of the triangle ABC by rotating with the angle measure of 45^0 and the reference point and justify this construction mathematically". They formed the construction on GeoGebra as in Figure 2c.



a. Example for Objective 1



b. Example for Objective 2



b. Example for Objective 3

Figure2. Examples for performed tasks on GeoGebra for three objectives of the HLT

Data Collection and Data Analysis

Data collection period took approximately eight weeks. In the first and eighth weeks, VHGT, pretest and posttest were conducted to the PMSMT and lasted 100 minutes (50 minutes for each test), the instructional sequence was enacted through the remaining weeks. The researcher of the current study read and scored all participants' answer sheets attained from VHGT and triangle tests. The quantitative part of the study included analysis of the scores of pre- and post-tests. Firstly, Usiskin's (1982) grading system in which each group of questions belonged to van Hiele geometric thinking levels was scored with the points of 1, 2, 4, 8, 16 respectively was used for VHGT test. This criterion was used to define the PMSMT's thinking levels. Additionally, 100 - point numerical scale developed by Gutierrez, Jaime, and Fortuny (1991) was used to show the participants' thinking levels in detail. This scale proposes that there are five qualitative scales between any two van Hiele levels. Gutierrez et al. (1991) explained that values in interval (0%, 15%) can be associated with *no acquisition* of the level, (15%, 40%) with *low acquisition*, (40%, 60%) with *intermediate acquisition*, (60%, 85%) with *high acquisition* and (85%, 100%) with *complete acquisition* in the scale. Secondly, the triangle test was awarded by four-point scale by the researchers. 0 points referring completely incorrect answers, 4 points for completely correct ones and points between the ranges of 1 to 3 for partially correct ones were used. In order to answer the initial two research problems focusing on the differentiation on scores of VHGT and triangle test applied before and after treatment, the technique of paired samples *t*-test was chosen because its assumptions of the independence of observations and the normality were provided. Based on the observed measurements for Kolmogorov-Smirnov tests, it was observed that the distribution of the pre-test and post-test scores acquired from VHGT, and triangle test was not different from normal distribution. Moreover, skewness and kurtosis values were measured between the range of 1 and -1 so that the normality assumptions were provided for the tests. Hence, a paired samples *t*-test was conducted in order to compare their pretest and posttest scores.

The qualitative part was composed of content analysis of the solutions of triangle problems in the triangle test. Six-step qualitative data analysis technique was performed, organizing the data, coding the data, generating categories and themes, testing the emergent understandings as considering individual differences, searching for alternative explanations, and writing the report. In this sequence, the researcher and an academician having PhD. In mathematics education formed a list of common codes by comparing their lists of codes, discussing and reaching consensus and all of the written documents of triangle test were coded in an iterative process based on 85% researchers' agreement. Finally, similar codes were collected under themes. In order to provide validity and reliability of the qualitative data, investigator triangulation and member checking strategies were used (Marshall & Rossman, 2006). A researcher and an academician coded the qualitative data of the current study independently by comparing their codes and themes. Also, the participants examined the results and interpretations of the analysis. Another academician having PhD. In mathematics education analyzed and evaluated the qualitative results based on consistency and coherence. These processes were performed twice. In the content analysis process, the PMSMT's problem solving processes were examined in order to identify the types of problem-solving strategies. In the analysis, four strategies were extracted; *solving by procedural operations* (making computations based on known values on the triangles), *solving by forming line segments* (drawing line segments randomly without any aim, randomly striving for a solution), *strategy of using main elements by new triangles* (drawing line segments consciously by planning to form new triangles) and *strategy of using related theorems* (forming new shapes by encouraging and explaining related theorems). Through the content analysis process, the PMSMT's problem solving processes and these solution strategies were identified and examined based on their flexibility in strategy use and emerged themes and codes were illustrated in Table 2.

Table 2. The list of the themes and codes for PMSMT's solutions of the problems

Strategies (Themes)	Explanations based on actions (Codes)
Solving by procedural operations	• Computations • Main properties of triangles and right triangles
Solving by forming line segments	• Forming auxiliary elements of triangles
Strategy of using main elements by new triangles	• Forming particular triangles such as equilateral, isosceles or congruent/similar triangles to specific ones • Using their main elements and properties
Strategy of using related theorems	• Forming particular triangles such as equilateral, isosceles or congruent/similar triangles to specific ones • Thinking about triangles formally in abstract way in detail and different perspectives • Using related theorems such as theorems of Stewart, Menelaus and Angle Bisector

Findings

Quantitative findings

Statistical procedure was made to the total scores of the participants on all of the tests. In order to analyze the difference between PMSMT's pretest and posttest scores, paired samples *t*-test was conducted because the assumptions of this test were provided. Based on Kolmogoroz-Simirnov test results showed that the PMSMT's pretest and posttest measurements for VHGT and triangle test were normally distributed (p>0,05). Furthermore, skewness and kurtosis measurements were observed between the range of 1 and -1 so that the normality assumptions were provided for the paired samples *t*-tests. There were statistically significant differences between test scores before and after conducting six-week instructional sequence as illustrated on Table 3.

Table 3. Paired Sample *t*-test Results for the PMSMT's VHGT and Triangle Test Scores

		Mean	**Std. Dev.**	***t*(45)**	***p***
VHGT	Pretest	2.29	0.46	-3.87	<0.001
	Posttest	2.71	0.46		
Triangle Test	Pretest	11.09	3.18	-16.10	<0.001
	Posttest	22.81	5.28		

There was a statistically significant increase in VHGT scores from the pre-test (M = 2.29, SD = 0.46) to the post-test (M = 2.71, SD = 0.46), $t(45)$ = -3.87, $p < .001$ in the favor of post-test. Six-week instructional sequence helped the participants improve their test scores through the instructional sequence. Initially, based on 100-point numerical scale for VHGT, their initial mean score of 2.29 represented *low acquisition* of the Level-II since this mean score was in interval of 0% - 15%. Then, at the end of the sequence, their last mean score of 2.71 meant *high acquisition* of the Level-II since this value was in the interval of 60% - 85%. Moreover, for the data of VHGT, eta squared value was calculated as approximately 0.25. Based on the guidelines [proposed by Cohen (1988)], there was a large effect with substantial difference. Instructional sequence including the tasks supported by GeoGebra could cause the difference on scores for geometric thinking to be consistently positive.

There was statistically significant increase in triangle test scores from the pre-test (M = 11.09, SD = 3.18) to the post-test (M = 22.81, SD = 5.28), $t(45)$ = -16.10, $p < .001$. These scores showed that instructional sequence designed based on tasks supported by GeoGebra improved the levels of problem-solving performance about triangles. For the data of triangle test, eta-squared value was calculated as approximately as 0.85 so there is a large effect with substantial difference. In other words, 85% of the variance in the PMSMT's triangle test scores can be explained by the effect of the treatment performed by the instructional sequence including the tasks by GeoGebra. More specifically, this treatment caused the difference on scores for problem solving performance to be consistently positive.

Qualitative findings

The qualitative data analysis was made by using written documents of the PMSMT's solutions for the problems about triangles in order to represent their development of flexibility and strategy use. How they produced solutions, solution strategies and geometric thinking were considered to analyze and represent their flexibility in the strategy use for the geometry problems in the triangle test. The number and percentage of the PMSMT's using the strategies established based on flexibility were illustrated in Table 4 to answer the third research question of the current study.

Table 4. PMSMT's Frequency Values for Problem Solving Strategies on Pre-test and Post-test

Solution Strategies	**Pre-test**	**Post-test**
	N (%)	**N(%)**
Solving by Procedural Operations	42(91)	10(22)
Solving by Forming Line Segments	15 (33)	42(91)
Strategy of using main elements by new triangles	0 (0)	30(65)
Strategy of using related theorems	0(0)	20 (43)

Solving by Procedural Operations

Although most of the PMSMT (91%) used procedural solution strategy for the problems in the pre-test, 10 of them (22%) used this strategy in some problems in the post-test. In this solution strategy, the participants understood the problem and analyzed the known and unknown parts of the problem accurately. However, they could not decide how to solve the problem, so they focused on procedural operations. They placed the known values such as the lengths of the edges or angle measures on triangles explained in problems. After forming particular triangles by explained information, they made operations on them. They did not draw new line segments and form new angles or edges except for right triangles, right angles or right edges. In this strategy, the PMSMT did not make reasoning about the process of formation of right triangles and focus on whether forming right angles or perpendicular edges were necessary to solve the problem. They formed them randomly and tried to reach the solution.

In the third problem, a PMSMT's incorrect solution by solving procedural operations can be represented as an example similar to Figure 3. He formed four right triangles and examined all interior angles' measures by new right triangles and angles. In the pre-test, he formed unnecessary triangles and focused on angle measures of these triangles focusing on 180^0 as sum of interior angle measures of triangles. In the post-test, the PMSMT using this strategy formed four right triangles and focusing on the measures of elements of triangles. They considered all elements separately and tried to find the angle measure of DAC as in Figure 3. In this strategy, the PMSMT formed right angles unnecessarily and found interior angles' measures of ABD and ADC inaccurately. Hence, they could not provide the accurate and necessary solution in this problem by this strategy.

Problem: IACI=IBDI, m(ACB)=68^0 and m(ABC)=34^0, find the measure of angle of BAD.

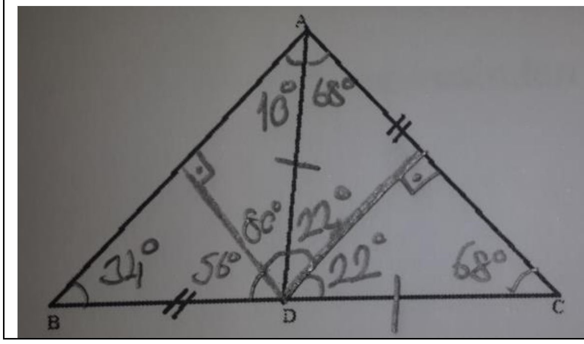


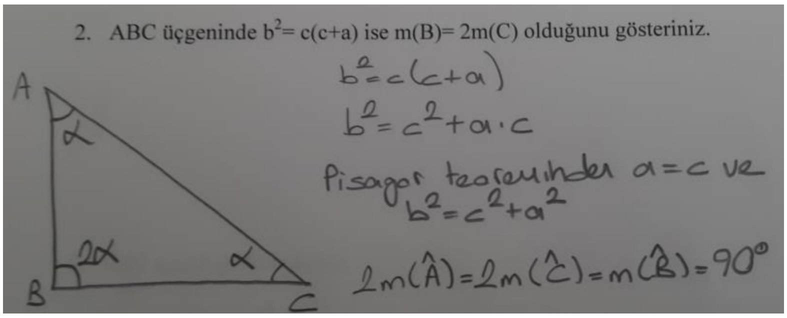
Figure 3. An incorrect solution for the third question of pretest

PMSMT focused on procedural operations by these knowledge without reasoning higher than Level-0 of van Hiele geometric thinking. Before the instructional sequence supported by the tasks on GeoGebra, the PMSMT focused on the types of triangles such as equilateral, right or isosceles and triangles and visualization of them in holistic perspective. After engaging in the tasks, the PMSMT using this strategy considered main elements of triangles. Hence, it was observed that this solution strategy was used by the PMSMT with Level-0 in pre-test and then, they represented low-level acquisition of Level-I of van Hiele geometric thinking about the concept in the post-test. The PMSMT focused on the main elements, and their properties and orientations strictly by ignoring their connection of auxiliary elements their changing orientations.

Solving by forming line segments

While 15 PMSMT (33%) solved pre-test problems by the strategy of forming line segments, 42 of them (91%) used it in the post-test. In this strategy, the participants usually focused on understanding problems and analyzing the known and unknown parts of them. They placed the known values and formed necessary auxiliary elements appropriately. By using the properties and related theorems of needed auxiliary elements, the participants formed solutions. The PMSMT used properties of triangles required for problems, but they could not produce accurate solutions. They were not able to make connection between these properties. A few PMSMT were able to form correct solution for problems by using this strategy.

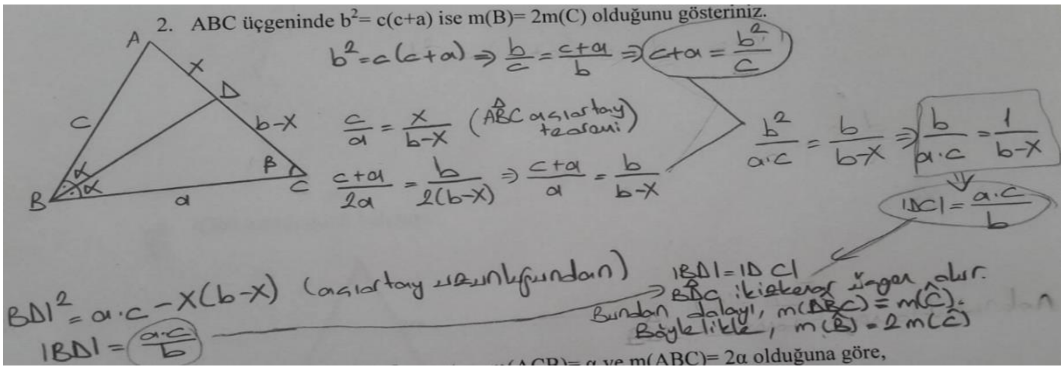
For example, in the pre-test, a PMSMT solved the second problem stating "Show that m(B) = 2m(C) when b^2 = c(c+a) in a triangle of ABC" by the strategy of procedural operations as in Figure 4. They focused on whole triangle and its main elements. Based on the lengths of edges, they made connection between Pythagorean Theorem and the expression of b^2 = c(c+a). Then, they formed an isosceles right triangle and showed m(B)=2m(C) insufficiently.



2. Based on the fact $b^2 = c(c+a)$ in the triangle of ABC, show that m(B) = 2m(C).

Figure 4. A correct solution by strategy of procedural operations

In the post-test, the same participant solved this problem by the strategy of forming line segments as in Figure 5. She formed the angle bisector of the angle of B since m(B) = 2m(C). Then, she focused on the relationship between the lengths of the edges of triangle and the parts of the edge of IACI. Also, she found the connection between the length of the angle bisector in Figure 4 and the length of the edges of triangles. She identified IBDI = IDCI and explained that the triangle of BDC was an isosceles triangle. Hence, she appropriately showed α = β and m(B) = 2m(C). By doing so, she could accurately solve the problem by using the strategy of forming line segments. The PMSMT using this strategy for this problem performed similar actions.



$IBDI^2$=a.c – x(b-x) (because of the length of angle bisector)
IBDI=IDCI, The triangle of BDC becomes isosceles triangle so that m(DBC) = m(C). Therefore, m(B)=2m(C)

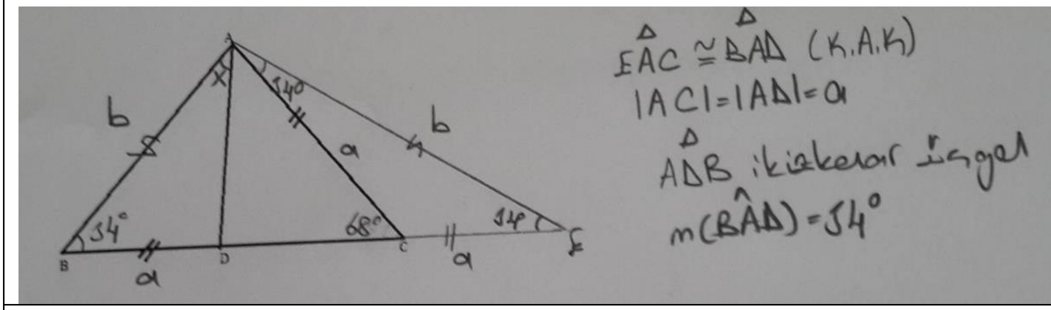
Figure 5. A correct solution by strategy of forming line segments

It was observed that the PMSMT tended to consider the roles and presence of auxiliary elements of triangles after engaging in the tasks through instructional sequence. They focused on the formation of them, their properties, and places by making connection between them so that they moved away from a more rigid point of view. Moreover, they began to think based on abstract definitions and explanations about the shapes. Hence, it can be explained that they used the high-level of acquisition of Level-I and low-level of acquisition of Level-II of van Hiele geometric thinking.

Strategy of using main elements by new triangles

The PMSMT did not use the strategy of using main elements by new triangles to solve problems in the pre-test but 30 of them (65%) used it in different problems in the post-test. They could clearly analyze the problem and form triangles by using the explained information in problems. By this strategy, the PMSMT formed new triangles adding line segments to produce particular triangles such as equilateral, isosceles or congruent/similar. After forming these new triangles, they focused on measures of main or auxiliary elements of them.

In the pre-test, the PMSMT solved the third problem by the strategy of using procedural operations as illustrated in Figure 3. In the post-test, some of the PMSMT solving this problem by the strategy of using main elements by new triangles formed a new exterior triangle of ACE as in Figure 6. Then, they focused on the angle measures and the lengths of the edges of the triangles as in Figure 5. Afterwards, they found that this new triangle was an isosceles triangle and congruent to the triangle of BAD. They could focus on and examine the particular triangles and their particuler properties by switching between them easily. By doing so, they solved the problem appropriately.



EAC and BAD triangles are similar triangles (Side-Angle-Side)
The triangle of ADB is isosceles triangle

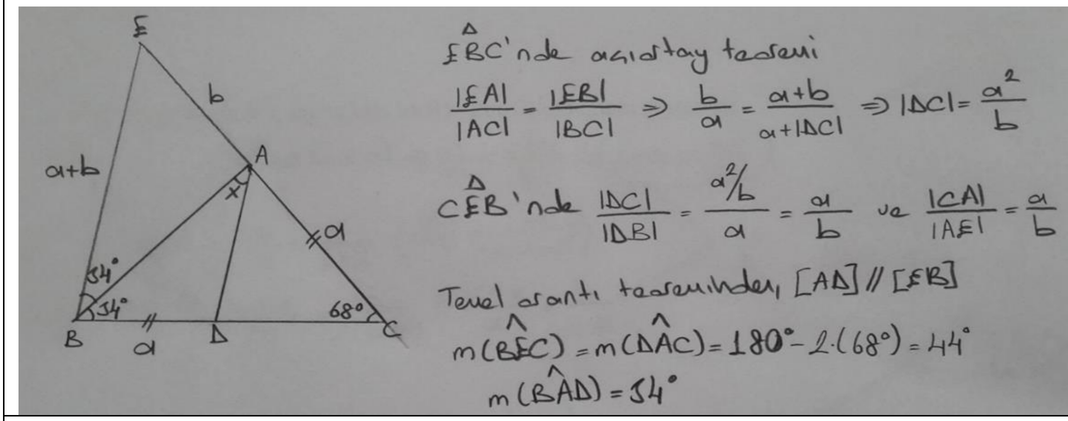
Figure 6. A correct solution the strategy of using main elements by new triangles

The PMSMT using this strategy of using main elements by new triangles in different problems could successfully make connections between elements and properties of triangles required for problems. Also, when the VHGT scores of them were considered, it was observed that they could acquire the properties of initial three van Hiele geometric thinking levels after engaging in the tasks on GeoGebra. Moreover, it can be explained that the PMSMT using this strategy could attain the properties of high-level acquisition of Level-II and low-level acquisition of van Hiele geometric thinking Level-III.

Strategy of using related theorems

The PMSMT solving problems by the strategy of using related theorems usually analyzed problems, known and unknown parts of them by understanding. They formed new triangles by drawing interior or exterior triangles produced based on some criteria facilitating appropriate solution processes by known and unknown parts of problems. They could easily form and use these triangles and related theorems by switching between them in no rigid way. Hence, these new triangles were particular triangles such as equilateral or isosceles triangles. Also, they were congruent or similar to triangles formed by known parts of problems. After forming these special new triangles, they used related theorems in order to solve problems.

In the pre-test, the PMSMT solved the third problem by the strategy of procedural operations as illustrated in Figure 3. The strategy of using related theorems can be exemplified by some PMSMT's solution for the third problem of the post-test as in Figure 7. Through the solution process by this strategy, they formed new exterior triangle of EBA. By this way, the PMSMT drew an isosceles triangle of EBC and made [AB] the angle bisector of the angle of EBC as in Figure 7. By using the angle bisector theorem, they found the length of [DC] based on the values of a and b. Then, they realized [AD] // [EB] on the new triangle of EBC. Hence, they solved the problem by following accurate solution process.



Angle Bisector Theorem on the triangle of EBC
Because of Basic Proportionality Theorem, [AD] // [EB]

Figure 7. A correct solution by the strategy of using related theorems

The PMSMT using the strategy of using related theorems were most likely to use the related theorems in the solution process when it was appropriate after taking the instructional sequence including the tasks of geometric constructions. They solved the problems by verifying the solutions, analyzing, and using the formal geometric statements. When the VHGT and the triangle tests scores of them used this strategy were examined, it was observed that these PMSMT got the highest scores. It can also be said that the PMSMT using this strategy represented the properties of Level-III of van Hiele geometric thinking.

Discussion and Conclusion

In the light of the findings, it was observed that the instructional sequence performed based on the designed HLT supported by GeoGebra (see Table 1) could enhance the improvement of the PMSMT's geometric thinking. This result can be confirmed by the previous research emphasizing the development of geometric thinking and problem solving with the help of the tasks (Ansah et al., 2022; De Villiers, 2003; Elchuck, 1992; Hayati & Ulya, 2022; Meng & Idris, 2012; Idris, 2009). Hence, it can be explained that they could acquire the properties of initial three van Hiele geometric thinking levels as the suggested level for their profession (Mayberry, 1983) through the instructional sequence performed based on the designed HLT in Table 1. At that point, the tasks supported by GeoGebra could be beneficial to be used in teaching geometry concepts effectively so that their thinking could be improved. Based on the pretest scores, it can be claimed that the PMSMT could not successfully form and use heuristic strategies based on geometrical thinking. In other words, the PMSMT could use geometric thinking at lower levels; however, in the posttest, they could improve their geometric thinking so that they could form new strategies by making connection between the elements and properties of different types of triangles. They tended to think about the triangles by transforming them into new triangles or forming new triangles using the prior triangle rather than static triangle given on the problem. In other words, differentiation on problem solving strategies happened and they could adapt and switch on the different triangles by considering their connection based on their elements and properties. This can be accepted as a signal for the development of geometric thinking because they began to use and represent the characteristics of higher level of geometric thinking. To line with this view, this development and the actions observed through the tasks in the instructional sequence by relating different triangles and their properties flexibly could enhance the formation of new strategies for solving problems on the posttest. Hence, it can be claimed that the development of flexibility and strategy use can be affected by geometric thinking so the HLT including the tasks enhancing geometric thinking could encourage the development of flexibility and strategy use.

In the findings of the present study, it was observed that the PMSMT improved their problem-solving performance through the instructional sequence performed based on the designed HLT including the tasks based on GeoGebra. In the pretest, it was observed that they could not use their geometrical knowledge and thinking to solve the problem. This finding is parallel to the previous research (De Bock et al., 1998; Schoenfeld, 1992). In the pretest, it was observed that the PMSMT did not tend to think about the elements and properties based on their connection. Therefore, they generally solved the problems focusing on procedural operations and adding some main or auxiliary elements without considering their connection or hierarchical relationship. They also failed to form and use alternative solution strategies. This finding representing the PMSMT's inflexible actions can be confirmed by the study of Muir and Beswick (2005). The tasks in the HLT performed through the instructional sequence could help them analyze the formation of triangles, their elements, related theorems and properties by performing the tasks. The HLT could affect their solutions and strategies since they solved the problems focusing on formation of new triangles, elements and properties in the posttest even though they did not use in the pretest. Hence, it can be stated that the designed tasks on DGE included by the HLT represented in Table 1 could enhance the PMSMT's forming new strategies for solving geometry problems (Fey et al., 2010; Hölzl, 2001; Olive et al., 2010; Wilson et al., 1993). In other words, in the present study, it was observed that the HLT supported by effective activities performed using GeoGebra encouraged problem solving since the PMSMT could improve their geometrical thinking skills and levels so that they could create new problem-solving strategies. There exist previous research whose findings encourage the development of problem solving skills (Sudarsono et al.,, 2021) and geometrical thinking through instructional sequences (A'yunet al.,2021; Baist et al., 2019; Hendroanto et al., 2019; Mulyatna et al., 2021; Rizki et al., 2018; Yudianto et al., 2018)With this motivation, it can be stated that the instructional sequence enacted based on the designed HLT could enhance the development of problem solving performance, and flexibility and strategy use. The reason behind this result might be affected by the tasks on GeoGebra. For example, they constructed altitude of a triangle by using the knowledge of circles, elements, and properties of different types of triangles. They could turn an obtuse triangle into acute triangle by dragging so that they could observe the change. These tasks improved their knowledge and geometric thinking and reflected on their problem solving strategies. For example, they formed solution strategies by forming exterior or interior new triangles. They could form other triangles and change the places and roles of the elements by dragging and drawing properties of the GeoGebra. In other words, by the engagement in GeoGebra through the instructional sequence performed based on the designed HLT, the PMSMT could think about the triangles and their elements and properties flexibly by reasoning about their properties and elements. Therefore, it can be claimed that these tasks in the HLT affected the PMSMT's strategies since they could form the *strategy of using main elements by new triangles* and *strategy of using related theorems* after the instructional sequence. Moreover, they could use other strategies identified in the study appropriately in the posttest. Hence, they became more successfully reason about the properties and elements of triangles. The results of the current study represented that the PMSMT could display flexibility, and form and use heuristic strategies based on the context and their geometric thinking. It was observed that they could produce problem solutions by flexibly and using and creating strategies effectively after the instructional sequence, Moreover, it can be stated that they could represent these behaviors and skills by developing their geometric thinking and creating problem solving strategies as encouraged in previous research (A'yun et al., 2021; Baist et al., 2019; Mulyatna et al., 2021). Also, it can be stated that the PMSMT became more successful geometry problem solvers by developing their flexibility. This finding is parallel to the results of the study of Elia, Heuvel-Panhuizen and Kolovou (2009).

To conclude, a possible explanation about the findings of the current study can be that flexibility in the context of geometry can be improved with the help of geometrical thinking in DGE. Also, they could understand the rationale of the strategies by reasoning about the properties and elements of the triangles so that they could modify the strategies and their thinking flexibly. This finding is in line with the study of Baroody (2003). Moreover, it can be stated that the GeoGebra could encourage this development with the help of its properties by supporting modifying and playing with the geometric shapes flexibly. This finding can suggest a practical implication for teacher training programs to develop preservice mathematics teachers' flexibility and problem-solving skills with the help of improvement on their geometric thinking. Moreover, the tasks supported by DGS in the HLT can provide a useful way of geometric thinking and solving problems by different solution strategies. This is parallel to the findings of various research emphasizing the positive effects of the tasks in the literature (Leikin & Levav-Waynberg, 2009; Sullivan et al., 2006; Zaslavsky & Sullivan, 2011). Also, preservice mathematics teachers can improve their flexibility based on their skills of problem solving and thinking needed for teaching concepts in the classrooms in the future. Hence, the tasks supported by DGS in the HLT suggest useful ways of educating effective mathematics teachers in teacher education programs. Based on the findings of the study, it is necessary to design HLTs for the courses including the tasks encouraging thinking and problem solving. Also, the claim that the more flexible the PMSMT are in strategy use, the more successful they are in problem solving should be examined in different contexts. This study is limited to participation and examination preservice teachers. Hence, this study can be replicated for inservice teachers as the experts of the real classrooms. This current study is also limited to the concept of triangles. Further study can be made about the other concepts of geometry so that the effects of DGS in the development of flexibility and strategy use by geometric thinking and geometrical problem-solving performance can be examined effectively. Moreover, this study is limited to examination of flexibility and strategy use in the context of geometry and geometrical thinking. Further studies can also be made in the context of other mathematical skills and mathematics learning areas.

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