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# THE FREE VIBRATION OF NON-HOMOGENEOUS TRUNCATED CONICAL SHELLS ON A WINKLER FOUNDATION

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**Abstract**

*In this work, free vibration of non-homogeneous truncated conical shells on a Winkler foundation is studied. After formed the fundamental relations and governing equations, the dimensionless frequency parameter of the non-homogeneous isotropic truncated conical shell with or without an elastic foundation are found. Finally, effects of variations of the shell characteristics, non-homogeneity and the Winkler foundation on minimum values of the dimensionless frequency parameter have been studied. The results are compared with other works in open literature.*

**Keywords:** Conical shells, elastic foundation, non-homogeneity, free vibration

## 1. Introduction

Shell structures are widely used in engineering fields because of their strength characteristics. They are used in the form of tanks, water ducts, process equipment, subsea/ground pipelines, and in many other applications. Shells supported on soft and light filaments in space vehicles and boilers and storage thanks on floor grid work in ships are some of the other instances of shells on elastic foundation. Most earthen soils can be appropriately represented by a mathematical model from Pasternak, whereas sandy soils and liquids can be represented by Winkler's model. Pipelines are often subjected to dynamic loads caused by seismic forces, sea waves and nuclear explosions. Similarly, tube bundles of heat exchangers are subjected to flow-induced vibrations as well as vibrations emanating from attached pumps and compressors. Therefore, the dynamic behavior of shells on elastic foundations is of substantial practical interest and it has been widely investigated [1–8]. Many of these studies are for homogenous cylindrical shells on elastic foundations.

In recent years non-homogeneous shell structures are being widely used in aerospace, automotive, marine and other technical applications. The non-homogeneity of materials stems from production techniques, surface and thermal polishing processes, effect of radiation, etc. Thus, the physical properties of materials change from point to point as random, piecewise continuous or continuous functions of coordinates. Notable contributions were made [9–12] dealing with various types of non-homogeneity considerations. Considering the effects of non-homogeneity and an elastic foundation complicate the problem considerably. Hence, the vibration problem of non-homogeneous structural elements resting on elastic foundations is rare in literature [13-15]. Studies on the free vibration of truncated conical shells made of non-homogeneous materials resting on a Winkler foundation have not been seen in the open literature. The present paper deals with the free vibration of non-homogenous truncated conical shells on a Winkler foundation.

## 2. Basic relations and governing equations

Consider a circular non-homogeneous truncated conical shell as shown in Fig. 1, where $\gamma$ is the semi-vertex cone angle, L is the length, h is the thickness, $R_1$ and $R_2$ are the radii at the ends. The reference surface of the conical shell is taken as the middle surface where an orthogonal coordinate system $(S,\theta,\zeta)$ is fixed. The $S$- axis lies on the curvilinear middle surface of the cone, $S_1$ and $S_2$ being the coordinates of the points where this axis intersects the small and large bases, respectively. Furthermore, the $\zeta$-axis is always normal to the moving $S$-axis, lying in the plane generated by the $S$-axis and the axis of the cone, and points inwards. The $\theta$-axis is in the direction perpendicular to the $S-\zeta$ plane.

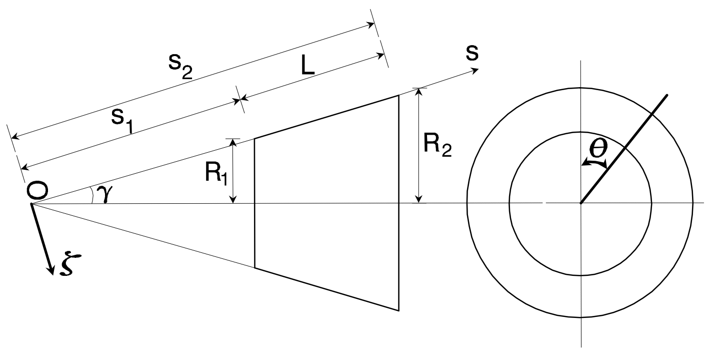


Fig.1. Geometry of the truncated conical shell

The stress-strain relation for the non-homogeneous isotropic truncated conical shell is:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = \frac{E_0\varphi_1(\bar{\zeta})}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 - \zeta \dfrac{\partial^2 w}{\partial S^2} \\ \varepsilon_2^0 - \zeta\left( \dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial \varphi^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S} \right) \\ \varepsilon_{12}^0 - \zeta\left( \dfrac{1}{S}\dfrac{\partial^2 w}{\partial S \partial \varphi} - \dfrac{1}{S^2}\dfrac{\partial w}{\partial \varphi} \right) \end{bmatrix} \tag{1}$$

where $\sigma_1,\sigma_2$ and $\sigma_{12}$ are the stresses, $\varepsilon_1^0$ and $\varepsilon_2^0$ are the normal strains in the curvilinear coordinate directions $S$ and $\theta$ on the middle surface, respectively, while $\varepsilon_{12}^0$ is the corresponding shear strain; $\varphi = \theta \sin\gamma$, $w$ is the displacement of the middle surface in the normal direction, positive towards the axis of the cone and assumed to be much smaller than the thickness; $E_0,\nu$ and $\rho_0$ are the Young's modulus, Poisson's ratio and density of the homogeneous material, respectively; $\varphi_1(\bar{\zeta}) = 1 + \mu\varphi_0(\bar{\zeta})$, $\varphi_0(\bar{\zeta})$ is continuous function of non-homogeneity defining the variations of the Young's modulus and density, respectively, satisfying the condition $\left|\varphi_0(\bar{\zeta})\right| \le 1$, and $\mu$ is a non-homogeneity coefficient, satisfying $0 \le \mu \le 1$;

The well-known force and moment resultants are expressed by

$$[(T_1, T_2, T_{12}), (M_1, M_2, M_{12})] = \int_{-h/2}^{h/2} (1, \zeta)(\sigma_1, \sigma_2, \sigma_{12})\, d\zeta \tag{2}$$

The relations between the forces $T_1, T_2$ and $T_{12}$ and the stress function $\Phi$ are given by

$$(T_1,\ T_2, T_{12}) = \left( \frac{1}{S^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{1}{S}\frac{\partial\Phi}{\partial S},\ \ \frac{\partial^2\Psi}{\partial S^2},\ -\frac{1}{S}\frac{\partial^2\Phi}{\partial S\partial\varphi} + \frac{1}{S^2}\frac{\partial\Phi}{\partial\varphi} \right) \tag{3}$$

The non-homogeneous truncated conical shell is resting on a Winkler foundation. The foundation interface pressure p may be expressed as

$$P = kw \tag{4}$$

where $k\,(N/m^3)$ is the modulus of subgrade reaction for the foundation [2].

Substituting Eq. (1) into (2) after some rearrangements, the relations found for moments and strains, being substituted into the vibration and compatibility equations of non-homogeneous truncated conical shells on a Winkler foundation [2, 17], together with relations (3) and (4), then considering the independent variables $S = S_1 e^z$ and $\Phi = \Phi_1 e^{2z}$, after lengthy computations, these equations for $w$ and $\Phi_1$ can be obtained as

$$\begin{aligned} & c_{12}e^{2z}\left( \frac{\partial^4\Phi_1}{\partial z^4} - 4\frac{\partial^3\Phi_1}{\partial z^3} + 4\frac{\partial^2\Phi_1}{\partial z^2} + \frac{\partial^4\Phi_1}{\partial\varphi^4} + 2\frac{\partial^2\Phi_1}{\partial\varphi^2} \right) + \\ & + 2(c_{11} - c_{31})e^{2z}\left( \frac{\partial^4\Phi_1}{\partial z^2\partial\varphi^2} - 2\frac{\partial^3\Phi_1}{\partial z\partial\varphi^2} + \frac{\partial^2\Phi_1}{\partial\varphi^2} \right) + \left( \frac{\partial^2\Phi_1}{\partial z^2} + 3\frac{\partial\Phi_1}{\partial z} + 2\Phi_1 \right) e^{3z} S_1 \cot\gamma - \\ & - c_{13}\left( \frac{\partial^4 w}{\partial\varphi^4} + \frac{\partial^4 w}{\partial z^4} - 4\frac{\partial^3 w}{\partial z^3} + 4\frac{\partial^2 w}{\partial z^2} + 2\frac{\partial^2 w}{\partial\varphi^2} \right) - \\ & - 2(c_{14} + c_{32})\left( \frac{\partial^4 w}{\partial z^2\partial\varphi^2} - 2\frac{\partial^3 w}{\partial z\partial\varphi^2} + \frac{\partial^2 w}{\partial\varphi^2} \right) - S_1^4 e^{4z} kw - S_1^4 e^{4z}\rho_1 h\frac{\partial^2 w}{\partial t^2} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} & b_{11}e^{2z}\left( \frac{\partial^4\Phi_1}{\partial z^4} + 4\frac{\partial^3\Phi_1}{\partial z^3} + 4\frac{\partial^2\Phi_1}{\partial z^2} + \frac{\partial^4\Phi_1}{\partial\varphi^4} + 2\frac{\partial^2\Phi_1}{\partial\varphi^2} \right) + \\ & + 2(b_{31} + b_{12})e^{2z}\left( \frac{\partial^4\Phi_1}{\partial z^2\partial\varphi^2} + 2\frac{\partial^3\Phi_1}{\partial z\partial\varphi^2} + \frac{\partial^2\Phi_1}{\partial\varphi^2} \right) + \\ & - b_{14}\left( \frac{\partial^4 w}{\partial\varphi^4} + \frac{\partial^4 w}{\partial z^4} - 4\frac{\partial^3 w}{\partial z^3} + 4\frac{\partial^2 w}{\partial z^2} + 2\frac{\partial^2 w}{\partial\varphi^2} \right) + \\ & + 2(b_{32} - b_{13})\left( \frac{\partial^4 w}{\partial z^2\partial\varphi^2} - 2\frac{\partial^3 w}{\partial z\partial\varphi^2} + \frac{\partial^2 w}{\partial\varphi^2} \right) + S_1 e^z \cot\gamma\left( \frac{\partial^2 w}{\partial z^2} - \frac{\partial w}{\partial z} \right) = 0 \end{aligned} \tag{6}$$

in which expressions $c_{ij},\ b_{ij}\ (i, j = 1 \div 6)$ are defined as follows:

$$c_{11}=a_{11}^{1}b_{11}+a_{12}^{1}b_{12},\ c_{12}=a_{11}^{1}b_{12}+a_{12}^{1}b_{11},\ c_{13}=a_{11}^{1}b_{13}+a_{12}^{1}b_{23}+a_{11}^{2},$$
$$c_{14}=a_{11}^{1}b_{14}+a_{12}^{1}b_{13}+a_{12}^{2},\ \ c_{31}=a_{66}^{1}b_{31},\ \ c_{32}=a_{66}^{1}b_{32}+a_{66}^{2},\ b_{11}=a_{11}^{0}L_0^{-1},$$
$$b_{12}=-a_{12}^{0}L_0^{-1},\ b_{13}=\left(a_{12}^{0}a_{12}^{1}-a_{11}^{1}a_{11}^{0}\right)L_0^{-1},\ b_{14}=\left(a_{12}^{0}a_{11}^{1}-a_{12}^{1}a_{11}^{0}\right)L_0^{-1},\ b_{31}=1/a_{66}^{0}, \tag{7}$$
$$b_{32}=-a_{66}^{1}/a_{66}^{0},\ \ L_0=a_{11}^{0}a_{11}^{0}-a_{12}^{0}a_{12}^{0};\ a_{11}^{k_1}=\frac{E_0h^{k_1+1}}{1-\nu^2}\int_{-1/2}^{1/2}\overline{\zeta}^{k}\varphi_1\left(\overline{\zeta}\right)d\overline{\zeta},\ a_{12}^{k_1}=\nu a_{11}^{k_1},$$
$$a_{66}^{k_1}=(1-\nu)a_{11}^{k_1},\ \rho_1=\rho_0\int_{-1/2}^{1/2}\varphi_1\left(\overline{\zeta}\right)d\overline{\zeta};\ \ k_1=0,1,2$$

**2. The solution of governing equations**

The truncated conical shell is simply supported at both ends. Then the solution of the equation (6) is sought in the following form [17]:

$$w=f(t)e^{z}\sin\beta_1 z\sin\beta_2\varphi \tag{8}$$

where f(t) is time dependent amplitude and the following definitions apply:

$$\beta_1=\frac{m\pi}{z_0},\ \beta_2=\frac{n}{\sin\gamma},\ z_0=\ln\frac{S_2}{S_1},\ z=\ln\frac{S}{S_1} \tag{9}$$

Substituting Eq. (8) into Eq. (6) and applying the superposition method to the resulting equation, the particular solution is obtained as follows:

$$\Phi_1=f(t)(K_1\sin\beta_1 z+K_2\cos\beta_1 z+K_3e^{-z}\sin\beta_1 z)\sin\beta_2\varphi \tag{10}$$

Where $K_i\,(i=1,2,3)$ are depending on the shell characteristics and material properties [12].

Substituting Eqs. (8) and (10) into Eq. (5) and applying Galerkin method, after integrating for the natural frequency of free vibration of non-homogeneous truncated conical shells resting on a Winkler foundation, the following equation is obtained:

$$\omega_k=\sqrt{\frac{X_1U_1+X_2U_2+X_3U_3+X_6+kX_9}{X_9\rho_1 h}} \tag{11}$$

where $X_i\,(i=1\div 9)$ are depending on the shell characteristics and material properties [12].

As $k=0$, from Eq. (11), for the natural frequency of free vibration of non-homogeneous truncated conical shells without a Winkler foundation, the following equation is obtained:

$$\omega=\sqrt{\frac{X_1U_1+X_2U_2+X_3U_3+X_8}{X_9\rho_1 h}} \tag{12}$$

The dimensionless frequency parameters $\omega_{1k}$ and $\omega_1$ of the non-homogeneous truncated conical shell with or without a Winkler foundation, respectively, are defined as

$$\omega_{1k} = \omega_k R_2 \sqrt{(1-\nu^2)\rho_0 / E_0} \tag{13}$$

$$\omega_1 = \omega R_2 \sqrt{(1-\nu^2)\rho_0 / E_0} \tag{14}$$

The minimum values of the natural frequency and dimensionless frequency parameter are obtained by minimizing Eqs. (11)-(14) with respect to m and n.

The truncated conical shell is transformed into the cylindrical shell when $\gamma \to 0$. If $\gamma \to 0$ are substituted in Eqs. (11)-(14) corresponding formulas for simply supported cylindrical shells with or without a Winkler foundation are obtained. In this case, $\omega_k$; $\omega_{1k}$; $\omega$; $\omega_1$ in Eqs (11)-(14) are transformed into $\omega_{cyl}^k$; $\omega_{cyl}^k$; $\omega_{cyl}$; $\omega_{1cyl}$, respectively.

When $\mu = 0$, the appropriate formulas for the dimensionless frequency parameters of cylindrical and conical shells made of homogeneous materials with or without a Winkler foundation are found as a special case.

## 4. Comparative studies and numerical results

### *4.1. Comparative studies*

In order to examine the accuracy of the present free vibration analysis, two comparisons are made with the results available in the literature.

Table 1 shows the dimensionless frequency parameters $\omega_1$ for a homogeneous isotropic truncated conical shell without an elastic foundation for $\nu = 0.3$, $h / R_2 = 0.01$, $L = 0.25 S_2$, $\gamma = 30^o$. The present results are compared with solutions given by Irie et al. [16] and Liew et al. [18], and a good agreement is obtained for all the modes.

Table 1. Comparisons of dimensionless frequency parameter $\omega_1$ of a homogeneous isotropic truncated conical shell with simply supported edges

| N | Irie et al. [16] | Liew et al. [18] | Present study |
|---|---|---|---|
| 2 | 0.7910 | 0.7904 | 0.7943 |
| 3 | 0.7284 | 0.7274 | 0.7085 |
| 4 | 0.6352 | 0.6339 | 0.6199 |
| 5 | 0.5531 | 0.5514 | 0.5437 |
| 6 | 0.4949 | 0.4930 | 0.4896 |
| 7 | 0.4643 | 0.4632 | 0.4623 |
| 8 | 0.4645 | 0.4623 | 0.4627 |
| 9 | 0.4892 | 0.4870 | 0.4882 |

Comparison of results in this study is made with those of Paliwal et al. [4] for free vibration analysis of homogeneous isotropic cylindrical shells resting on a Winkler foundation. The comparison is shown in Table 2. Present results are obtained from Eq.(14) as $\mu = 0$; $\gamma \to 0$; $R_2 = R_1 = R$; $L = L_1$.

This comparison is to ensure that elastic foundation effects have been correctly integrated into the present formulation.

Table 2. Comparison of dimensionless frequency parameter $\omega_{1cyl}^{k}$ for a cylindrical shell resting on a Winkler foundation $(R/h = 100; L_1/R = 2; k = 10^{-4}\ N/m^3)$

| | $\omega_{1cyl}^{k}$ | |
|---|---|---|
| (m,n) | Paliwal et al. [4] | Present study |
| (1,1) | 0.6788227178 | 0.6792138004 |
| (1,2) | 0.3639407237 | 0.3646346369 |
| (1,3) | 0.2052558042 | 0.2080412884 |
| (1,4) | 0.1274543541 | 0.1382361504 |

Based on the above comparisons the accuracy of the present study is validated.

*4.2. Numerical results*

Numerical computations, for homogeneous and non-homogeneous isotropic truncated conical shells with or without a Winkler foundation have been carried out using expressions (11)-(14). The results are presented in Tables 3-4. The non-homogeneity functions of the material of the truncated conical shell are assumed to be linear and quadratic functions which $\varphi_0(\overline{\zeta}) = \overline{\zeta}$ and $\overline{\zeta}^2$ (see [8, 13, 17]. Material properties of the shell are given below [17]: $E_0=2.11\times10^{11}$ N/m$^2$; $\nu_0=0.3$; $\rho_0=8000$ kg/m$^3$.

The results given in all tables below for the dimensionless frequency parameter, corresponding to longitudinal wave number m=1, only the number of circumferential waves $(n)$ is presented in parentheses. H and NH are show homogenous and non-homogenous cases, respectively, in below tables.

Table 3 shows variations of the minimum values of the dimensionless frequency parameter and corresponding circumferential wave numbers for homogeneous and non-homogeneous truncated conical shells with different non-homogeneity functions, versus the Winkler foundation modulus k. As the foundation modulus k increases the values of $\omega_{1k}$ increase continuously for homogeneous and non-homogeneous cases. The effect of the non-homogeneity on the minimum values of dimensionless frequency parameter is little and for high values of the foundation modulus k is insignificantly. When the Young's modulus is keep constant and the Winkler foundation modulus is changed, the effect on the dimensionless frequency parameter are 1.40% ; 2.78%; 13.2%, 25.01%; 95.28%; 157.43% for $k = 5\times10^5;\ 10^6;\ 5\times10^6;\ 1\times10^7; 5\times10^7; 1\times10^8\ N/m^3$, respectively with the homogeneous isotropic truncated conical shell. As the values of the foundation modulus k is little the non-homogeneity has a considerable influence on the dimensionless frequency parameter.

Table 3. Variations of $\omega_{1k}$, $\omega_1$, $n_k$ and $n$ for the H and NH isotropic truncated conical shells versus the k(N/m$^3$), ($R_1$=1m; h=0.01m; L=2$R_1$; $\gamma = 30^o$; $\mu = 0.9$)

| $\varphi_0(\bar{\zeta})$ / k (N/m$^3$) | $\omega_1$, (n) 0 | $\bar{\zeta}$ | $\bar{\zeta}^2$ | $\omega_{1k}$, ($n_k$) 0 | $\bar{\zeta}$ | $\bar{\zeta}^2$ |
|---|---|---|---|---|---|---|
| $5\times10^5$ | 0.1751(7) | 0.1714(7) | 0.1775(6) | 0.1775(7) | 0.1739(7) | 0.1797(6) |
| $1\times10^6$ | | | | 0.1800(7) | 0.1764(7) | 0.1819(6) |
| $5\times10^6$ | | | | 0.1982(7) | 0.1949(7) | 0.1988(6) |
| $1\times10^7$ | | | | 0.2189(7) | 0.2159(7) | 0.2181(6) |
| $5\times10^7$ | | | | 0.3419(7) | 0.3400(7) | 0.3342(6) |
| $1\times10^8$ | | | | 0.4507(7) | 0.4493(7) | 0.4381(6) |

The variation of minimum values of the dimensionless frequency parameters of homogeneous and non-homogeneous truncated conical shells with or without a Winkler foundation with different non-homogeneity functions, versus the semi-vertex angle $\gamma$, as the $k = 10^6\,N/m^3$, are given in Fig. 2. As the semi-vertex angle $\gamma$ increases, the values of $\omega_{1k}$ and $\omega_1$ increase as $\gamma \le 45^o$ and decrease as $\gamma > 45^o$ for homogeneous and non-homogeneous truncated conical shells with or without a Winkler foundation. As the semi-vertex angle $\gamma$ is increased, the effect of a Winkler foundation on the minimum values of $\omega_{1k}$ increase.

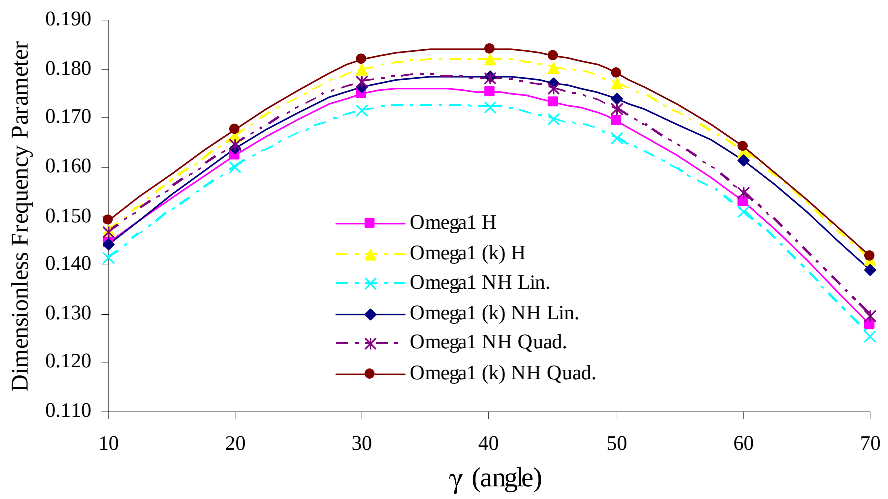


Fig. 2. Variations of $\omega_{1k}$ and $\omega_1$ for the N and NH truncated conical shells versus the $\gamma$ ($k = 10^6\,N/m^3$; $R_1$=1m; h=0.01m; L=2$R_1$; $\mu = 0.9$)

**5. Conclusions**

In this study, the free vibration of non-homogeneous truncated conical shells resting on a Winkler foundation is presented. The basic relations and governing equations have been obtained for the truncated conical shell, the Young's modulus and density of which vary continuously in the thickness direction. The dimensionless frequency parameter of non-homogeneous truncated conical shells with or without an elastic foundation is obtained.

Finally, carrying out some numerical calculations, effects of the variation of truncated conical shell characteristics, non-homogeneity and a Winkler foundation on minimum values of the dimensionless frequency parameter have been studied.

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