

THE DUAL RODRIGUES PARAMETERS

İ. Karakılıç

Dokuz Eylül University, Faculty of Arts and Sciences, Department of Mathematics,
Tınaztepe Kampüsü, Buca 35140, İzmir, Turkey

Accepted Date: 19 September 2009

Abstract

The development of Rodrigues parameters in the first half of 19th century has attracted much attention in the field of theoretical kinematics. The importance of the Rodrigues formulae depends on the use of the tangent of the half rotation angle being integrated with the components of the rotation axis.

In this paper Rodrigues parameters of the dual spherical motion are obtained, which are called the dual Rodrigues parameters. The dual Rodrigues parameters contain the rotation angle and the distance parameter of the straight lines (the shortest distance between the straight lines) of the corresponding spatial motion.

Keywords: Study mapping; Dual spherical motion; Rotation angle; Dual Rodrigues parameters

1. Introduction

Line geometry investigates the set of lines in R^3 . There exists a vast literature on this branch of classical geometry, for example [4-7]. Line geometry possesses a close relation to spatial kinematics [2,8-11] and has therefore found application in mechanism and robotic kinematics.

Any motion on the D.U.S. (Dual Unit Sphere) can be represented by the rotation of a moving D.U.S. K on a fixed D.U.S. K' with the same center, which correspond to the line spaces H and H' in R^3 respectively. The relation between the points of K and K' is given by a dual orthogonal matrix \hat{A} (hat over an alphabet denotes the dual form of a quantity), such that $\hat{A}\hat{x} = \hat{X}$. In this paper we derive the Cayley's Formula and obtain the Rodrigues parameters for the dual case.

It is well known by the Study mapping that the points on the D.U.S. correspond to the straight lines in space. If \hat{a} and \hat{b} are the vectors of the D.U.S. corresponding to the straight lines l_a and l_b in the real space R^3 and if $\hat{f} = f + ef^*$ is the angle between \hat{a} and \hat{b} , then f defines the angle between l_a and l_b and f^* defines the distance between l_a and l_b . A dual number is a formal sum $\hat{a} = a + ea^*$, where a and a^* are real numbers and e is the dual unit with $e^2 = 0$. Addition and multiplication are given by

$$(a_1 + ea_1^*) + (b_1 + eb_1^*) = (a_1 + b_1) + e(a_1^* + b_1^*)$$
$$(a_1 + ea_1^*) \cdot (b_1 + eb_1^*) = (a_1b_1) + e(a_1b_1^* + a_1^*b_1)$$

Karakılıç

If we denote the set of dual numbers by D , for some non zero $\hat{a}, \hat{b} \in D$ (e.g., $3e, 5e \in D$) we have $\hat{a}\hat{b} = 0$ (i.e., $3e \cdot 5e = 15e^2 = 0$). Hence D is not a field, D is a commutative ring with identity.

For a given analytic function F we can extend its definition to the set D by

$$\begin{aligned} f(x + ex^*) &= \sum_{k=0}^{\infty} a_k (x + ex^* - x_0)^k \\ &= \sum_{k=0}^{\infty} a_k (x - x_0)^k + e \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1} x^* \\ &= f(x) + ex^* f'(x) \end{aligned}$$

For instance,

$$\begin{aligned} \text{Sin}\hat{x} &= \text{Sin}(x + ex^*) = \text{Sin}x + ex^* \text{Cos}x \\ \text{Cos}\hat{x} &= \text{Cos}(x + ex^*) = \text{Cos}x - ex^* \text{Sin}x \\ e^{\hat{x}} &= e^x + ex^* e^x \end{aligned}$$

A dual vector \hat{v} in three dimensional dual space D^3 is defined by $\hat{v} = \mathbf{v} + e\mathbf{v}^*$, where $\mathbf{v}, \mathbf{v}^* \in R^3$. D^3 satisfies all axioms of the vector space, since its domain D is only a ring not a field, D^3 is a D -module. However the elements of D^3 are also called (dual) vectors.

D^3 is a linear space over the real numbers with dimension 6. This bilinear form defines a kind of degenerate scalar product (see also [14]). Hence the norm of \hat{v} , denoted by $\|\hat{v}\|: D^3 \rightarrow D$ is;

$$\begin{aligned} \|\hat{v}\| &= (\hat{v} \cdot \hat{v})^{\frac{1}{2}} = (\mathbf{v} \cdot \mathbf{v} + 2e\mathbf{v} \cdot \mathbf{v}^*)^{\frac{1}{2}} \\ &= \|\mathbf{v}\| \left[\left(1 + e \frac{\mathbf{v} \cdot \mathbf{v}^*}{\|\mathbf{v}\|^2}\right)^2 \right]^{\frac{1}{2}} \\ &= \|\mathbf{v}\| + e \frac{\mathbf{v} \cdot \mathbf{v}^*}{\|\mathbf{v}\|} = \left(\|\mathbf{v}\|, \frac{\mathbf{v} \cdot \mathbf{v}^*}{\|\mathbf{v}\|} \right) \end{aligned}$$

The dual vector with the norm $1 = (1, 0)$ is called a dual unit vector. If $\hat{v} = \mathbf{v} + e\mathbf{v}^*$ is a dual unit vector, then by above expansion $\|\hat{v}\| = 1$ and $\mathbf{v} \cdot \mathbf{v}^* = 0$. (For detail algebraic properties of dual numbers, reader may be referred to [15])

2. Cayley's Formula for the dual spherical motion

The dual spherical motion maintain the constant lengths of the dual vectors, that is

$$\left\| \hat{X} \right\| = \left\| \hat{x} \right\| ,$$

which yields

$$(\hat{X} - \hat{x})^T (\hat{X} + \hat{x}) = 0 .$$

Since $\hat{X} = \hat{A} \hat{x}$,

$$\hat{X} + \hat{x} = (\hat{A} + I) \hat{x} \quad \text{or} \quad \hat{x} = (\hat{A} + I)^{-1} (\hat{X} + \hat{x}) \quad \text{and} \quad (\hat{X} - \hat{x}) = (\hat{A} - I) \hat{x} ,$$

thus we get

$$\hat{X} - \hat{x} = (\hat{A} - I)(\hat{A} + I)^{-1} (\hat{X} + \hat{x}) .$$

Let us denote $(\hat{A} - I)(\hat{A} + I)^{-1}$ by \hat{B} . Since $\hat{X} - \hat{x}$ is orthogonal to $\hat{X} + \hat{x}$, $\hat{B}(\hat{X} + \hat{x})$ is orthogonal to $\hat{X} + \hat{x}$. That is any vector \hat{z} , which can be written as the sum of two dual vectors on the D.U.S., is orthogonal to $\hat{B}\hat{z}$. Then we have,

$$\hat{z}^T \hat{B} \hat{z} = \sum (\hat{b}_{ij} + \hat{b}_{ji}) \hat{z}_i \hat{z}_j = 0 .$$

This relation holds for every \hat{z} hence $\hat{b}_{ii} = 0$ and $\hat{b}_{ij} = -\hat{b}_{ji}$, which implies that \hat{B} is skew-symmetric, that is $\hat{B} = -\hat{B}^T$.

On the other hand, skew-symmetry of \hat{B} provides that $(I - \hat{B})$ is not singular. A simple computation yields,

$$\begin{aligned} B &= (\hat{A} - I)(\hat{A} + I)^{-1} \\ \hat{B}(\hat{A} + I) &= (\hat{A} - I) \\ \hat{B} + I &= \hat{A} - \hat{B}\hat{A} \\ (I + \hat{B})(I - \hat{B})^{-1} &= \hat{A} \end{aligned}$$

Thus we obtain the Cayley Formula for the dual case;

$$\hat{A} = (I + \hat{B})(I - \hat{B})^{-1} .$$

Using that \hat{B} is skew symmetric, we compute \hat{A}^T as follows:

$$\begin{aligned} \hat{A}^T &= (I + \hat{B})^T ((I - \hat{B})^{-1})^T \\ &= (I + \hat{B}^T)(I - \hat{B}^T)^{-1} \\ &= (I - \hat{B})(I + \hat{B})^{-1} , \end{aligned}$$

Karakılıç

From which we get

$$\hat{A}\hat{A}^T = \hat{A}^T\hat{A} = I.$$

Hence every skew-symmetric dual matrix \hat{B} determines an orthogonal dual matrix \hat{A} .

If we define the skew-symmetric dual matrix \hat{B} by

$$\hat{B} = \begin{bmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{bmatrix},$$

then instead of $\hat{B}\hat{y}$ (\hat{y} is a dual vector centered at the D.U.S.) one can use $\hat{b} \times \hat{y}$ where $\hat{b} = (b, b^*) = (b_1 + eb_1^*, b_2 + eb_2^*, b_3 + eb_3^*) = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$. Hence

$$\hat{B}\hat{y} = \hat{b} \times \hat{y}.$$

3. Rodrigues Equations

In the above calculations for a given orthogonal dual matrix \hat{A} we obtain a skew-symmetric dual matrix \hat{B} by the Cayley's formula. It is clear that the relation,

$$\hat{X} - \hat{x} = \hat{B}(\hat{X} + \hat{x}),$$

between the fixed and the moving frame coordinates can be written in the form,

$$\hat{X} - \hat{x} = \hat{b} \times (\hat{X} + \hat{x}).$$

This is analogous to the Rodrigues equations in the real case [8]. Let us call \hat{b} the dual Rodrigues vector. Now we define a dual plane perpendicular to \hat{b} , and denote the projections of \hat{X} and \hat{x} on this dual plane by \hat{X}' and \hat{x}' . Let \hat{f} be the angle between \hat{X}' and \hat{x}' (\hat{f} is the vertex angle of the rhombus formed by \hat{X}' and \hat{x}' , so \hat{f} is the rotation angle).

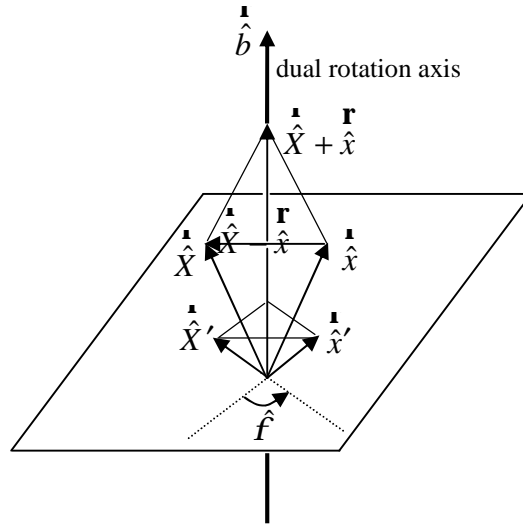


Figure 1. The rhombus formed by the diagonals
 $\hat{X}' + \hat{x}'$ and $\hat{X} - \hat{x}$.

It is easy to see that $\hat{X} - \hat{x} = \hat{b} \times (\hat{X} + \hat{x})$ implies $\hat{X}' - \hat{x}' = \hat{b} \times (\hat{X}' + \hat{x}')$ and
 $\|\hat{X}' - \hat{x}'\| = \|\hat{b}\| \|\hat{X}' + \hat{x}'\|$, from which we get

$$\|\hat{b}\| = \frac{\|\hat{X}' - \hat{x}'\|}{\|\hat{X}' + \hat{x}'\|}.$$

From the fig. 1 we have

$$\frac{\|\hat{X}' - \hat{x}'\|}{\|\hat{X}' + \hat{x}'\|} = \frac{\frac{\|\hat{X}' - \hat{x}'\|}{2}}{\frac{\|\hat{X}' + \hat{x}'\|}{2}} = \tan \frac{\hat{f}}{2};$$

therefore

$$\|\hat{b}\| = \frac{\|\hat{X}' - \hat{x}'\|}{\|\hat{X}' + \hat{x}'\|} \tag{1}$$

Using the algebraic properties of dual numbers and (1) we obtain

Karakılıç

$$\left\| \hat{\mathbf{b}} \right\| = \left\| \mathbf{b} \right\| + e \frac{\mathbf{r} \cdot \mathbf{r}^*}{\left\| \mathbf{b} \right\|} \quad (2)$$

and

$$\text{Tan} \frac{\hat{f}}{2} = \text{Tan} \frac{f}{2} + e \frac{f^*}{2} (1 + \text{Tan}^2 \frac{f}{2}) \quad (3)$$

The equations (2) and (3) imply,

$$\left\| \mathbf{b} \right\| + e \frac{\mathbf{r} \cdot \mathbf{r}^*}{\left\| \mathbf{b} \right\|} = \text{Tan} \frac{f}{2} + e \frac{f^*}{2} (1 + \text{Tan}^2 \frac{f}{2}). \quad (4)$$

Hence we have from (4) $\left\| \mathbf{b} \right\| = \text{Tan} \frac{f}{2}$ (the norm of the real Rodrigues vector) and

$$\frac{\mathbf{r} \cdot \mathbf{r}^*}{\left\| \mathbf{b} \right\|} = \frac{f^*}{2} (1 + \text{Tan}^2 \frac{f}{2}) \quad (5)$$

Let us denote the unit vector which is in the same direction of \mathbf{b} by $\hat{\mathbf{s}}$, that is $\hat{\mathbf{s}} = \frac{\mathbf{b}}{\left\| \mathbf{b} \right\|}$ and $\hat{\mathbf{s}} = (s_1, s_2, s_3)$ is called the unit Rodrigues vector. So (5) yields,

$$\hat{\mathbf{s}} \cdot \mathbf{b}^* = \frac{f^*}{2} (1 + \text{Tan}^2 \frac{f}{2}) \quad (6)$$

On the other hand, let us define the dual Rodrigues vector by $\hat{\hat{\mathbf{s}}}$, $\hat{\hat{\mathbf{s}}} = \frac{\hat{\mathbf{b}}}{\left\| \hat{\mathbf{b}} \right\|}$. Using the algebraic

properties of the dual numbers we have,

$$\hat{\hat{\mathbf{s}}} = \hat{\mathbf{s}} + e \hat{\mathbf{s}}^* = \frac{\hat{\mathbf{b}}}{\left\| \hat{\mathbf{b}} \right\|} = \frac{\mathbf{b}}{\left\| \mathbf{b} \right\|} + e \left(\frac{\mathbf{b}^*}{\left\| \mathbf{b} \right\|} - \frac{\mathbf{r} \cdot \mathbf{r}^*}{\left\| \mathbf{b} \right\|^3} \right) \quad (7)$$

where $\hat{\mathbf{s}} = (s_1, s_2, s_3)$ and $\hat{\mathbf{s}}^* = (s_1^*, s_2^*, s_3^*)$.

From (7),

$$s_1^* = \frac{b_1^* \left\| \mathbf{b} \right\|^2 - b_1 (b_1 b_1^* + b_2 b_2^* + b_3 b_3^*)}{\left\| \mathbf{b} \right\|^3}, \quad (8)$$

$$s_2^* = \frac{b_2^* \left\| \mathbf{b} \right\|^2 - b_2 (b_1 b_1^* + b_2 b_2^* + b_3 b_3^*)}{\left\| \mathbf{b} \right\|^3}, \quad (9)$$

Karakılıç

$$s_3^* = \frac{b_3^* \|\mathbf{b}\|^2 - b_3(b_1 b_1^* + b_2 b_2^* + b_3 b_3^*)}{\|\mathbf{b}\|^3} . \quad (10)$$

We can easily obtain b_1^*, b_2^*, b_3^* from the equations (8),(9),(10) as follows

$$b_1^* = \frac{s_1^* \|\mathbf{b}\|^3 + b_1 b_2 b_2^* + b_1 b_3 b_3^*}{\|\mathbf{b}\|^2 - b_1^2}, \quad (11)$$

$$b_2^* = \frac{s_2^* \|\mathbf{b}\|^3 + b_2 b_1 b_1^* + b_2 b_3 b_3^*}{\|\mathbf{b}\|^2 - b_2^2}, \quad (12)$$

$$b_3^* = \frac{s_3^* \|\mathbf{b}\|^3 + b_3 b_1 b_1^* + b_3 b_2 b_2^*}{\|\mathbf{b}\|^2 - b_3^2}. \quad (13)$$

Since the real Rodrigues parameters are;

$$\begin{aligned} b_1 &= s_1 \tan \frac{f}{2} \\ b_2 &= s_2 \tan \frac{f}{2} \quad \text{and} \quad \|\mathbf{b}\| = \tan \frac{f}{2}, \\ b_3 &= s_3 \tan \frac{f}{2} \end{aligned}$$

we can rewrite the equations (11),(12) and (13) as,

$$b_1^* = \frac{s_1^* \tan^3 \frac{f}{2} + s_1 s_2 \tan^2 \frac{f}{2} b_2^* + s_1 s_3 \tan^2 \frac{f}{2} b_3^*}{\tan^2 \frac{f}{2} - s_1^2 \tan^2 \frac{f}{2}},$$

Simplifying the above equation we get,

$$b_1^* = \frac{s_1^* \tan \frac{f}{2} + s_1 s_2 b_2^* + s_1 s_3 b_3^*}{1 - s_1^2}, \quad 0 < f < 2p \quad (14)$$

Similarly,

Karakılıç

$$b_2^* = \frac{s_2^* \tan \frac{f}{2} + s_1 s_2 b_1^* + s_2 s_3 b_3^*}{1 - s_2^2} \quad (15)$$

$$b_3^* = \frac{s_3^* \tan \frac{f}{2} + s_3 s_1 b_1^* + s_3 s_2 b_2^*}{1 - s_3^2} \quad (16)$$

Since $\hat{s}^{\mathbf{I}}$ is a unit dual vector, $\|\hat{s}^{\mathbf{I}}\| = 1$, we get

$$\begin{aligned} s_1^2 + s_2^2 + s_3^2 &= 1, \\ s_1 s_1^* + s_2 s_2^* + s_3 s_3^* &= 0. \end{aligned} \quad (17)$$

For the general case let us assume that $s_i \neq 0$ for all $i=1,2,3$. If one or two of them are equal to zero then one can obtain the solutions of b_i^* 's from (14), (15), (16). Now we take $s_1 \neq 0, s_2 \neq 0, s_3 \neq 0$. So the denominators, $1 - s_i^2$, are also different from zero by (17).

If we expand (6) we get,

$$s_1 b_1^* + s_2 b_2^* + s_3 b_3^* = \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}).$$

Since $s_1 \neq 0$,

$$b_1^* = \frac{\frac{f^*}{2} (1 + \tan^2 \frac{f}{2}) - s_2 b_2^* - s_3 b_3^*}{s_1}. \quad (18)$$

The equations (14) and (18) yields,

$$s_2 b_2^* + s_3 b_3^* = (1 - s_1^2) \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}) - s_1 s_1^* \tan \frac{f}{2}. \quad (19)$$

If we substitute (18) into (15) we get,

$$b_2^* = s_2^* \tan \frac{f}{2} + s_2 \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}). \quad (20)$$

If we substitute (20) into (19) we get,

$$b_3^* = s_3^* \tan \frac{f}{2} + s_3 \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}). \quad (21)$$

If we substitute (20) and (21) into (15) we get,

$$b_1^* = s_1^* \tan \frac{f}{2} + s_1 \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}). \quad (22)$$

4. Conclusion

For the first time we obtain the dual Rodrigues parameters b_1^*, b_2^*, b_3^* as follows;

$$b_1^* = s_1^* \tan \frac{f}{2} + s_1 \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}) \quad (22)$$

$$b_2^* = s_2^* \tan \frac{f}{2} + s_2 \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}) \quad (20)$$

$$b_3^* = s_3^* \tan \frac{f}{2} + s_3 \frac{f^*}{2} (1 + \tan^2 \frac{f}{2}) \quad (21)$$

which are similar to well known Rodrigues parameters b_1, b_2, b_3 in the real case (see [8]). As seen from (22), (20), (21) these parameters depend on the angle f and the distance f^* of the corresponding rigid body motion in space. f denotes the angle between straight lines l_a and l_b in the space and f^* is the shortest distance between l_a and l_b . l_a and l_b defines the instant positions of the rigid body see fig. 2.

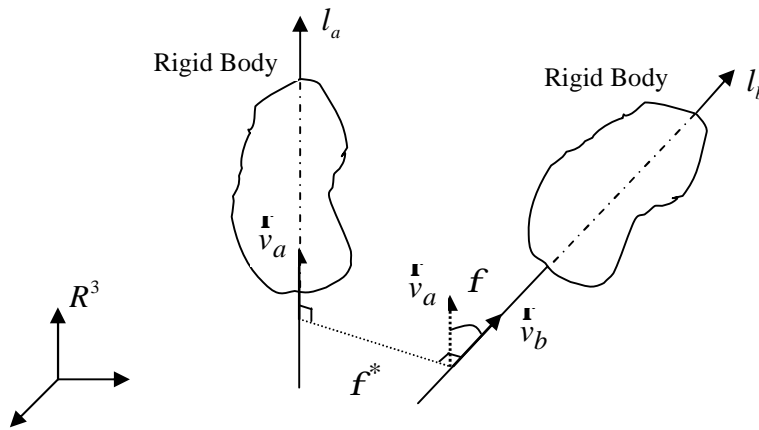


Figure 2. The angle f and the shortest distance f^* .
 \mathbf{v}_a and \mathbf{v}_b denote the unit directions of l_a and l_b respectively.

References

- [1] W. K. Clifford Proc. London Mathematic Society, 4, 1873, p. 381.
- [2] F. M. Dimentberg, The Screw Calculus and its Applications in Mechanics, (Izdat, "Nauka", Moscow, USSR, 1965) English translation: AD680993, Clearinghouse for Federal and Scientific Information, Virginia).
- [3] E. Study, Die Geometrie der Dynamen, Leibzig, 1901.
- [4] W.L. Edge, The Theory of Ruled Surfaces, Cambridge University Press, Cambridge, 1931.

- [5] V. Hlavaty, Differential Line Geometry, P. Nothoft Ltd., Groningen, 1953.
- [6] E. A. weis, Einführung in die Linengeometri und Kinematic, Leipzig, Berlin, 1935.
- [7] K. Zidler Linengeometri mit Anwendungen, 2 volumes, De Gruyter, Berlin, 1902.
- [8] J. M. McCarthy, An Introduction to Theoretical Kinematics, The MIT Press, Cambridge.
- [9] O. Bottema, B. Roth, Theoretical Kinematics, North Holland, Amsterdam, 1979.
- [10] J. M. McCarthy, B. Roth, The Curvature Theory of Line Trajectories in Spatial Kinematics, ASME Journal of Mechanical Design 103, 4, 1981.
- [11] Y. Kirson, Curvature Theory in Space Kinematics, PhD dissertation 140, University of California Berkley, 1975.
- [12] G. R. Veldkamp, On the Use of Dual Numbers, Vectors and Matrices in Instantaneous Spatial Kinematics, Mechanism and Machine Theory 11(2), (1976), 141-156.
- [13] Ö. Köse, C. C. Sarioğlu, B. Karabey, İ. Karakılıç, Kinematic Differential Geometry of a Rigid Body in Spatial Motion Using Dual Vector Calculus: Part II, Applied Mathematics and Computation 182 (2006) 333-358.
- [14] H. Potmann, J. Wallner, Computational line geometry, Springer-Verlag , Berlin, 2001.
- [15] I. S. Fischer, Dual-Number Methods in Kinematics, Statics and Dynamics, CRC Press LLC. Florida, 1999.