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A NEW PROOF OF CHAMPERNOWNE'S NUMBER IS TRANSCENDENTAL

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Abstract

In this study, a series representation of the number 0,1234...9101112... , which is proved by Kurt MAHLER that it is transcendental, is given and a program which gives the number on an arbitrary digit of 0,1234...9101112... is written. Moreover we proved in a different way that this number is a transcendental one.

Keywords: Algebraic numbers; transcendental numbers; digits.

1. Introduction

The set of real numbers is divided among 'algebraic' and 'Transcendental' numbers. A number x is called 'algebraic' if it satisfies an polynomial equation with integer coefficients, such that

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0,\ n \geq 1. \qquad (1)$$

For example, $\sqrt{2}$ is an algebraic number of degree 2 because it is solution of the polynomial equation $x^2-2=0$. However, not every real number is algebraic. Any real number that is not algebraic is said to be transcendental [1] . In fact, Georg Cantor proved that the set of algebraic numbers is countable and there are so numbers that they are not algebraic but transcendental, like π and e [2,3]
The existence of a transcendental number is first proved in 1851 by Liouville, he showed that 0,110001000... is transcendental [2]. Up to day many transcendental number has been found. The most important fifteen of them are given here [4].

1. $\pi = 3.1415$ ...
2. e = 2.718 ...
3. Euler's constant, sometimes called gamma or the Euler-Mascheroni constant, has the mathematical value of .577215664901532860606512090082... .Euler's constant (gamma) is defined as the limit of the expression (1 + 1/2 + 1/3 + 1/4 + ... + 1/ *n*) - ln (*n*), as *n* approaches infinity (Not proven to be transcendental, but generally believed to be by mathematicians).
4. Catalan's constant, $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ = 1 - 1/9 + 1/25 - 1/49 + ... (Not proven to be transcendental, but generally believed to be by mathematicians.)
5. Liouville's number, $\sum_{n=1}^{\infty} 10^{-n!}$ =0.110001000000000000000001000 ...

6. Chaitin's "constant", the probability that a random algorithm halts. (Noam Elkies of Harvard notes that not only is this number transcendental but it is also incomputable.)
7. Champernowne's number, 0.12345678910111213141516171819202122232425... This is constructed by concatenating the digits of the positive integers.
8. Special values of the zeta function, such as zeta (3). (Transcendental functions can usually be expected to give transcendental results at rational points.)
9. ln(2).
10. Hilbert's number, $2^{\sqrt{2}}$. (This is called Hilbert's number because the proof of whether or not it is transcendental was one of Hilbert's famous problems. In fact, according to the Gelfond-Schneider theorem, any number of the form $a^{\sqrt{b}}$ is transcendental where a and b are algebraic (a≠0, a≠1) and b is not a rational number. Many trigonometric or hyperbolic functions of non-zero algebraic numbers are transcendental.)
11. $e^{\sqrt{p}}$
12. $p^{\sqrt{e}}$ (Not proven to be transcendental, but generally believed to be by mathematicians.)
13. Morse-Thue's number, 0.01101001 ...
14. $i^{\sqrt{i}}$ (Here i is the imaginary number $\sqrt{-1}$. If a is algebraic and b is algebraic but irrational then $a^{\sqrt{b}}$ is transcendental. Since i is algebraic but irrational, the theorem applies. Note also: $i^{\sqrt{i}}$ is equal to $e^{\sqrt{\frac{-p}{2}}}$ and several other values. Consider $i^{\sqrt{i}} = e^{\sqrt{i.\log i}}$. Since log is multivalued, there are other possible values for $i^{\sqrt{i}}$.
15. Feigenbaum numbers, e.g. 4.669 ... . (These are related to properties of dynamical systems with period-doubling. The ratio of successive differences between period-doubling bifurcation parameters approaches the number 4.669 ... , and it has been discovered in many physical systems before they enter the chaotic regime. It has not been proven to be transcendental, but is generally believed to be.)

One of these numbers is Champernowne’s number. Champernowne's constant 0.12345678910111213... is the number obtained by concatenating the positive integers and interpreting them as decimal digits to the right of a decimal point. It is normal in base 10 [5,6]. It is named after mathematician D. G. Champernowne. Mahler [7] showed it to also be transcendental. Transcendental numbers have been studied in many fields [8-12].

2. Series representation of Champernowne’s Number

Lemma: Let b be the number of digit of n, than

$$0,123...9101112....=\sum_{n=1}^{9} n.10^{-n} + \sum_{n=10}^{\infty} n.10^{-n-\left(\sum_{k=2}^{b}(n-10^{k-1}+1)\right)} \qquad (2)$$

Proof: Let b be the number of digit of n , we may write

$$0,123456789101112... = 0.123456789 + 0,000000000101112... \qquad (3)$$

Since

$$0,123456789 = \sum_{n=1}^{9} n.10^{-n} \qquad (4)$$

we show that

$$0,000000000101112... = \sum_{n=10}^{\infty} n.10^{-n-\left(\sum_{k=2}^{b}(n-10^{k-1}+1)\right)}. \qquad (5)$$

Let us use induction method.

$$0,000000000101112... = 10.10^{-11}+11.10^{-13}+12.10^{-15}+...+99.10^{-189}+100.10^{-192}+... \qquad (6)$$

Let

$$f(n) = n + \sum_{k=2}^{b}\left(n-10^{k-1}+1\right), \qquad (7)$$

than

$$\sum_{n=10}^{\infty} n.10^{-n-\left(\sum_{k=2}^{b}(n-10^{k-1}+1)\right)} = \sum_{n=10}^{\infty} n.10^{-f(n)} \qquad (8)$$

i) For n = 10 we have;

$$10.\ 10^{-f(10)} = 10.10^{-10-\sum_{k=2}^{2}\left(10-10^{k-1}+1\right)} = 10.10^{-10-(10-10+1)} = 10.10^{-11} \qquad (9)$$

ii) Suppose that for n = T ,

$$T.10^{-f(T)} = T.10^{-T-\sum_{k=2}^{b}\left(T-10^{k-1}+1\right)} \qquad (10)$$

is true.

iii) We shall show that for n = T+1 the expression is true. It is easily see that (T+1).th term is equals to $(T+1).10^{-f(T)-m}$, here m is the number of digit of (T+1) .

If $T = 10^s-1$;$\exists s\in N$ we get;
$T+1 = 10^s$ and m = s+1 and b = s
One can see that

$$(T+1).10^{-f(T)-m} = (T+1).10^{-T-\left(\sum_{k=2}^{s}\left(T-10^{k-1}+1\right)\right)-(s-1)} = \qquad (11)$$

$$(T+1).10^{-(T+1)-\left(\sum_{k=2}^{s}\left(T-10^{k-1}+1\right)\right)-s} = (T+1).10^{-(T+1)-\left((T-10+1)+(T-100+1)+...+\left(T-10^{s-1}+1\right)\right)-s} = \qquad (12)$$

$$(T+1).10^{-(T+1)-\left(((T+1)-10+1)+((T+1)-100+1)+...+\left((T+1)-10^{s-1}+1\right)\right)-1}. \qquad (13)$$

Since (T+1) = 10^s than 1= [(T+1)-10^s+1] and than

$$(T+1).10^{-(T+1)-\left(((T+1)-10+1)+((T+1)-100+1)+...+\left((T+1)-10^{s-1}+1\right)\right)-1} = \tag{14}$$

$$(T+1).10^{-(T+1)-\left(((T+1)-10+1)+((T+1)-100+1)+...+\left((T+1)-10^{s-1}+1\right)+\left((T+1)-10^{s}+1\right)\right)} = \tag{15}$$

$$(T+1).10^{-(T+1)-\left(\sum_{k=2}^{s+1}\left((T+1)-10^{k-1}+1\right)\right)} = (T+1).10^{-(f(T+1))} \tag{16}$$

where s+1 is number of digit of (T+1).
This completes proof.
If T≠10^s-1 for every s∈N than the number of digits of T and (T+1) are equal and m = b
In this case for n=T+1 (T+1).th term is

$$(T+1).10^{-f(T)-b} = (T+1).10^{-T-\left(\sum_{k=2}^{b}\left(T-10^{k-1}+1\right)\right)-b} = \tag{17}$$

$$(T+1).10^{-(T+1)-\left((T-10+1)+(T-100+1)+...+\left(T-10^{b-1}+1\right)\right)-(b-1)} = \tag{18}$$

$$(T+1).10^{-(T+1)-\left(((T+1)-10+1)+((T+1)-100+1)+...+\left((T+1)-10^{b-1}+1\right)\right)} = \tag{19}$$

$$(T+1).10^{-(T+1)-\left(\sum_{k=2}^{b}\left((T+1)-10^{k-1}+1\right)\right)} = (T+1).10^{-(f(T+1))} \tag{20}$$

it shows that the assartion is true for (T+1).

3. Programing About Champernowne's Number

It is well known that irrational numbers do not contain cyclic blocks in their decimals and they have infinite decimal representation without a rule. There it is very difficult to find an arbitrary digit. In this section, a program which computes an arbitrary digit of 0,123...9101112... is given.

```
#include<stdio.h>
#include<math.h>
#include<conio.h>
#include<ctype.h>
main()
{
long unsigned int n,R,k,k1,l,a,b,c,m,t,d,s,e,f,d1,d2,d3,d4,d5,d6,bs,i,j;
char g;
do{
clrscr();
printf("                    Input the number....:  ");
scanf("%d",&n);
if(n<10)
printf("            \n The number is %d.",n);
```

```
else
{
k=2;
do{
l=pow(10,k)-1;
a=0;
for(b=2;b<=k;b++){
c=l-pow(10,b-1)+1;
a+=c;}
a=a+l;
f=k;
k++;
printf("\n                %d",a);
}while(n>=a);
printf("\n\n               f=%d",f);
m=0;
for(b=2;b<=f;b++){
c=1-pow(10,b-1);
m=m+c;}
m=n-m;
printf("\n\n               m=%d",m);
s=m%f;
printf("\n\n               s=%d",s);
t=(m-s)/f;
printf("\n\n               t=%d",t);
if(s==0){
e=t%10;
printf("\n\n               e=%d",e);
R=e;
printf("\n\n        The number is %d.",R);}
else{
d=t+1;
for(i=1;i<d;i++){
j=pow(10,i)-1;
printf("J=%d",j);
if(d<=j){
bs=i;
break;}}
printf("\n the number of digit: %d",bs);
s=bs-s+1;
printf("\n\n               d=%d",d);
d5=pow(10,s);
d1=d%d5;
printf("\n\n               d1=%d",d1);
d6=pow(10,s-1);
d2=d1%d6;
printf("\n\n               d2=%d",d2);
d3=d1-d2;
printf("\n              d3=%d",d3);
d4=d3/d6;
```

```
printf("\n\n                    d4=%d",d4);
printf("\n\n      The number is %d.",d4);
} }
printf("\n\n      Do you want to try again ( Y/N)  ?");
g=toupper(getch());
}while(g=='Y');
return 0;
}
```

4. A Proof of 0.123...9101112... is Transcendental

There are many transcendental number on the real line but it is difficult to show that if a given number is transcendental. In this section a different proof of the number 0.123...9101112... from the proof of Kurt Mahler is introduced that considered number is transcendental.

Proof:
Let

$$B = \sum_{n=1}^{9} n.10^{-n} + \sum_{n=10}^{\infty} n.10^{-n-\left(\sum_{k=2}^{b}(n-10^{k-1}+1)\right)} . \qquad (21)$$

Suppose B is algebraic, so that it satisfies some equation

$$f(x) = \sum_{j=0}^{m} c_j.x^j = 0 \qquad (22)$$

with integral coefficients. For any x satisfying 0 < x < 1, we have by the triangle inequality

$$\left|f^{'}(x)\right| = \left|\sum_{j=1}^{m} j.c_j.x^{j-1}\right| < \sum\left|j.c_j\right| = \mathrm{C}, \qquad (23)$$

where C is a constant and it is defined by the right side equation of the inequlity and it depends on only the coefficients of f(x). Define

$$B_t = \sum_{n=1}^{9} n.10^{-n} + \sum_{n=10}^{t} n.10^{-n-\left(\sum_{k=2}^{b}(n-10^{k-1}+1)\right)} \qquad (24)$$

so that

$$B - B_t = \sum_{n=t+1}^{\infty} n.10^{-n-\left(\sum_{k=2}^{b}(n-10^{k-1}+1)\right)} < 2.(t+1).10^{-(t+1)-\left(\sum_{k=2}^{b}((t+1)-10^{k-1}+1)\right)} \qquad (25)$$

For sufficiently large t we obtain

$$B-B_t < 2.(t+1).10^{-(t+1)-\left(\sum_{k=2}^{b}\left((t+1)-10^{k-1}+1\right)\right)} < 2.(t+1)^m.10^{(m+1)\left(-(t+1)-\left(\sum_{k=2}^{b}\left((t+1)-10^{k-1}+1\right)\right)\right)} \quad (26)$$

By the mean value theorem,

$\left|f(B)-f(B_t)\right|=\left|B-B_t\right|.\left|f^{'}(f)\right|$ for some number f between B and B_t. But we can see that for sufficiently large t;

$$\left|B-B_t\right|.\left|f^{'}(f)\right| < 2.\text{C}.\ (t+1)^m.10^{(m+1)\left(-(t+1)-\left(\sum_{k=2}^{b}\left((t+1)-10^{k-1}+1\right)\right)\right)} \quad (27)$$

$$\left|f(B)-f(B_t)\right|=\left|f(B_t)\right|=\left|\sum_{j=o}^{m}c_j.B_t^{\ j}\right| > (t)^m.10^{m.\left(-(t)-\left(\sum_{k=2}^{b}\left((t)-10^{k-1}+1\right)\right)\right)}, \quad (28)$$

because $c_j B_t^{\ j}$ is rational number with denominator $10^{j.\left((t)+\left(\sum_{k=2}^{b}\left((t)-10^{k-1}+1\right)\right)\right)}$.

Finally we observe that

$$2.\text{C}.(t+1)^m.10^{(m+1)\left(-(t+1)-\left(\sum_{k=2}^{b}\left((t+1)-10^{k-1}+1\right)\right)\right)} < (t)^m.10^{m.\left(-(t)-\left(\sum_{k=2}^{b}\left((t)-10^{k-1}+1\right)\right)\right)} \quad (29)$$

if t is sufficiently large. It is contradiction. This shows that 0,123456789101112... is transcendental.

5. Conclusions

Mahler proved that the Champernowne constant 0.1234567891011121314151617181920... is transcendental number in 1937 [13]. This proof made him well-known. In this study, we have gave different proof of the Champernowne's number is transcendental via using the series representation of the number. We belive that same series representation of numbers can use to proof of transcendence of the numbers.

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