International Journal of Engineering & Applied Sciences (IJEAS)
Vol.3, Issue 1(2011)71-79

VARIOUS CALCULATION METHODS FOR LARGE DEFLECTIONS OF NONLINEAR ELASTIC MATERIAL SUBJECTED TO A MOMENT

Eren, I.
Yıldız Technical University, Mechanical Engineering Department, 34349, İstanbul, Turkey
e-mail: er@yildiz.edu.tr

Received Date: 21 September 2010; Accepted Date: 25 October 2010

Abstract

Large deflections were calculated for nonlinear Ludwick type cantilever beams made of Ludwick type rectangular cross-sectional material subjected to a moment at the free end, by employing approximate and numerical methods for trial functions satisfying boundary conditions. Adequately approximate results were obtained for calculations employing single and binominal constant, second and fourth degree polynomial type trial functions. Increasing number of constant term and degree of trial functions resulted in more approximate values.

Keywords: Large Deflections, Geometrical Non-linearity, Material Non-linearity, Approximate and Numerical Methods.

1. Introduction

A large deflection of bearer systems under different load conditions is a well-known subject and various studies had been conducted about the subject. Due to the importance of the issue, today studies still continue. In many cases encountered in different engineering practices, results obtained by linearization are satisfactorily approximate. However, well-known curvature expression for elastically curve is not linear and also real materials do not have linear stress-strain relationship. When this fact is considered, deflections could not be calculated with analytical methods. Instead, approximate and numerical methods should be employed. Large deflections of uniform and non-uniform, concentrated or distributed loaded linear elastic cantilever beams were investigated in many studies [1-8]. Prathap and Varadan investigated large deflections of cantilever beams made of Ramberg-Osgood type nonlinear material, subjected to concentrated load at the free end. Same problem was solved by Varadan and Joseph for cantilever beams, for moment at the free end. Large deflections of cantilever beams made of Ludwick type nonlinear material subjected to concentrated load at the free end were investigated by Lewis and Monasa [11]. Same authors [12], solved the same problem for the moment at the free end. Lo and Gupta [13] investigated large deflections of rectangular cross-sectional beams, for deflection problems. Their method was to consider material stress-strain relationship as logarithmical beyond elastic limit. Lee [14] calculated large deflections of cantilever beams made of Ludwick type nonlinear material, for both uniform distributed load and concentrated load at the free end. Lately, Güven, Baykara and Bayer [15] calculated large deflections of free end of cantilever beams made of nonlinear Ludwick type bimodulus material (stress-strain relationship different for tension and compression) on which moment affecting on the free end. They defined these results in closed form and tabulated the numerical results depending on material constants.

2. Cantilever Beams Subjected to Moment at the Free End

In this study, vertical and horizontal deflections of uniform cantilever beams for L length, Shown in Figure 1, was calculated by selecting trial function satisfying boundary conditions and thus employing various numerical methods and Euler-Bernoulli curvature expression. Beam was assumed to be thin and non-elongating. Both material and geometrical nonlinearity were also assumed. The objective of this study is to determine the effect of geometrical and material nonlinearity on the large deflections and, show the difference between large deflections calculated with different methods for different moment values. Deflections obtained by assumed trial functions was compared to Reference values in order to determine the efficiency of trial functions employed in the study.

2.1. Methods and Formulation

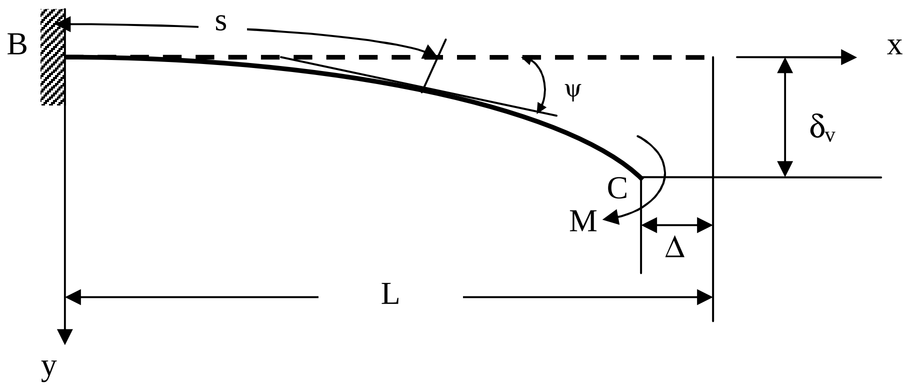

Fig. 1. Uniform cantilever beam moment affected on the free end.

In figure 1 Δ is the horizontal deflection, δ_v is the largest vertical deflection, M is the moment, ψ is the slope angle and s is the arc length.

For trial functions y(x), two different assumptions satisfying y(0)=0 and y'(0)= 0 boundary conditions, were employed;

$$y(x) = ax^2 \tag{1}$$

$$y(x) = cx^2 + ex^4 \tag{2}$$

Stress-strain relationship for Ludwick type material is as shown in below,

$$\sigma = B\varepsilon^{\frac{1}{n}} \tag{3}$$

Euler-Bernoulli curvature-moment expression for Ludwick type material,

$$\frac{d\psi}{ds} = \frac{y''(x)}{(1+(y'(x))^2)^{\frac{3}{2}}} = \frac{M^n}{K_n} \tag{4}$$

In this expression, for rectangular cross-section, K_n is assumed to be;

$$K_n = \frac{B^n h^{2n+1} b^n n^n}{2^{n+1}(1+2n)^n} \tag{5}$$

For total arc length, equation which is shown below, is employed

$$\int_0^{(L-\Delta)} \sqrt{(1+(y'(x))^2)} = L \tag{6}$$

when trial function (1) is written in the Equation (6), then arc length was

$$\int_0^{(L-\Delta)} \sqrt{(1+(2ax)^2)} = L \tag{7}$$

If this equation is integrated,

$$\frac{2a\sqrt{1+4a^2(L-\Delta)^2}(L-\Delta)+\text{ArcSinh}[2a(L-\Delta)]}{4a} = L \tag{8}$$

is obtained. When trial function (2) is written in the Equation (6), then arc length is;

$$\int_0^{(L-\Delta)} \sqrt{(1+(2ax)^2)} = L \tag{9}$$

Simpson method was employed for integration of Equation 9.

If right side of the Equation 4 was employed and $y(x) = ax^2$ trial function in Equation 1 and $y(x) = cx^2 + ex^4$ trial function in Equation 2 were considered, error function was obtained as shown below;

$$\varepsilon_\Omega = \frac{2a}{(1+(2ax)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n} \tag{10}$$

$$\varepsilon_\Omega = \frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n} \tag{11}$$

Number of equations that should be obtained by weighted residual methods; depend on number of constant terms in the trial function, for calculation of large deflections. Constant terms in horizontal deflection and trial functions are unknown, thus, 2 equations for single constant term trial function and 3 equations for double constant term trial functions, are required. One of the equations is to be evaluated from arc length equation and other will be derived from other methods that were employed in the study. Equations obtained for single constant term trial function was shown below only for Moments Method and Least-Squares Method.

3. Moment Method

If $y(x) = ax^2$ trial function from Equation 1 is employed and zero moment of error function in Equation 10 is integrated over the region according to moment method, then;

$$\int_{x=0}^{(L-\Delta)} \left(\frac{2a}{(1+(2ax)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n}\right)dx = 0 \tag{12}$$

If $y(x) = cx^2 + ex^4$ trial function from Equation 2 is employed and zero and first moment of error function in Equation 11 is integrated over the region according to moment method, then equations shown below are obtained:

$$\int_{x=0}^{(L-\Delta)} \left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n}\right)dx = 0 \tag{13}$$

$$\int_{x=0}^{(L-\Delta)} \left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n}\right)xdx = 0 \tag{14}$$

4. Least-Squares Method

If trial $y(x) = ax^2$ function from Equation 1 is employed and multiplication of partial derivation of constant term with error function in Equation 10, is integrated over the region according to least-squares method, then equation shown below is obtained:

$$\int_{x=0}^{(L-\Delta)} \left(\frac{2a}{(1+(2ax)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n}\right)\left(-\frac{24a^2x^2}{(1+4a^2x^2)^{\frac{5}{2}}} + \frac{2}{(1+4a^2x^2)^{\frac{3}{2}}}\right)dx = 0 \tag{15}$$

If trial $y(x) = cx^2 + ex^4$ function from Equation 2 is employed and multiplication of partial derivations of constant terms with error function in Equation 11, are integrated over the region according to least-squares method, then equations shown below are obtained:

$$\int_{x=0}^{(L-\Delta)} \left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n}\right)\left(-\frac{6x(2c+12ex^2)(2cx+4ex^3)}{(1+(2cx+4ex^3)^2)^{\frac{5}{2}}} + \frac{2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}}\right)dx = 0 \tag{16}$$

$$\int_{x=0}^{(L-\Delta)} \left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}} - \frac{M^n}{K_n}\right)\left(-\frac{12x^3(2c+12ex^2)(2cx+4ex^3)}{(1+(2cx+4ex^3)^2)^{\frac{5}{2}}} + \frac{12x^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}}\right)dx = 0 \tag{17}$$

5. Galerkin Method

Base functions in $y(x)=cx^2+ex^4$ trial function is assumed to be weight function and then multiplying error functions in Equation 11 with weight function gives the equations below;

$$\int_{x=0}^{(L-\Delta)}\left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}}-\frac{M^n}{K_n}\right)x^2dx=0 \tag{18}$$

$$\int_{x=0}^{(L-\Delta)}\left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}}-\frac{M^n}{K_n}\right)x^4dx=0 \tag{19}$$

6. Subregion Collocation Method

When error function in Equation 11 is examined in two sub regions and equaled to zero, the equations shown below can be written;

$$\int_{x=0}^{\frac{(L-\Delta)}{2}}\left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}}-\frac{M^n}{K_n}\right)dx=0 \tag{20}$$

$$\int_{x=\frac{(L-\Delta)}{2}}^{(L-\Delta)}\left(\frac{2c+12ex^2}{(1+(2cx+4ex^3)^2)^{\frac{3}{2}}}-\frac{M^n}{K_n}\right)dx=0 \tag{21}$$

7. Point Collocation Method

In this method, error function in Equation 11 is equaled to zero in two points selected over the region.

For $x=\frac{(L-\Delta)}{3}$,

$$\frac{2c+\frac{4}{3}e(L-\Delta)^2}{(1+(\frac{2}{3}c(L-\Delta)+\frac{4}{27}e(L-\Delta)^3)^2)^{\frac{3}{2}}}-\frac{M^n}{K_n}=0 \tag{22}$$

is obtained.

For $x=\frac{2(L-\Delta)}{3}$,

$$\frac{2c+\frac{16}{3}e(L-\Delta)^2}{\left(1+\left(\frac{4}{3}c(L-\Delta)+\frac{32}{27}e(L-\Delta)^3\right)^2\right)^{\frac{3}{2}}}-\frac{M^n}{K_n}=0 \qquad (23)$$

is obtained.

Vast majority of the integrations conducted in various methods, were executed according to Simpson method. By employing these equations and arc length equation, constant terms of trial function and horizontal deflection value, Δ, are calculated by Newton method. $y(L-\Delta)$ value for $x=L-\Delta$ condition in trial function gives largest deflection value at the free end.

8. Numerical Results

$\sigma = 66,1\varepsilon^{0.209}$, [12] (24)

In Table 1, large deflections of a NP8 aluminum alloy cantilever beam which have the dimensions of 50,8 cm , 2,54 cm , 0,635 cm by means of length, width and height respectively and behaving according to Equation 24, were given.

σ, stress unit is ksi. During the construction of Table 1, instead of value of 66.1 ksi for B, unit conversion was carried out and 0.455 10^5 N/cm^2 (0.455 GPa) was employed. Calculations in this study were conducted by Mathematica 5.2 software.

In Table 1, large deflection values calculated with various methods for different moment values can be seen. It is also possible to observe the deviations of the calculated values according to these methods compared to Reference values and effect of number of selected constant terms over the approximation to Reference result.

Table 1. Comparison of horizontal and vertical deflections calculated various methods for cantilever beams subjected to a moment at the free end

METHODS		M (Ncm)	2259,70	2485,67	2711,64	2937,60	3163,57	3389,54	3615,51	3841,48	3954,47
REFERENCE RESULTS		**Δ (cm)**	0,0843	0,2096	0,4811	1,0315	2,0833	3,9848	7,2390	12,4836	16,0579
		δv (cm)	2,5321	3,9901	6,0345	8,8024	12,4168	16,9395	22,2809	28,0495	30,8381
MOMENT METHOD (MM1)	**$y(x)=ax^2$**	**Δ (cm)**	0,0846	0,2118	0,4924	1,0823	2,2846	4,6682	9,0190	15,3405	18,7747
		δv (cm)	2,5396	4,0167	6,1227	9,0711	13,1606	18,7508	25,8685	33,2645	36,4608
Deviation for δv MM1 and Reference (%)			0,30	0,66	1,44	2,96	5,65	9,66	13,87	15,68	15,42
MOMENT METHOD (MM2)	**$y(x)=cx^2+ex^4$**	**Δ (cm)**	0,0843	0,2097	0,4815	1,0328	2,0891	4,0226	7,4857	13,4778	17,2452
		δv (cm)	2,5333	3,9921	6,0372	8,8061	12,4226	16,9551	22,3534	28,2715	31,0832
Deviation for δv MM2 and Reference (%)			0,05	0,05	0,04	0,04	0,05	0,09	0,32	0,79	0,79
LEAST-SQUARE METHOD (LSM1)	**$y(x)=ax^2$**	**Δ (m)**	0,0846	0,2116	0,4904	1,0619	2,1183	3,7225	5,6267	7,5517	8,4817
		δv (m)	2,5396	4,0167	6,1227	9,0711	13,1606	18,7508	25,8685	33,2645	36,4608
Deviation for δv LSM1 and Reference (%)			0,30	0,66	1,44	2,96	5,65	9,66	13,87	15,68	15,42
LEAST-SQUARE METHOD (LSM2)	**$y(x)=cx^2+ex^4$**	**Δ (cm)**	0,0843	0,2097	0,4815	1,0325	2,0862	3,9754	6,7543	9,4852	10,6736
		δv (cm)	2,5333	3,9920	6,0369	8,8043	12,4116	16,8702	21,5044	25,0277	26,3784
Deviation for δv LSM2 and Reference (%)			0,05	0,05	0,04	0,02	-0,04	-0,41	-3,61	-12,07	-16,91
GALERKIN METHOD (GM)	**$y(x)=cx^2+ex^4$**	**Δ (cm)**	0,0843	0,2097	0,4808	1,0259	2,0285	3,5789	5,1408	5,8826	6,0468
		δv (cm)	2,5332	3,9915	6,0326	8,7738	12,2246	15,9068	18,1679	17,3727	15,9357
Deviation for δv GM and Reference (%)			0,04	0,04	-0,03	-0,33	-1,57	-6,49	-22,64	-61,46	-93,52
SUBREGION METHOD (SM)	**$y(x)=cx^2+ex^4$**	**Δ (cm)**	0,0843	0,2097	0,4816	1,0331	2,0918	4,0377	7,5381	13,5203	17,2541
		δv (cm)	2,5333	3,9921	6,0375	8,8080	12,4330	17,0130	22,4948	28,4633	31,1569
Deviation for δv SM and Reference (%)			0,05	0,05	0,05	0,06	0,13	0,43	0,95	1,45	1,02
POINT COLLOCATION METHOD (PCM)	**$y(x)=cx^2+ex^4$**	**Δ (cm)**	0,0843	0,2097	0,4813	1,0309	2,0735	3,9105	6,8174	10,7048	12,8348
		δv (cm)	2,5332	3,9919	6,0364	8,8003	12,9609	16,7918	21,6928	26,3255	28,2676
Deviation for δv PCM and Reference (%)			0,04	0,05	0,03	-0,02	4,20	-0,88	-2,71	-6,55	-9,09

9. Conclusions

In this paper, large deflections of cantilever beams made of Ludwick type nonlinear material subjected to a moment at the free end, were investigated by employing polynomial, single and binominal termed approximate trial functions that satisfy boundary conditions.

Single termed trial function assumed to be $y(x)=ax^2$, is a very simplified expression. It is logic to think that this simple trial function would not give good results for calculated deflections for such a complex problem involving both geometrical and material non-linearity. However, vertical deflection deviations are below %6 compared to Reference value, even decreasing to %0.3, for Moment and Least Squares Methods. Assuming trial function as $y(x)=ax^2$, deflections were calculated by employing Moment and Least Squares Methods. When $y(x)=ax^2$ trial function employed for Galerkin, Subregion Collocation and Point Collocation Methods, beyond determined moment values, appropriate roots giving the constant terms and horizontal deflections, could not be obtained. Thus, calculations were not made for these three methods.

For double termed trial function $y(x)=cx^2+ex^4$, best results were achieved by employing Moment and Subregion Collocation Methods for obtaining vertical deflection values that agree with Reference values. Minimum deviation for Moment method was %0.05 and maximum deviation was 0.79 and for Subregion Collocation Method minimum deviation was %0.05 and maximum deviation was %1.45. Other methods also were given deviations below %1, for vast majority of the moment values. However, for some massive moment values, deviations were too high for some methods and smaller for other methods, but increasing with the value of the moment.

$y(x)=cx^2+ex^4$ trial function is actually a simple polynomial. Approximation of calculated deflection values to %0 levels is really interesting and notable. Double constant termed trial function gives better results than single constant termed trial functions in aspect of calculated deflections. Increasing number of constant terms would give better results. However, increasing number of components of trial function would result in more complex equations, which would also complicate the calculations.

Another important highlight is the close results of Point Collocation Method compared to Reference values. What makes this result interesting is the simplicity of method. The method gives the simplest results without employing any complex integration. For vast majority of the moment values, deviations are below %1 and decrease down to %0.02

References

1. Bisshopp, K. E. and Drucker, D.C. Large Deflections of Cantilever Beams, *Q. Appl. Math.*,**3**: 272-275. (1945).
2. Scott, E. J. Carver, D. R. and Kan M. On the Linear Differential Equation for Beam Deflection, *J. Appl. Mech.*, **22**: 245-248. (1955).
3. Lau, J. H. Large Deflections of Beams with Combined Loads, *J. Eng. Mech.*, **108**: 180-185. (1982).
4. Rao, B. N. and Rao, G. V. On the Large Deflection of Cantilever Beams with End Rotational Load, *Z. Angew-Math. Mech.*, **66**: 507-509. (1986).

5. Baker, G. On the Large Deflections of Non- prismatic Cantilevers with a Finite Depth, *Comput. Struct.*, **46**: 365-370. (1993).
6. Lee, B. K., Wilson, J.F. and Oh, S.J. Elastica of Cantilevered Beams with Variable Cross Sections, *Int. J. Non-Linear Mech*., **28**: 579-589. (1993).
7. Frisch-Fay, R. *Flexible Bars*, Butter Worths, London. (1962).
8. Fertis, D. G. *Nonlinear Mechanics*, CRC Pres, New York. (1999).
9. Prathap, G. and Varadan, T. K. The Inelastic Large Deformation of Beams, *J. Appl. Mech*., **43**: 689-690. (1976).
10. Varadan, T. L. and Joseph, D. Inelastic Finite Deflections of Cantilever Beams, *J. Aeronaut Soc. India*, **39**: 39-41. (1987).
11. Lewis, G. And Monosa, F. Large Deflections of Cantilever Beams of Non-linear Materials, *Comput. Strct.*, **14**: 357-360. (1981).
12. Lewis, G. And Monosa, F. Large Deflections of Cantilever Beams of Non-linear Materials of the Ludwick Typr Subjected to an End Moment, *Int. J. Non-Linear Mech.*, **17**: 1-6. (1982).
13. Lo, C.C. and Gupta, S.D. Bending of a Nonlinear Rectangular Beam in Large Deflection, J. Appl. Mech., 45: 213-215. (1978).
14. Lee, K. Large Deflections of Cantilever Beams of Non-linear Elastic Material under a Combined Loading, Int. J. Non-Linear Mech., 37: 439-443. (2002).
15. Güven, U., Baykara, C., Bayer, İ., Large deflections of a cantilever beam of nonlinear bimodulus material subjected to an end moment, J.Reinforced Plastics and Composites, Vol.24, 12:1321-1326. (2005).