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ANALYTICAL SOLUTIONS FOR THE TORSIONAL VIBRATIONS OF VARIABLE CROSS-SECTION RODS

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Abstract

The objective of this paper is to present exact analytical solutions for the torsional vibration of rods with non-uniform cross-section. Using appropriate transformations the equation of motion of torsional vibration of a rod with varying cross-section is reduced to analytically solvable standard differential equations whose form depends upon the specific area variation. Solutions are obtained for a rod with for a polynomial area variation. The solutions are obtained in terms of special functions such as Bessel and Neumann functions. Simple formulas to predict the natural frequencies of non-uniform rods with various end conditions are presented. The natural frequencies of variable cross-section rods for these end conditions are calculated and their dependence on taper is discussed.

Keywords: Torsional Vibration, Isotropic, Rod, Variable Cross-Section, Analytical Solution

1. Introduction

The vibration of beams and rods has been studied extensively and is still receiving attention in the literature. Non-uniform beams and rods may provide a better or more suitable distribution of mass and strength than uniform beams and therefore can meet special functional requirements in architecture, robotics, aeronautics and other innovative engineering applications. Thus, it is necessary to determine the natural frequencies and mode shapes in the vertical direction for high-rise structures at the design stage for certain cases. When analyzing the free vibrations of high-rise structures, it is possible to regard such structures as a cantilever rod with varying cross-sections. However, in general, it is not possible or, at least, very difficult to get the exact analytical solutions of differential equations for free vibrations of rods with variably distributed mass and stiffness. These exact rod solutions are available only for certain rod shapes and boundary conditions. Nagaraj and Sahu [1] studied the torsional vibrations of non-uniform pre-twisted rotating blades by using finite element methods based on both the Rayleigh-Ritz and Galerkin formulations. Rezeka [2] presented several cases of variable cross-section circular shaft with variable wall thickness. Eisenberger [3] gives exact solutions for the torsional vibration frequencies of symmetric variable and open cross-section bars. He derived an analytical method to form the dynamic stiffness matrix of the bar, including the effect of warping. Li [4] investigated the torsional vibration of multi-step non-uniform rods with various concentrated elements. He obtained the exact solutions for the free torsional vibration of non-uniform rods whose variations of cross-section were described by exponential functions and power functions.

Previous studies clearly show that vibration characteristics of isotropic rods with continuously changing cross-section have significant features and are not yet fully addressed. The present study investigates free torsional vibrations of isotropic rods with varying cross-section. The object is to obtain analytical solutions describing the vibration behavior of the rods under different boundary conditions and to determine the effects of continuously variable cross-section on the natural frequencies and mode shapes.

2. Analysis

Consider an isotropic rod with a variable cross-section. Dimensionless variables are defined according to

$$t = \frac{t^*}{L}\sqrt{\frac{G}{\rho}} \qquad x = \frac{x^*}{L} \qquad \psi = \frac{\psi^*}{\Psi} \qquad I = \frac{I^*}{I_0^*} \tag{1}$$

where t^* is the dimensional time, x^* is the dimensional coordinate measured from the left end of the rod along its length, I^* is the dimensional polar moment of inertia of the cross-section of the rod respectively, ψ^* is the dimensional twist angle, ρ is the mass density per unit are of the rod, E is the Young's modulus, L is the length of the beam, Ψ is any reference twist angle and J_0 is the polar moment of inertia of the cross-section of the rod at the left end of the rod where $x = 0$ that is $\square_0^* = \square_0^*(0)$. Governing equation in the dimensionless form can be written as follows:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{I'(x)}{I(x)}\frac{\partial \psi}{\partial x} - \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{2}$$

Solution of the Eq. (2) can be assumed in the following form:

$$\psi(x,t) = \Phi(x)\Upsilon(t) \tag{3}$$

Substitution of Eq. (3) into Eq. (2) yields two ordinary differential equations.

$$\Phi'' + \frac{I'(x)}{I(x)}\Phi' + \Omega^2\Phi = 0 \tag{4}$$

$$\Upsilon'' - \Omega^2\Upsilon = 0 \tag{5}$$

Here Ω is a real constant and defined as $\square^2 = \square^2\square\square^2/\square$ and $\square$ is circular frequency. Solution of Eq. (5) is well known and can be written as

$$\Upsilon(t) = c_1 \sin(\Omega t) + c_2 \cos(\Omega t) \tag{6}$$

Equation (4) has variable coefficients. Therefore, exact solutions of this equation for a general polar moment of inertia $I(x)$ cannot be obtained. However, for certain specific area variations, exact solutions can be obtained. In the following sections, using appropriate transformations, equation (4) will be reduced to analytically solvable differential equations for $I(x) = (1 + \frac{\alpha x}{L})^n$ (polynomial variation).

2.1. Solution for polynomial area variations

In order to obtain an exact solution, equation (4) is rewritten with $I(x)$ as the independent variable [27, 28], yielding

$$\left(\frac{dI}{dx}\right)^2\left(\frac{d^2\Phi}{dI^2}\right)+\left(\frac{1}{I}\right)\left(\frac{d}{dx}\right)\left[I\frac{dI}{dx}\right]\left(\frac{d\Phi}{dI}\right)+\Omega^2\Phi=0 \tag{7}$$

The above equation is solved for a rod with a polar moment of inertia variation that is given by the following expression

$$I(x)=\left(1+\frac{ax}{L}\right)^n \tag{8}$$

Equation (7) can be re-written as

$$\frac{d^2\Phi}{dI^2}+\left(\frac{1}{I}\right)(n)(2n-1)\left(\frac{d\Phi}{dI}\right)+\left(\frac{\Omega^2}{a^2n^2}\right)\left(\frac{1}{J^{2-2/n}}\right)\Phi=0 \tag{9}$$

To simplify equation (9) the following variables ξ and ζ which replace Φ and J respectively are introduced

$$\frac{\Phi}{\xi}=I^\alpha \qquad \frac{\zeta}{\gamma}=I^\beta \tag{10,11}$$

$$\alpha=\frac{1-n}{2n} \qquad \beta=\frac{1}{n} \qquad \gamma=\frac{\Omega L}{a} \tag{12}$$

Transforming equation (9) from Φ-I space to ξ-ζ space yields the Bessel equation

$$\frac{d^2\xi}{d\zeta^2}+\left(\frac{1}{\zeta}\right)\left(\frac{d\xi}{d\zeta}\right)+\left(1-\frac{\nu^2}{\zeta^2}\right)\xi=0 \tag{13}$$

where *n* is given by

$$\nu=\frac{1-n}{2} \tag{14}$$

$$\xi=c_1J_\nu(\zeta)+c_2Y_\nu(\zeta) \qquad \textit{When } \upsilon \textit{ is an integer} \tag{15a}$$

$$\xi=c_1J_\nu(\zeta)+c_2J_{-\nu}(\zeta) \qquad \textit{When } \upsilon \textit{ is not an integer} \tag{15b}$$

Therefore the twist angle can be written as

$$\Phi=I^\alpha[c_1J_\nu(\gamma I^\beta)+c_2Y_\nu(\gamma I^\beta)] \qquad \textit{When } \upsilon \textit{ is an integer} \tag{16a}$$

$$\Phi=I^\alpha[c_1J_\nu(\gamma I^\beta)+c_2J_{-\nu}(\gamma I^\beta)] \qquad \textit{When } \upsilon \textit{ is not an integer} \tag{16b}$$

2.2. Numerical results

The natural frequencies of a rod with its polar moment of inertia varying according to the equation

$$I(x)=\left(1+\frac{ax}{L}\right)^2 \tag{17}$$

is discussed in this section.

For the case of $I(x) = (1 + \frac{ax}{L})^2$ the solution is

$$\Phi = (\frac{1}{I^{\frac{1}{4}}})[c_1 J_{1/2}\left(\frac{\Omega L}{a}\sqrt{I}\right) + c_2 J_{-1/2}\left(\frac{\Omega L}{a}\sqrt{I}\right)] \quad (18)$$

It can easily be shown that for the case of n=2 the solution given by equation (18) can be re-written as

$$\Phi = \left(\frac{1}{\sqrt{I}}\right)[c_1 sin\left(\frac{\Omega L}{a}\sqrt{I}\right) + c_2 cos\left(\frac{\Omega L}{a}\sqrt{I}\right)] \quad (19)$$

The natural frequencies of a rod with its polar moment of inertia varying according to the equation
$I(x) = (1 + \frac{ax}{L})^4$
is discussed in this section.

Classical boundary conditions to the rod are considered here, as follows.

Free-Free	$\Phi'(0) = 0$	$\Phi'(1) = 0$	(20)
Fixed-Free	$\Phi(0) = 0$	$\Phi'(1) = 0$	(21)
Fixed-Fixed	$\Phi(0) = 0$	$\Phi(1) = 0$	(22)

For a fixed-fixed rod, the boundary conditions yields the set of two homogeneous algebraic equations:

$$\left(\frac{1}{I_0^{\frac{3}{8}}}\right)[c_1 J_{-\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_0^{\frac{1}{4}}\right) + c_2 J_{\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_0^{\frac{1}{4}}\right) = 0 \quad (23)$$

$$\left(\frac{1}{I_1^{\frac{3}{8}}}\right)[c_1 J_{-\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_1^{\frac{1}{4}}\right) + c_2 J_{\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_1^{\frac{1}{4}}\right) = 0 \quad (24)$$

Where

$$I_1 = I(L) = (aL + b)^4 \quad (25)$$

Since these equations are homogeneous, they are solvable only when their determinant vanishes, which yields the relationship for the non-dimensional eigenvalue β:

$$J_{-\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_0^{\frac{1}{4}}\right) J_{\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_1^{\frac{1}{4}}\right) - J_{\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_0^{\frac{1}{4}}\right) J_{-\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_1^{\frac{1}{4}}\right) = 0 \quad (26)$$

Table 1 shows the eigenvalues for uniform rods (a=0) and for tapered rods (fixed-fixed) with a=0,1,2, b=1 and L=1. The natural frequencies are presented in terms of β where $\beta = \omega\sqrt{\frac{\rho}{E}}$ .For uniform rods $\beta L = j\pi$, where j is an integer, the mode number. Table 1 indicates that the lowest natural frequencies are affected most by the taper. For higher modes, the natural frequencies are close to that of a uniform rod. The mode shape is given by

$$\psi = c_1 I^{-3/8}[J_{-\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I^{\frac{1}{4}}\right) - [\frac{J_{-\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_0^{\frac{1}{4}}\right)}{J_{\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I_0^{\frac{1}{4}}\right)}] J_{\frac{3}{8}}\left(\left[\frac{\beta}{a}\right] I^{\frac{1}{4}}\right) \qquad (27)$$

For Fixed-free rods, the boundary conditions results in the transcendental equation for the eigenfrequency.

$$J_{-\frac{3}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right)\left[\frac{-3}{aL+b}J_{\frac{3}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) - \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{11}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) + \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{-5}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right)\right] - J_{\frac{3}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right)\left[\frac{-3}{aL+b}J_{\frac{-3}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) - \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{5}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) + \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{-11}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right)\right] = 0 \qquad (28)$$

Table 1 shows the eigenvalues for uniform rods (a=0) and for tapered rods (fixed-free) with *a*=0,1,2, *b*=1 and *L*=1.For uniform rods $\beta L = \frac{1}{2}(2j-1)\pi$ where *j* is an integer, the mode number. Table 1 indicates that, for fixed-free rods, the lowest natural frequencies are affected most by the taper, For higher modes, the natural frequencies are close to that of a uniform rod. It is also interesting to note that taper reduces the natural frequency, and the first mode disappears. (The first mode is present until *a*=0.97)

For free-free rods, the boundary conditions results in the transcendental equation for eigenfrequency

$$PS - QR = 0$$

where

$$P = -\frac{3}{b}J_{-\frac{3}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right) - \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{5}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right) + \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{-11}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right),$$

$$Q = -\frac{3}{aL+b}J_{-\frac{3}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) - \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{5}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) + \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{-11}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right),$$

$$R = -\frac{3}{b}J_{\frac{3}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right) - \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{11}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right) + \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{-5}{8}}\left(\beta\frac{I_0^{\frac{1}{4}}}{a}\right), \qquad (29)$$

$$S = -\frac{3}{aL+b}J_{\frac{3}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) - \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{11}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right) + \beta\frac{I_0^{\frac{1}{4}}}{a}J_{\frac{-5}{8}}\left(\beta\frac{I_1^{\frac{1}{4}}}{a}\right).$$

Table 1 shows the eigenvalues for uniform rods (a=0) and for tapered rods (free-free) with *a*=0,1,2, *b*=1 and *L*=1.For uniform rods $\beta L = j\pi$ where *j* is an integer, the mode number. Table 1 indicates that, like the cases of fixed-fixed and fixed-free rods, the lowest natural frequencies are affected most by the taper. For higher modes, the natural frequencies are close to that of a uniform rod.

3. Discussion

The authors have been able to obtain the solution to the problem for the case of a polynomial polar moment of inertia variation, by transforming the differential equation such that the inertia I is the independent variable. The solutions are obtained as direct functions of inertia. It may be possible that other problems can be solved by using this approach.

4. Conclusion

Exact analytical solutions describing the torsional vibration of rods were obtained by transforming the equation of motion to standard differential equations which are analytically solvable in terms of special functions. Solutions are obtained for a rod with a polynomial polar moment of inertia variation. The solutions are obtained in terms of special functions such as Bessel and Neumann as well as trigonometric functions. Simple formulas to predict the natural frequencies of non-uniform beams with various end conditions are presented. It is shown that the lowest natural frequencies are affected most by the taper.

Table 1. Non-dimensional natural frequencies of rod with three different boundary conditions

a	Mode Number	Non-dimensional natural frequencies		
		Free-Free	Fixed-Fixed	Fixed-Free
0	1	3.141593	3.141593	1.570796
	2	6.283185	6.283185	4.712389
	3	9.424778	9.424778	7.853982
	4	12.566371	12.566371	10.995574
	5	15.707963	15.707963	14.137167
1	1	3.378458	3.133487	-
	2	6.425906	6.278921	4.487482
	3	9.524152	9.421905	7.721747
	4	12.642120	12.564210	10.901630
	5	15.769030	15.706230	14.064260
2	1	3.286891	3.125646	-
	2	6.614998	6.272251	4.404069
	3	9.671519	9.417264	7.672932
	4	12.759890	12.560670	10.866970
	5	15.983120	15.703370	14.037360

For higher modes, the natural frequencies are close to that of a uniform rod. The expressions obtained in this analysis are in terms of Bessel and trigonometric functions and are easy to evaluate. These closed form expressions presented herein can be used also as benchmarks for checking the results obtained from numerical or approximate methods.

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