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ROUTH-PADÉ APPROXIMANTS OF CONTINUOUS TIME SISO SYSTEMS BASED ON MULTI-OBJECTIVE OPTIMIZATION BY GRAPHICAL PROCEDURE

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Abstract

A multi-objective optimization based graphical procedure to derive a reduced order (rth-order) approximant for given (stable) SISO linear continuous-time system is presented. In this method, stability and the first r time moments/Markov parameters are preserved as well as the errors between last r time moments/Markov parameters of the system and those of the model are minimized. The method is useful as it alleviates the problems of deciding the values of number of error functions to be minimized and values of weights on the errors in arriving at good approximants.

Keywords: Model reduction, Padé approximation, Routh criterion, VEGA.

1. Introduction

The usefulness of techniques for deriving low-order approximations of high-order systems has already been accepted due to the advantages of reduced computational effort and increased understanding of the original system. Consequently, a large number of time-domain and frequency-domain systems. Simplification techniques have been developed to suit different requirements. Amongst them, a frequency domain method is Padé approximation in which $2r$ terms of the power series expansion (time moments/Markov parameters) are fully retained in low-order (rth-order) model $G_r(s)$ of the high-order (nth-order) transfer function $G_n(s)$. The Padé approximation does not guarantee the stability of the reduced-order model. To overcome the problem of stability, several stable reduction methods such as Routh approximation, the Hurwitz polynomial approximation, the stability equation method and the method using Michailov stability criterion have been proposed. Recently, geometric programming based (computer-oriented) methods [11-13] for the solution of the Routh-Padé approximation problem are presented. In these methods, first r time moments/Markov parameters are fully retained and the weighted sum of squares of the errors between a set of subsequent time moments/Markov parameters of the system and those of the model are minimized while preserving stability. These methods have the drawback that the question of finding some means (free of hit and trial) of deciding the values of the number of time moments/Markov parameters (say m) to be matched or near-matched and the weights to correspond to assured substantial improvement in system approximation as well as the question of establishing the existence of such values are left unresolved.

In this note, a nonlinear programming, Vector Evaluated Genetic Algorithm (VEGA) based graphical procedure for the solution of Routh-Padé approximation problem is presented. The method is essentially a multi-objective optimization procedure in which not only stability is preserved and the first *r* terms of the power series expansion of $G_n(s)$ are fully retained but also the errors between last *r* time moments/Markov parameters of the system and those of the model are minimized.

2. Model reduction problem viewed as a non-linear optimization problem and review of the existing methods

The system under consideration for ready reference is reproduced here:

$$G_n(s) = \frac{a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}{s^n + b_1 s^{n-1} + \ldots + b_n} \quad (1)$$

$$= t_1 + t_2 s + \ldots + t_n s^{n-1} + \ldots \quad (2)$$

(expansion around $s = 0$)

$$= M_1 s^{-1} + M_2 s^{-2} + \ldots + M_n s^{-n} + \ldots \quad (3)$$

(expansion around $s = 0$)

Assume that a reduced order (r th-order) model of the form

$$\hat{G}_r(s) = \frac{\hat{a}_1 s^{r-1} + \hat{a}_2 s^{r-2} + \ldots + \hat{a}_r}{s^r + \hat{b}_1 s^{r-1} + \ldots + \hat{b}_r}, r < n \quad (4)$$

$$= t_1 + t_2 s + \ldots + t_n s^{n-1} + \ldots \quad (5)$$

$$= M_1 s^{-1} + M_2 s^{-2} + \ldots + M_n s^{-n} + \ldots \quad (6)$$

is required.

3. Formulation of the Multi-Objective Optimization Problem (MOOP)

It is easy to verify that the following equations hold true:

$$\hat{t}_1 = \begin{cases} \dfrac{\hat{a}_r}{\hat{b}_r} & i = 1 \\ \left(\hat{a}_{r-i} \sum_{j=1}^{i-1} \hat{t}_j \hat{b}_{r+j-1}\right) \hat{b}_r^{-1} & i = 2,3,\ldots \end{cases} \quad (7)$$

($\hat{a}_i = \hat{b}_i = 0$ for $i = r+1, r+2, \ldots$)

$$\hat{M}_i = \begin{cases} \hat{a}_i & i = 1 \\ \hat{a}_i - \sum_{j=1}^{i-1} \hat{M}_j \hat{b}_{i-j} & i = 2,3,\ldots \end{cases} \quad (8)$$

($\hat{a}_i = 0$ for $i \le 0$; $\hat{b}_0 = 1$; $\hat{b}_i = 0$ for $i \le -1$)

We seek a stable model for which *r* equations given by

$$\left.\begin{array}{l}\hat{t}_i - t_i = 0 \\ \hat{M}_j - M_j = 0\end{array}\right\} \quad \begin{array}{l} i = 1,\ldots,\lambda \\ j = 1,\ldots,r-\lambda\end{array} \tag{9}$$

are satisfied, which implies, from (7),

$$\left.\begin{array}{r}\hat{a}_{r+1-i} = \sum_{j=1}^{i} t_j \hat{b}_{r-i+j} \\ \hat{a}_i = \sum_{j=1}^{r-\lambda} M_j \hat{b}_{i-j}\end{array}\right\} \quad \begin{array}{l} i = 1,\ldots,\lambda \\ j = 1,\ldots,r-\lambda\end{array} \tag{10}$$

There exist an infinite number of stable models for which (10) is satisfied. This arbitrariness in stability preservation can be exploited to minimize square of the errors of matching of next *r* time-moments and Markov-parameters of the system with those of the model (a total of *2r* terms). This problem is converted to MOOP and VEGA is used to generate Pareto-optimal solutions by minimizing objective functions $z^t_{\lambda+k}$, $z^M_{r-\lambda+l}$ given by:

$$\left.\begin{array}{l} z^t_{\lambda+k} = (1 - \frac{\hat{t}_{\lambda+k}}{t_{\lambda+k}})^2 \\ z^M_{r-\lambda+l} = (1 - \frac{\hat{M}_{r-\lambda+l}}{M_{r-\lambda+l}})^2 \end{array}\right\} \quad k = 1,\ldots,\overline{m} \;\; ; l = 1,\ldots r - \overline{m} \tag{11}$$

using (8) subject to. (9), (11) takes the form

$$\left.\begin{array}{r} z^t_{\lambda+k}(\hat{\mathbf{b}}) = f(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r) \\ z^M_{r-\lambda+l}(\hat{\mathbf{b}}) = f(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_r) \end{array}\right\} \quad k = 1,\ldots,\overline{m}; l = 1,\ldots r - \overline{m} \tag{12}$$

4. Formulation of the Stability Constraints

Now the denominator polynomial of (4) can be expressed as:

$$\left.\begin{array}{c} s^r + \hat{d}_1 s^{r-1} + (\hat{d}_2 + \hat{d}_3 + \ldots + \hat{d}_r) s^{r-2} + \hat{d}_1(\hat{d}_3 + \hat{d}_4 + \ldots + \hat{d}_r) s^{r-3} \\ + [\hat{d}_2(\hat{d}_4 + \hat{d}_5 + \ldots + \hat{d}_r) + \hat{d}_3(\hat{d}_5 + \hat{d}_6 + \ldots + \hat{d}_r) + \hat{d}_4(\hat{d}_6 + \hat{d}_7 + \ldots + \hat{d}_r) + \ldots \\ + \hat{d}_{r-2}\hat{d}_r] s^{r-4} + \ldots + \hat{d}_{1+q}\hat{d}_{3+q}\ldots\hat{d}_{r-2}\hat{d}_r \end{array}\right\} \tag{13}$$

which is constructed by taking the coefficients of the first two rows of the Routh array with the elements of its first column given by:

$$\hat{d}_1, \hat{d}_2, \hat{d}_1\hat{d}_3, \hat{d}_2\hat{d}_4, \hat{d}_1\hat{d}_3\hat{d}_5, \ldots, \hat{d}_{1+q}\hat{d}_{3+q}\ldots\hat{d}_{r-2}\hat{d}_r \tag{14}$$

where $q=1$ for r even and $q=0$ for r odd. By setting:

$$\hat{d}_1\hat{b}_1^{-1}=1,(\hat{d}_2+\hat{d}_3+...+\hat{d}_r)\hat{b}_2^{-1}=1,$$
$$\hat{d}_1(\hat{d}_3+\hat{d}_4+...+\hat{d}_r)\hat{b}_3^{-1}=1,...,(\hat{d}_{1+q}\hat{d}_{3+q}...\hat{d}_{r-2}\hat{d}_r)\hat{b}_r^{-1}=1 \tag{15}$$

is matched with the denominator polynomial of the model in (4), namely, with

$$\hat{b}_r+\hat{b}_{r-1}s+...+\hat{b}_1s^{r-1}+s^r \tag{16}$$

and the necessary and the sufficient condition that all the roots of (16) be strictly in the left half plane is

$$\hat{d}_1>0,\hat{d}_2>0,...,\hat{d}_r>0 \tag{17}$$

which, of course, implies

$$\hat{b}_1>0,\hat{b}_2>0,...,\hat{b}_r>0\ . \tag{18a}$$

Note that, for a given *r*, $\hat{b}_i$ $i=1,...,r$, can easily be expressed in terms of $\hat{d}_i$, $i=1,...,r$, by constructing an inverse Routh array (i.e., with the element of its first column given by (14)). Thus, pertaining to *r*= 4, (15) becomes

$$\hat{b}_1=\hat{d}_1,\ \hat{b}_2=\hat{d}_3+\hat{d}_3+\hat{d}_4,\ \hat{b}_3=\hat{d}_1(\hat{d}_3+\hat{d}_4),\ \hat{b}_4=\hat{d}_2\hat{d}_4 \tag{18b}$$

5. Problem statement

The problem is to minimize (12) subject to (15) and (17), (18).

6. Application of VEGA

The vector evaluated genetic algorithm (VEGA) [14] is proposed herein for solving the above stated problem. VEGA is the simplest possible multi-objective GA [14] and is straightforward extension of a single-objective extension of multi-objective optimization. Since a number of objectives (say Q) have to be handled, GA population is divided at every generation into Q equal subpopulations randomly. Each subpopulation is assigned a fitness value based on different objective function. After each solution is assigned a fitness value, the selection operator restricted among solutions of each subpopulation, is applied until the complete subpopulation is filled [14].

7. Examples

The step-by-step procedure to obtain reduced-order model is explained with the help of the examples presented below.

Example 1

Consider a stable third-order system [22]

$$G_s(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2} \tag{19}$$
$$(t_1 = 1, t_2 = 0.5, t_3 = 0.75;$$
$$M_1 = 8, M_2 = -26, M_3 = 66).$$

Suppose a second-order approximant ($r = 2$) is required. The system considered in this example is critically damped. For obtaining better reduced-order model of this type of the system, equal number of time-moments and Markov parameters of the system are to be retained or near-retained in the model. The Routh-Padé approximants can systematically be arrived at, by following steps given below:

Step 1: From the requirement of the first *r* terms matching (with $\lambda = 1$), one has

$$\left.\begin{aligned}\hat{a}_2 &= t_1\hat{b}_2\\ \hat{a}_1 &= M_1\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\hat{a}_2 &= \hat{b}_2\\ \hat{a}_1 &= 8.\end{aligned}\right. \tag{20}$$

Step 2: From (8) together with (9), one obtains

$$\left.\begin{aligned}\hat{t}_2 &= (-\hat{b}_1 + 8)\hat{b}_2^{-1}\\ \hat{M}_2 &= \hat{b}_2 - 8\hat{b}_1\end{aligned}\right\} \tag{21}$$

while (15) takes the following form

$$\left.\begin{aligned}\hat{b}_1 &= \hat{d}_1\\ \hat{b}_2 &= \hat{d}_2\end{aligned}\right\} \tag{22}$$

Step 3: Using (22), objective function (12) and (17a) take the following forms respectively (for $\overline{m}$ =1)

$$\left.\begin{aligned}z_2^t &= (1 - \frac{\hat{t}_2}{t_2})^2 &= (1 - \frac{8\hat{d}_2^{-1} - \hat{d}_1\hat{d}_2^{-1}}{0.5})^2\\ z_2^M &= (1 - \frac{\hat{M}_2}{M_2})^2 &= (1 - \frac{\hat{d}_2 - 8\hat{d}_1}{-26})^2\end{aligned}\right\} \tag{23}$$

subject to constraints $\hat{d}_1 > 0, \hat{d}_2 > 0$. (24)

Step 4: The problem is to minimize (23) subject to (24). Following VEGA [14] parameters has been used to obtain the optimal values of $\hat{d}_1$ and $\hat{d}_2$ (refer Table1)

Table 1. Assigned Fitness

Solution	b_1	b_2	f_1	f_2	Partition	Assigned Fitness
1	2.951056	5.201056	0.0886436	0.08527766	1	0.886436
2	2.951056	4.951056	1.0806434	0.07975278	1	1.0806434
3	4.451056	4.951056	0.1880176	0.0320378	1	0.1880176
4	3.951056	4.951056	0.403972	0.000693	2	0.000693
5	4.201056	5.201056	0.212369	0.00857	2	0.00857
6	3.951056	4.951056	0.40397	0.000693	2	0.000693

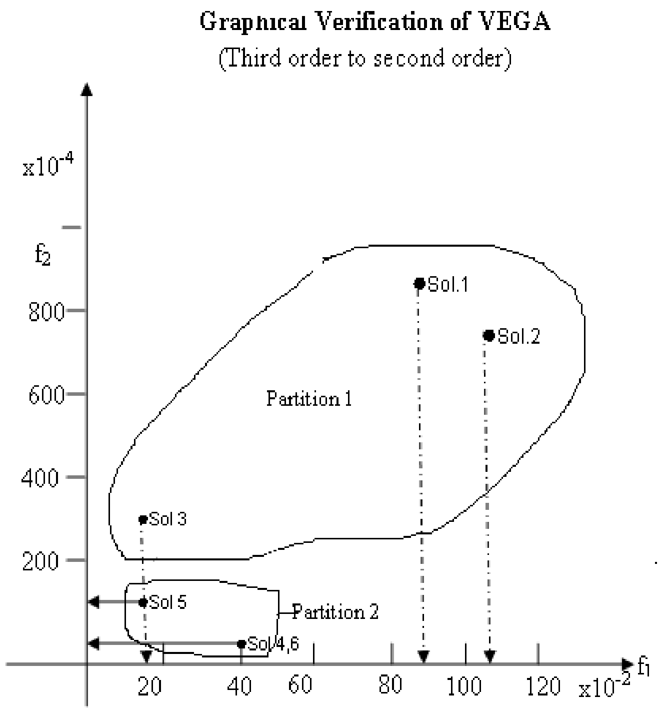

Fig 1. Graphical verification by VEGA (Third order to second order)

$$\hat{d}_1^* = 3.951056, \quad \hat{d}_2^* = 4.951056 \tag{25}$$

and using (22), $\hat{b}_1$ and $\hat{b}_2$ are evaluated as

$$\hat{b}_1 = 3.951056, \quad \hat{b}_2 = 4.951056 \tag{26}$$

Step 5: From (20), the numerator parameters of the model turn out to be

$$\hat{a}_1 = 8, \;\; \hat{a}_2 = 4.951056 \tag{27}$$

Step 6: Thus, the model is identified as

$$\hat{G}_1(s) = \frac{8s + 4.951056}{s^2 + 3.951056s + 4.951056} \tag{28}$$

On the other hand, by applying the method of [22] the model is obtained as

$$\hat{G}_2(s) = \frac{0.7477s + 1.0}{s^2 + 0.2477s + 1.0} \tag{29}$$

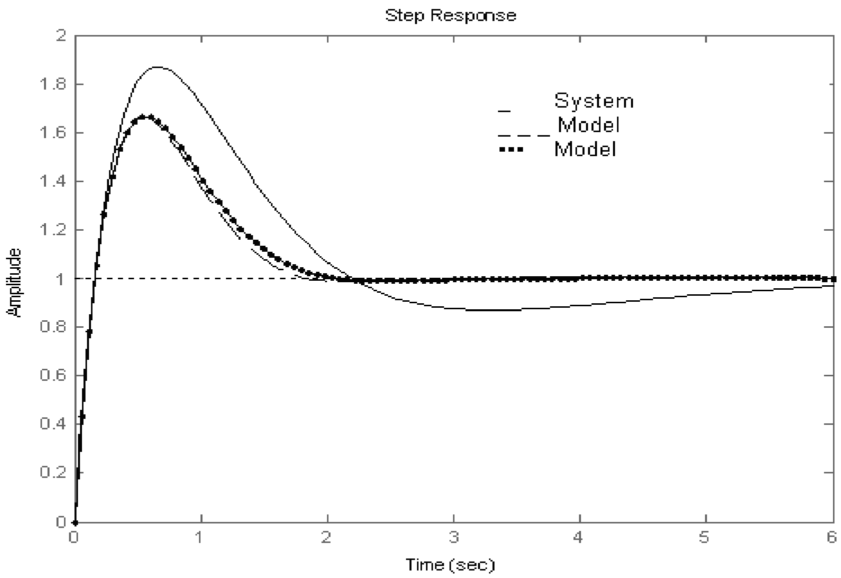

Fig. 2. Step response of system (19) and models (28) and (29)

The integral square error (ISE) is given by $ISE = \int_0^{\infty} [y(t) - \hat{y}(t)]^2 \, dt$, which can be calculated using Parseval's theorem of the overall time response of system & model, where $y(t)$ and $\hat{y}(t)$ denote output response of $G_n(s)$ and $\hat{G}_r(s)$ corresponding to (28) and (29) are focus to be 0.1404 & 29.8684 respectively, confirming the applicability of the present technique to realize improvement in system approximation. The problems of finding the values of $\overline{m}$ and weights (w_i) are also eliminated in arriving at (28).

Example 2

Suppose for a fourth-order system given by Lepschy and Viaro [21]:

$$G_4(s) = \frac{267s^3 + 527s^2 + 385s + 100}{s^4 + 4s^3 + 6s^2 + 4s + 1} \tag{30}$$

$$(t_1 = 100, \quad t_2 = -1, \quad t_3 = -13, t_4 = 9$$
$$M_1 = 267, M_2 = -541, M_3 = 947, M_4 = -1510)$$

a second-order approximant is to be obtained.

Step 1: From (10),

$$\left.\begin{aligned} \hat{a}_2 &= t_1 \hat{b}_2 \\ \hat{a}_1 &= M_1 \end{aligned}\right\} \Rightarrow \quad \begin{cases} \hat{a}_2 = 100\hat{b}_2 \\ \hat{a}_1 = 267. \end{cases} \tag{31}$$

Step 2: From (8) together with (9), one obtains (taking $\overline{m}$ =1):

$$\hat{t}_1 = t_1, \hat{M}_1 = M_1, \hat{t}_2 = t_2, \hat{t}_2 = (-100\hat{b}_1 + 267)\hat{b}_2^{-1},$$
$$\hat{M}_2 = t_1\hat{b}_2 - M_1\hat{b}_1 = 100\hat{b}_2 - 267\hat{b}_1. \tag{32}$$

Step 3: The problem, using (11) and (15) is stated as:

$$\text{Minimize} \quad \left.\begin{aligned} z_2^t &= (1 - \frac{\hat{t}_2}{t_2})^2 \\ z_2^M &= (1 - \frac{\hat{M}_2}{M_2}) \end{aligned}\right\} \tag{33}$$

subject to constraints $\hat{d}_1 > 0$ and $\hat{d}_2 > 0$.

Table 2. Assigned Fitness

Solution	b_1	b_2	f_1	f_2	Partition	Assigned Fitness
1	3..051056	2.851056	1.011875	0.000449789	2	0.000449789
2	3..051056	2.851056	1.011875	0.000449789	2	0.000449789
3	2.971056	3.201056	0.139134	0.015721567	2	0.015721567
4	2.751056	3.451056	0.71135388	0.078496968	1	0.71135388
5	2.951056	3.951056	0.2764348	0.0750186	1	0.2764348
6	2.951056	3.951056	0.2764348	0.0750186	1	0.2764348

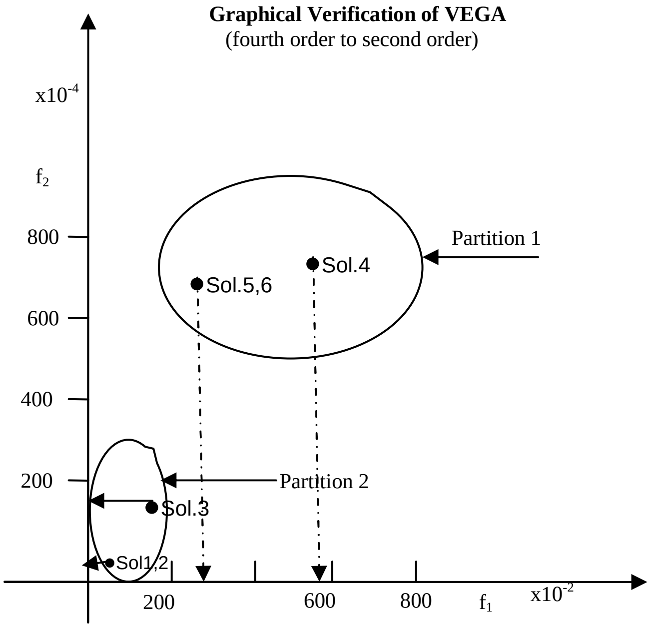

Fig 3. Graphical Verification of VEGA (fourth order to second order)

Step 4: The VEGA converges to the following optimal solution:

$$\hat{d}_1 = \hat{b}_1 = 3.051056, \qquad \hat{d}_2 = \hat{b}_2 = 2.851056. \tag{34}$$

Step 5: From (21) the numerator parameters turns out as

$$\hat{a}_1 = 267, \qquad \hat{a}_2 = 2.851056 \tag{35}$$

and the second order model of the form:

$$\hat{G}_2(s) = \frac{267s + 285.1056}{s^2 + 3.051056s + 2.851056} \tag{36}$$

For comparison, the model as obtained by technique of Lepschy and Viaro [21] is presented below:

$$\hat{G}_2(s) = \frac{267s + 321.82}{s^2 + 3.1738s + 3.2182} \tag{37}$$

However, the ISE which is corresponding to (36) and (37) are respectively 1.446 and 3.128, reveals that (36) is an improvement over (37)

8. Conclusions

In this paper, the problem of finding Routh-Padé approximants has been viewed as a multi-objective optimization problem. It is shown that, using Pareto Optimality and VEGA, the denominator of the model can be chosen so as to minimize errors between the $(r+1)$th and the subsequent time moments and Markov parameters of the model and the corresponding time moments and/or Markov parameters of the system while preserving stability.

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