Advances in the Theory of Nonlinear Analysis and its Applications 6 (2022) No. 2, 246–257.
https://doi.org/10.31197/atnaa.1053966
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Existence and stability results for a nonlinear implicit fractional differential equation with a discrete delay

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Abstract

In this paper, we are concerned with a class of nonlinear implicit fractional differential equation with a discrete delay. By means of the contraction mapping principle, we prove the existence of a unique solution. Then, we investigate the continuous dependence of the solution upon the initial delay data and the Ulam stability.

Keywords: Fractional differential equation, discrete delay, existence and continuous dependence of solution on initial data, Ulam stability, Banach fixed point theorem.
2010 MSC: 34A08, 34A12, 34K37, 34K20.

1. Introduction

Fractional differential equations are a generalization of ordinary differential equations to arbitrary non-integer orders. During the last decades it has been shown that fractional differential equations arise naturally in a number of fields, when dealing with memory or hereditary properties, such as in viscoelasticity, biology, engineering, biophysics, medicine, control theory, etc. For more details about the theory and widespread applications of fractional differential equations, one can see [15, 18] and the references therein.

On the other hand, time delays appear in many phenomena in diverse domains such as in engineering, biology, medicine, etc., where information transmission or responses in control systems are not instantaneous, for example see [16].

Thus, fractional differential equations with delays have more adequate applications in some cases. Subsequently, a large number of mathematicians investigated this class of equation and obtained several results on the existence and uniqueness of solutions, one can consult [3, 4, 6, 9, 10].

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Received January 5, 2022, Accepted February 23, 2022, Online February 25, 2022

One of main qualitative properties of solutions of differential equations is their data dependence which is studied by many methods. Recently, Ulam's type stabilities have attracted more and more attention. The classical concept of Ulam stability posed by Ulam in 1940, was obtained for functional equation by Hyers in 1941, see [11]. Then, the Hyers result was extended by replacing functional equations with differential equations and this approach, which guarantees the existence of an $\varepsilon-$solution, is quite useful in many applications where finding the exact solution is impossible. It is important to notice that there are many applications for Ulam-Hyers stability in realistic problems from different topics such as population dynamic, biology , economics, etc. To know many see [11, 12, 14, 20, 21].

With the expansion of the fractional calculus, more and more papers concerned with problems involving various stabilities of solutions of fractional differential equations with or without delays are published, as example [1, 5, 9, 10, 17] and for that of Ulam type, see [3, 7, 13, 22].

In [7], results on the existence and uniqueness of the solution and two types of Ulam's stability for an initial value problem for nonlinear implicit fractional differential equation with Riemann-Liouville fractional derivative,

$$\begin{cases} D_{0+}^{\alpha} y\left(t\right) = f\left(t, y\left(t\right), D_{0+}^{\alpha} y\left(t\right)\right); \quad t \in \left(0, T\right], 0 < \alpha < 1 \\ \left. t^{1-\alpha} y\left(t\right)\right|_{t=0} = y_0 \in \mathbb{R}, \end{cases}$$

were obtained by using the Banach contraction mapping principle and Schaefer's fixed point theorem.

Motivated by the above work, we will investigate, the existence and uniqueness of the solution and some stability properties, namely the continuous dependence upon the initial data, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities for problem involving a more general nonlinear implicit fractional differential equation with a discrete delay

$$D_{0+}^{\alpha} x\left(t\right) + \lambda x\left(t\right) = F\left(t, x\left(t\right), x\left(t-\eta\right), D_{0+}^{\alpha} x\left(t\right)\right); \quad t \in \left[0, T\right], \tag{1}$$

for $0 < \alpha < 1$, subject to the initial condition:

$$\left. I_{0+}^{1-\alpha} x\left(t\right)\right|_{t=0^{+}} = 0 \tag{2}$$

and to be in accordance with the given delay, we need the following data on the delay interval

$$x\left(t\right) = \varphi\left(t\right); \quad t \in \left[-\eta, 0\right) \quad \text{with} \lim_{t \to 0^{-}} \varphi\left(t\right) = 0, \tag{3}$$

D_0^{α} denotes the Riemann-Liouville derivative of order $\alpha \in (0,1)$; λ is a real positive constant; F and φ are given functions; $\eta > 0$ is the time delay .

The difficulties of this problem come from the implicit form of the equation (1) as well as the time delay. It's possible that (1) becomes of explicit form in some cases of weak nonlinearity of F, then the results obtained for (1) will remain true with lightened conditions. To ensure the implicit form of (1), we assume that $F\left(t, x, y, z\right)$ is totally nonlinear precisely with its last variable.

The manuscript is structured as follows. In Section 2, we give some basic results from fractional calculus, which will be used throughout the paper. In Section 3, we will employ the Banach contraction mapping principle to show the existence and uniqueness of the solution for the problem (1)-(3) by transforming the problem into an equivalent integral equation. In Section 4, we will study the continuous dependence upon the initial data and two types of Ulam's stability.

2. Preliminaries

In this section, we introduce definitions and preliminary results which can be found in the books of Kilbas et *al* [15] and Podlubny [18].

Definition 2.1. *The left-sided Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ for an integrable function $f : \mathbb{R}^+ \to \mathbb{R}$, is defined by*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \text{ for almost every } t > 0,$$

$\Gamma(\alpha)$ is the Euler's gamma function and $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$.

Lemma 2.2. *When $f\in C[0,T]$, then $I_{0+}^{\alpha}f\in C[0,T]$.*

Definition 2.3. *The left-sided Riemann-Liouville fractional derivative of order $\alpha\in(0,1)$ is defined by*

$$D_{0+}^{\alpha}f(t):=\frac{d}{dt}I_{0+}^{1-\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(s)}{(t-s)^{\alpha}}ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 2.4. *If f is an integrable function such that $I_{0+}^{1-\alpha}f\in AC[0,T]$, then $D_{0+}^{\alpha}f\left(t\right)$ exists for almost every $t\in[0,T]$. Furthermore, the following equalities*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}f(t)=f(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I_{0+}^{1-\alpha}f(0^{+})\ \text{and}\ D_{0+}^{\alpha}I_{0+}^{\alpha}f(t)=f(t),$$

hold almost everywhere on $\left[0,T\right]$.

Recall that the space of absolutely continuous functions $AC[0,T]$, coincides with the space of primitives of Lebesgue summable functions i. e.

$$f\left(t\right)=c+\int_0^t\theta\left(s\right)ds\ \text{ on }\ [0,T]\quad\text{for }\theta\in L^{1}\left[0,T\right].$$

Let us define the **Mittag-Leffler Function** which is an important tool in the fractional calculus.

Definition 2.5. *A two-parameter Mittag-Leffler function is defined by the series expansion*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)};\ z,\alpha,\ \beta\in\mathbb{C}\text{ with Re}\alpha>0; \tag{4}$$

in particular $E_{\alpha,1}(z)=E_{\alpha}(z)$ and $E_{1,1}(z)=\exp(z)$

Corollary 2.6. *The following properties hold for $0<\alpha\leq\beta\leq1$, $\lambda\in\mathbb{R}^{+}$ and $t\in[0,T)$ (some $T\leq\infty$) :*

(i) $E_{\alpha,\beta}(-\lambda t^{\alpha})$ is a completely monotonic function and

$0<E_{\alpha,\beta}(-\lambda t^{\alpha})\leq\dfrac{1}{\Gamma\left(\beta\right)}$;

(ii) $t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})$ is a completely monotonic function and

$$\int_0^t(t-s)^{\beta-1}E_{\alpha,\beta}(-\lambda(t-s)^{\alpha})ds=t^{\beta}E_{\alpha,\beta+1}(-\lambda t^{\alpha})<\infty. \tag{5}$$

Now, we give the integral form of the solution related to a linear Cauchy problem.

Lemma 2.7. *The linear Cauchy fractional differential problem*

$$\begin{cases} D_{0+}^{\alpha}x(t)+\lambda x(t)=H(t),\quad t\geq0,\quad \lambda>0,\ 0<\alpha<1,\\ I_{0+}^{1-\alpha}x(0^{+})=c,\end{cases} \tag{6}$$

where H is an integrable function, has the following integral representation of the solution for $t\in(0,T]$,

$$x(t)=ct^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})H(s)ds. \tag{7}$$

Finally, we give a type of Gronwall's inequality for singular kernels which can find in [23].

Corollary 2.8. *Suppose $\alpha>0$, $a\left(t\right)$ is a nonnegative nondecreasing locally integrable function on $[0,T)$ (some $T\leq\infty$) , $g\left(t\right)$ is a nonnegative, nondecreasing continuous function defined on $[0,T)$ such that $g\left(t\right)\leq M$ (a constant) and suppose $u\left(t\right)$ is nonnegative and locally integrable on $[0,T)$ with*

$$u\left(t\right)\leq a\left(t\right)+g\left(t\right)\int_0^t\left(t-s\right)^{\alpha-1}u\left(s\right)ds,\ \text{ on }\ [0,T).$$

Then

$$u\left(t\right)\leq a\left(t\right)E_{\alpha}\left(g\left(t\right)\Gamma\left(\alpha\right)t^{\alpha}\right),\ t\in[0,T).$$

3. Main Results

3.1. Existence and uniqueness results

The existence result below is based on the well-known Banach fixed point theorem.

Let us introduce the following hypotheses on F:

(H1) 1- $F \in C([0,T] \times \Omega)$ where Ω is an open set in $\mathbb{R}^3$ which contains 0 with $\sup_{t\in[0,T]} |F(t,0,0,0)| = F_0$;

2- F satisfies the Lipschitz condition i. e there exist constants $L_1 > 0$, $L_2 > 0$, $0 < L_3 < 1$ such that:

$$|F(t,x_1,y_1,z_1) - F(t,x_2,y_2,z_2)| \leq L_1 |x_1 - x_2| + L_2 |y_1 - y_2| + L_3 |z_1 - z_2|,$$

for all $(x_1,y_1,z_1), (x_2,y_2,z_2) \in \Omega$ and each $t \in [0,T]$;

For convenience, we shall give the following definition and notations.

Definition 3.1. *A real function $x \in C[-\eta,T] \cap AC[0,T]$ is said to be a solution of problem (1)-(3) if it satisfies (1) on $(0,T]$, (2) for $t = 0$ and (3) on $[-\eta,0)$ for a given continuous function φ with $\lim_{t\to 0^-} \varphi(t) = 0$.*

Let us denote

$$\Sigma_1 = \frac{T^\alpha}{\Gamma(\alpha+1)(1-L_3)}; \tag{8}$$

and $C[a,b]$ the Banach space of continuous real functions on $[a,b]$ endowed with the sup norm$\|x\|_{C[a,b]} = \sup_{t\in[a,b]} |x(t)|$.

Define the space

$$C_0[-\eta,0] = \{\psi \in C[-\eta,0] : \psi(0) = 0\}$$

which is a Banach space endowed with the sup norm $\|\psi\|_{C_0[-\eta,0]} = \sup_{t\in[-\eta,0]} |\psi(t)|$.

Theorem 3.2. *Assume that (H1) is satisfied and $\varphi \in C_0[-\eta,0]$. If*

$$0 < \Sigma_1 [L_1 + L_2 + \lambda L_3] < 1, \tag{9}$$

then, implicit delayed fractional differential problem (1)-(3) has a unique solution $x \in C[-\eta,T] \cap AC[0,T]$.

Proof. First, we define $\mathcal{B}$ on $C[-\eta,T]$ satisfying the functional equation

$$\mathcal{B}x(t) = \begin{cases} F(t,x(t),x(t-\eta),\mathcal{B}x(t) - \lambda x(t)), t \in (0,T], \\ 0, \quad t \in [-\eta,0]. \end{cases} \tag{10}$$

$\mathcal{B}$ is well defined, since we have for each $x \in C[-\eta,T]$

$$\begin{aligned} |\mathcal{B}x(t)| \leq\ & |F(t,x(t),x(t-\eta),\mathcal{B}x(t) - \lambda x(t)) - F(t,0,0,0)| \\ & + |F(t,0,0,0)|, \ t \in (0,T]. \end{aligned} \tag{11}$$

Then,

$$|\mathcal{B}x(t)| \leq \frac{\lambda L_3 + L_1 + L_2}{(1-L_3)} \|x\|_{C[-\eta,T]} + \frac{F_0}{(1-L_3)} < \infty, \ t \in [-\eta,T].$$

Next, using Lemma 2.7 with $c = 0$ and $H(t) = \mathcal{B}x(t)$ for $t \in (0,T]$, we obtain that $x \in C[-\eta,T]$ the solution of problem (1)-(3) satisfies

$$x(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \mathcal{B}x(s)\, ds, \quad t \in (0,T] \\ 0, \quad t = 0 \\ \varphi(t), \quad t \in [-\eta,0); \end{cases} \tag{12}$$

for a given $\varphi \in C_0[-\eta, 0]$. Now, let us define the following Banach spaces:

$$\begin{aligned} C_0[-\eta, T] &= \{y \in C([-\eta, T], \mathbb{R}) : \ y(t) = 0; \quad t \in [-\eta, 0]\}, \\ C_{0,T}[-\eta, T] &= \{\psi \in C([-\eta, T], \mathbb{R}) : \psi(t) = 0, \ t \in [0, T]\} \end{aligned}$$

endowed with the sup norm $\|x\|_{C[-\eta,T]} = \sup\limits_{t\in[-\eta,T]} |x(t)|$. This allows us to rewrite the solution of (1)-(3) *as follows*

$$x(t) = \widetilde{x}(t) + \widetilde{\varphi}(t), \ t \in [-\eta, T]$$

where $\widetilde{x} \in C_0[-\eta, T]$ is defined from (12) *for* $t \in (0, T]$ and $\widetilde{\varphi} \in C_{0,T}[-\eta, T]$ is defined from (12) *for* $t \in [-\eta, 0]$.

Next, we define the operator P on $C_0[-\eta, T]$ by

$$P\widetilde{x}(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\mathcal{B}\widetilde{x}(s)\, ds, & t \in (0, T] \\ 0, & t \in [-\eta, 0]. \end{cases} \tag{13}$$

and reduce problem (1)-(3) to a fixed point problem $\widetilde{x} = P\widetilde{x}$. Now, we prove that P makes $C_0[-\eta, T]$ into itself. First, for $t \in (0, T]$,

$$|P\widetilde{x}(t)| \le \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) |\mathcal{B}\widetilde{x}(s)|\, ds. \tag{14}$$

From *(11),* we have on $C_0[-\eta, T]$

$$\begin{aligned} |\mathcal{B}\widetilde{x}(s)| &\le \frac{\lambda L_3 + L_1}{(1 - L_3)} |\widetilde{x}(s)| \\ &\quad + \frac{L_2}{(1 - L_3)} |\widetilde{x}(s - \eta)| + \frac{F_0}{(1 - L_3)}. \end{aligned} \tag{15}$$

Using (15) in (14), we obtain

$$\begin{aligned} |P\widetilde{x}(t)| &\le \frac{(\lambda L_3 + L_1)}{(1 - L_3)} \sup_{s\in[0,t]} |\widetilde{x}(s)| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) ds \\ &+ \frac{L_2}{(1 - L_3)} \sup_{z\in[-\eta,t-\eta]} |\widetilde{x}(z)| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) ds \\ &+ \frac{F_0}{(1 - L_3)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) ds. \end{aligned}$$

By (5), the monotonicity of $E_{\alpha,\alpha+1}(-\lambda t^\alpha)$ on $[0, T]$ which gives

$$\sup_{0<t\le T} t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{T^\alpha}{\Gamma(\alpha + 1)};$$

and the fact that $P\widetilde{x}(t) = 0$ for $t \in [-\eta, 0]$, we get

$$\begin{aligned} \|P\widetilde{x}\|_{C[-\eta,T]} &\le [L_1 + L_2 + \lambda L_3] \frac{T^\alpha}{\Gamma(\alpha + 1)(1 - L_3)} \|\widetilde{x}\|_{C[-\eta,T]} \\ &\quad + F_0 \frac{T^\alpha}{\Gamma(\alpha + 1)(1 - L_3)} \\ &\le \left[[L_1 + L_2 + \lambda L_3] \|\widetilde{x}\|_{C[-\eta,T]} + F_0 \right] \Sigma_1 < \infty. \end{aligned}$$

Then, P makes $C_0[-\eta, T]$ into itself. Now, for each $\widetilde{x}$ and $\widetilde{y}$ in $C_0[-\eta, T]$,

$$|P\widetilde{x}(t) - P\widetilde{y}(t)| \le \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) |\mathcal{B}\widetilde{x}(s) - \mathcal{B}\widetilde{y}(s)|\, ds.$$

Using the Lipschitz property of F and as already seen in (15), we get

$$\begin{aligned} \left|\mathcal{B}\widetilde{x}(s)-\mathcal{B}\widetilde{y}(s)\right| &\leq \frac{\lambda L_3+L_1}{(1-L_3)}\left|\widetilde{x}(s)-\widetilde{y}(s)\right| \\ &+\frac{L_2}{(1-L_3)}\left|\widetilde{x}(s-\eta)-\widetilde{y}(s-\eta)\right|. \end{aligned} \tag{16}$$

Then, by the fact that $\left|P\widetilde{x}(t)-P\widetilde{y}(t)\right|=0=\left|\widetilde{x}(t)-\widetilde{y}(t)\right|$ for $t\in[-\eta,0]$, we obtain

$$\left\|P\widetilde{x}-P\widetilde{y}\right\|_{C[-\eta,T]}\leq\left[L_1+L_2+\lambda L_3\right]\Sigma_1\left\|\widetilde{x}-\widetilde{y}\right\|_{C[-\eta,T]}.$$

From (9), we conclude that P is a contraction and in view of the Banach fixed point theorem, P has a unique fixed point in $C_0[-\eta,T]$ which is the unique solution of (1) with the initial condition $\widetilde{x}(t)=0, t\in[-\eta,0]$. Thus, for a unique given $\widetilde{\varphi}\in C_{0,T}[-\eta,T]$ the solution of (1)-(3), $x(t)=\widetilde{x}(t)+\widetilde{\varphi}(t)$ is unique in $C([-\eta,T],\mathbb{R})$. Furthermore, $x(t)$ is absolutely continuous for $t\in[0,T]$. This yields from (12), for $t\in(0,T]$ where $\mathcal{B}x(s)$ satisfying (10) is continuous and $(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})$ is integrable on $[0,T]$ with $\lim_{t\to0+}x(t)=0$ then $x\in AC([0,T],\mathbb{R})$. Furthermore, $D_{0+}^{\alpha}x(t)$ is continuous by the continuity of $-\lambda x(t)+\mathcal{B}x(t)$, for $t\in(0,T]$. □

3.2. Continuous dependence upon the initial data

Next, we study the continuous dependence of the solution of equation (1) upon the initial data $\varphi(t)$ due to the discrete delay.

Theorem 3.3. *Assume that assumption (H1) and condition (9) hold. Then, the solution of problem (1)-(3) depends continuously for each $t\in[0,T]$ upon the initial data $\varphi\in C_0[-\eta,0]$.*

Proof. Let $x_j, j=1,2$ be solutions of equation (1) corresponding to the initial data $\varphi_j(t), j=1,2$, respectively. Then, we have

$$\begin{aligned} &\left|x_1(t)-x_2(t)\right| \\ \leq\ & \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})\left|\mathcal{B}x_1(s)-\mathcal{B}x_2(s)\right|ds. \end{aligned}$$

Using (16), we get

$$\begin{aligned} &\left|x_1(t)-x_2(t)\right| \\ \leq\ & \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})\frac{L_1+\lambda L_3}{(1-L_3)}\left|x_1(s)-x_2(s)\right|ds \\ &+\frac{L_2}{(1-L_3)}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})\left|x_1(s-\eta)-x_2(s-\eta)\right|ds. \end{aligned}$$

Putting $(s-\eta)=z$ in the last integral, with $\sup\limits_{0\leq(t-s)\leq T}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})=\dfrac{1}{\Gamma(\alpha)}$, we get

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_{-\eta}^{t-\eta}(t-\eta-z)^{\alpha-1}\left|x_1(z)-x_2(z)\right|dz \\ \leq\ & \frac{1}{\Gamma(\alpha)}\sup_{z\in[-\eta,0]}\left|x_1(z)-x_2(z)\right|\int_{-\eta}^{0}(t-\eta-z)^{\alpha-1}dz \\ &+\frac{1}{\Gamma(\alpha)}\sup_{z\in[0,t-\eta]}\left|x_1(z)-x_2(z)\right|\int_0^{t-\eta}(t-\eta-z)^{\alpha-1}dz \\ &\qquad\leq\frac{t^{\alpha}-(t-\eta)^{\alpha}}{\alpha\Gamma(\alpha)}\left\|\varphi_1-\varphi_2\right\|_{C[-\eta,0]}+\frac{(t-\eta)^{\alpha}}{\alpha\Gamma(\alpha)}\left\|x_1-x_2\right\|_{C[0,T]}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned}|x_1(t)-x_2(t)| &\le \frac{T^\alpha}{\alpha\Gamma(\alpha)(1-L_3)}L_2\|\varphi_1-\varphi_2\|_{C[-\eta,0]}\\ &+\frac{T^\alpha}{\alpha\Gamma(\alpha)(1-L_3)}(L_1+\lambda L_3+L_2)\|x_1-x_2\|_{C[0,T]}.\end{aligned}$$

Then, with $x_1(0)=0=x_2(t)$, we have

$$\begin{aligned}\sup_{t\in[0,T]}|x_1(t)-x_2(t)| &\le [L_1+L_2+\lambda L_3]\,\Sigma_1\sup_{t\in[0,T]}|x_1(t)-x_2(t)|\\ &\quad+\Sigma_1 L_2\|\varphi_1-\varphi_2\|_{C[-\eta,0]}.\end{aligned}$$

In view of (9), we have for each $t\in[0,T]$

$$|x_1(t)-x_2(t)| \le \frac{\Sigma_1 L_2}{1-[L_1+L_2+\lambda L_3]\,\Sigma_1}\|\varphi_1-\varphi_2\|_{C[-\eta,0]}. \tag{17}$$

This implies the continuous dependence of $x(t)$ for each $t\in[0,T]$ upon the initial data of the delay $\varphi(t)$. This completes the proof. □

3.3. Ulam-Hyers stability

As already said in the introduction, in 1941 Hyers gave an answer to the problem posed by Ulam for functional equation in the following result called the Ulam-Hyers stability:

Let $X_1;X_2$ be a Banach spaces, $\varepsilon>0$ and a mapping $f:X_1\to X_2$ such that $|f(x+y)-f(x)-f(y)|\le\varepsilon$ for $x,y\in X_1$. Then, there exists a unique additive mapping $g:X_1\to X_2$ such that $|g(x)-f(x)|\le\varepsilon$, $x\in X_1$.

The generalization of Ulam's type stability was proposed by replacing functional equations with differential equations of integer order and later of fractional order. In [2], Alsina and Ger seem to be the first authors who investigated the Hyers-Ulam stability of linear differential equation :$y'(t)=y(t)$. They obtained this result:

Let $\varepsilon>0$, I an open subinterval of $\mathbb{R}$ and $f:I\to\mathbb{R}$ a differentiable function. If f satisfies the differential inequality $|y'(t)-y(t)|\le\varepsilon$, $t\in I$, then there exists a differentiable function $g:I\to\mathbb{R}$ solution of the differential equation $y'(t)-y(t)=0$ such that $|f(t)-g(t)|\le 3\varepsilon$ for any $t\in I$.

Following this approach, we will embrace the definitions of Ulam-Hyers stability and Ulam-Hyers-Rassias stability presented by Rus in [19] .

Let ε be a positive real number and $\psi(t)$ be a positive continuous function on $[0,T]$ and consider the following inequalities:

$$|D_0^\alpha y(t)+\lambda y(t)-\mathcal{H}(t)|\le\varepsilon,\ t\in[0,T]; \tag{18}$$

and

$$|D_0^\alpha y(t)+\lambda y(t)-\mathcal{H}(t)|\le\varepsilon\psi(t),\ t\in[0,T]; \tag{19}$$

where $\mathcal{H}$ is defined for $t\in[0,T]$ by

$$\mathcal{H}(t)=F\left(t,y(t),y(t-\eta),D_{0^+}^\alpha y(t)\right). \tag{20}$$

Recall the definition of the solution of the inequality (18).

Definition 3.4. *A function $y\in C[-\eta,T]\cap AC[0,T]$ is a solution of inequality (18) with the initial conditions (2)-(3) if and only if there exists a function $h\in C[0,T]$ such that $|h(t)|\le\varepsilon$ for every $t\in[0,T]$ and*

$$\begin{cases} D_{0^+}^\alpha y(t)+\lambda y(t)=F\big(t,y(t),y(t-\eta),D_{0^+}^\alpha y(t)\big)+h(t),t\in(0,T]\\ I_{0^+}^{1-\alpha}y(t)\big|_{t=0^+}=0 \text{ and } y(t)=\varphi(t);t\in[-\eta,0),\end{cases} \tag{21}$$

for a given $\varphi\in C_0[-\eta,0]$.

Remark 3.5. *One can have similar definition for inequality (19) with* $|h(t)| \leq \varepsilon\psi(t)$.

Definition 3.6. *Fractional differential equation (1) is said to be Ulam-Hyers stable if there exists a real number* $k > 0$ *such that for each* $\epsilon > 0$ *and each absolutely continuous function* y *solution of inequality (18) there exists some absolutely continuous function* x *solution of (1), such that*

$$|y(t) - x(t)| \leq k\varepsilon, \quad t \in [0, T]. \tag{22}$$

Definition 3.7. *Fractional differential equation (1) is said to be Ulam-Hyers-Rassias stable with respect to* $\psi(t)$ *if there exists a real number* $\gamma > 0$ *such that for each absolutely continuous function* y *solution of inequality (18) there exists some absolutely continuous function* x *solution of (1), such that*

$$|y(t) - x(t)| \leq \gamma\varepsilon\psi(t), \qquad t \in [0, T]. \tag{23}$$

Lemma 3.8. *If* $y \in C[-\eta, T] \cap AC[0, T]$ *is a solution of (18), (2) and (3), then* y *satisfies for* $t \in (0, T]$

$$\left| y(t) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\mathcal{B}_h y(s)\, ds \right| < \varepsilon \frac{T^\alpha}{\Gamma(\alpha+1)}, \tag{24}$$

where $\mathcal{B}_h y(t)$ *satisfies for each* $y \in C[-\eta, T]$

$$\mathcal{B}_h y(t) = F(t, y(t), y(t-\eta), \mathcal{B}_h y(t) - \lambda y(t) + h(t)). \tag{25}$$

Proof. In view of Lemma 2.7 with $H(t) = \mathcal{B}_h y(t)$, the solution of (21) satisfies

$$y(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)(\mathcal{B}_h y(s) + h(s))\, ds, \quad t \in (0, T]. \tag{26}$$

First, we have to check that $\mathcal{B}_h$ is well defined. Then, for each $y \in C[-\eta, T]$, as already done with $\mathcal{B}$ in (15), we obtain with one more term

$$\begin{aligned}
|\mathcal{B}_h y(t)| &\leq (L_1 + L_2 + \lambda L_3) \frac{\|y\|_{C[-\eta,T]}}{1 - L_3} \\
&\quad + \frac{F_0}{1 - L_3} + \frac{L_3}{1 - L_3} |h(t)| < \infty.
\end{aligned}$$

Then, we get for $t \in (0, T]$

$$\begin{aligned}
I(t) &= \left| y(t) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\mathcal{B}_h y(s)\, ds \right| \\
&\leq \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)|h(s)|\, ds \\
&\leq \varepsilon \int_0^t z^{\alpha-1} E_{\alpha,\alpha}(-\lambda z^\alpha) dz \leq \varepsilon \sup_{0 \leq z \leq T} t^\alpha E_{\alpha,\alpha+1}(-\lambda z^\alpha),
\end{aligned}$$

which leads to (24). The proof is complete. □

Theorem 3.9. *Assume that* $\varphi \in C_0[-\eta, 0]$, *(H1) and condition (9) hold. Then, equation (1) has Ulam-Hyers stability.*

Proof. Under the above assumptions, problem (1)-(3) has a unique solution in $C[-\eta, T] \cap AC[0, T]$ satisfying (12). Let $y \in C[-\eta, T] \cap AC[0, T]$ be a solution of inequality (18) and (2)-(3), then we have for each $t \in (0, T]$

$$|y(t) - x(t)| = \left| y(t) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\mathcal{B}x(s)\, ds \right|$$

where $\mathcal{B}x(t)$ satisfies (10). Then,

$$\begin{aligned}|y(t)-x(t)| &\leq \left|y(t)-\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\mathcal{B}_h y(s)\,ds\right| \\ &\quad +\left|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\left(\mathcal{B}_h y(s)-\mathcal{B}x(s)\right)ds\right|.\end{aligned}$$

Expressions (10) and (20) with hypothesis (H1) give

$$\begin{aligned}|\mathcal{B}_h y(s)-\mathcal{B}x(s)| &\leq L_1|y(s)-x(s)|+L_2|y(s-\eta)-x(s-\eta)| \\ &\quad +L_3|\mathcal{B}_h y(s)-\mathcal{B}x(s)|+\lambda L_3|y(s)-x(s)|+L_3|h(s)|.\end{aligned}$$

As a result, we have

$$\begin{aligned}&|\mathcal{B}_h y(s)-\mathcal{B}x(s)| \leq \frac{\lambda L_3+L_1}{1-L_3}|y(s)-x(s)| \\ &+\frac{L_2}{1-L_3}|y(s-\eta)-x(s-\eta)|+\frac{L_3}{1-L_3}|h(s)|\end{aligned} \tag{27}$$

which implies

$$\begin{aligned}&|y(t)-x(t)|<I(t) \\ &+\left(\frac{\lambda L_3+L_1}{(1-L_3)}\right)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)|y(s)-x(s)|\,ds \\ &+\frac{L_2}{(1-L_3)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)|y(s-\eta)-x(s-\eta)|\,ds \\ &+\frac{L_3}{(1-L_3)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)|h(s)|\,ds.\end{aligned}$$

In view of (24), we get

$$\begin{aligned}\|y-x\|_{C[0,T]} &< \varepsilon\frac{T^\alpha}{\Gamma(\alpha+1)}+\varepsilon\frac{L_3T^\alpha}{\Gamma(\alpha+1)(1-L_3)} \\ &\quad +[L_1+\lambda L_3+L_2]\frac{T^\alpha}{\Gamma(\alpha+1)(1-L_3)}\|y-x\|_{C[0,T]}.\end{aligned}$$

Then, there exists a real positive constant

$$k=\frac{L_3\Sigma_1}{1-[L_1+L_2+\lambda L_3]\Sigma_1} \tag{28}$$

where Σ_1 is defined by (8), such that

$$|y(t)-x(t)|<k\varepsilon, \quad t\in(0,T],$$

with $x(t)=y(t)$ on $[-\tau,0]$ and this completes the proof. □

Theorem 3.10. *Assume that (H1) and condition (9) hold. If*

(H2) there exist a nondecreasing positive function $\psi\in C[0,T]$ and some positive constant β_ψ such that

$$I_{0+}^\alpha\psi(t)\leq\beta_\psi\psi(t), \quad \text{for any } t\in(0,T].$$

Then, equation (1) has Ulam–Hyers-Rassias stability with respect with $\psi(t)$.

Proof. Let $y \in C[-\eta, T] \cap AC[0, T]$ be a solution of the inequality (19) with initial conditions (2)-(3). Then, for $t \in (0, T]$

$$\begin{aligned}
&\left| y(t) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \mathcal{B}_h y(s)\, ds \right| \\
\le\ & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) |h(s)|\, ds \\
\le\ & \varepsilon \sup_{0 \le t-s \le T} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \int_0^t (t-s)^{\alpha-1} \psi(s)\, ds \\
& \le \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s)\, ds \le \varepsilon \beta_\psi \psi(t). \qquad (29)
\end{aligned}$$

Under (H1) and condition (9), problem (1)-(3) has a unique solution in $C[-\eta, T] \cap AC[0, T]$ satisfying (12). Then, we obtain for each $t \in (0, T]$

$$\begin{aligned}
|y(t) - x(t)| \ <\ & \left| y(t) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \mathcal{B}_h y(s)\, ds \right| \\
& + \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) (\mathcal{B}_h y(s) - \mathcal{B} x(s))\, ds \right|.
\end{aligned}$$

From (27) and (29) , we conclude

$$\begin{aligned}
&|y(t) - x(t)| < \varepsilon \beta_\psi \psi(t) + \frac{(L_1 + \lambda L_3)}{(1 - L_3)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)|\, ds \\
&+ \frac{L_2}{(1 - L_3)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s-\eta) - y(s-\eta)|\, ds \\
&+ \frac{L_3}{(1 - L_3)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s)|\, ds.
\end{aligned}$$

Letting $v(t) = \sup_{\beta \in [0,\eta]} |x(t-\beta) - y(t-\beta)|$, we can see that

$$v(t) < \frac{\varepsilon \beta_\psi \psi(t)}{(1 - L_3)} + \frac{[L_1 + \lambda L_3 + L_2]}{(1 - L_3)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds.$$

By Corollary 2.8, with $\psi(t)$ an increasing function, we get

$$v(t) < \frac{\varepsilon \beta_\psi \psi(t)}{(1 - L_3)} E_\alpha \left(\frac{[L_1 + \lambda L_3 + L_2]}{(1 - L_3)} t^\alpha \right).$$

From the monotonicity of E_α, it follows that there exists a real number $\gamma > 0$ given by

$$\gamma = \frac{\varepsilon \beta_\psi}{(1 - L_3)} E_\alpha \left(\frac{T^\alpha}{1 - L_3} [L_1 + \lambda L_3 + L_2] \right) \qquad (30)$$

such that

$$|y(t) - x(t)| \le \varepsilon \gamma \psi(t), \quad t \in (0, T], \qquad (31)$$

with $x(t) = y(t)$ on $[-\tau, 0]$. This is the wanted result. □

Example 3.11. *We consider the following nonlinear fractional differential equation with discrete delay:*

$$\begin{aligned}
D_{0+}^{1/2} x(t) + 2x(t) \ =\ & \frac{\exp(-4t)}{10} \frac{|x(t)|}{1 + |x(t)|} \\
& + \frac{t}{10} \cos x\left(t - \frac{2}{5}\right) + \frac{1}{t^2 + 10} \left| D_{0+}^{1/2} x(t) \right|; \quad t \in (0, 1], \qquad (32)
\end{aligned}$$

the initial condition $I_{0+}^{1/2}x(0) = 0$ *and the the delay initial data*

$$\varphi(t) = 3t, \qquad t \in [-0.4, 0) \tag{33}$$

where $\alpha = \frac{1}{2}, \lambda = 2, \eta = 0.4, T = 1$. F *given by*

$$F(t, x, y, z) = \frac{\exp(-4t)}{10}\frac{|x|}{1+|x|} + \frac{t}{10}\cos y + \frac{1}{t^2+10}|z|; \quad x, y, z \in \mathbb{R}$$

is Lipschitz continuous with

$$L_1 = L_2 = L_3 = \frac{1}{10} \text{ and } \sup_{t\in[0,T]} |F(t, 0, 0, 0)| = \frac{1}{10}.$$

This permits us to calculate Σ_1

$$\Sigma_1 = \frac{T^\alpha}{\alpha\Gamma(\alpha)(1-L_3)} = \frac{2}{\sqrt{\pi}\left(1-\frac{1}{10}\right)} \simeq 1.2538$$

which satisfies condition (9) as we have

$$\Sigma_1[L_1 + L_2 + \lambda L_3] = 1.2538 \times \frac{4}{10} \simeq 0.50152 < 1.$$

Also, $\varphi(t) = 3t \in C[-0.4, 0]$ *with* $\lim_{t\to 0^-}\varphi(t) = 0$. *This ensures the existence and uniqueness of the solution in view of Theorem 3.2.*

Moreover, from Theorem 3.9, we conclude the Ulam-Hyers stability of equation (32) on $[0, 1]$ *with the Ulam constant* k

$$k = \frac{L_3\Sigma_1}{1 - [L_1 + L_2 + \lambda L_3]\Sigma_1} \simeq 0.2515.$$

Next, for $\psi(t) = \exp(nt)$, $n > 0$ *which is an increasing positive continuous function which satisfies for* $t \in (0, 1]$

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1}\psi(s)\,ds &< 2t^{1/2} + 2n\exp(nt)\int_0^t (t-s)^{1/2}ds \\ &< 2\left(t^{1/2} + 2n\frac{t^{3/2}}{3}\right)\exp(nt) \\ &< 2\left(1 + \frac{2}{3}n\right)\psi(t).\end{aligned}$$

By setting $\beta_\psi = 2\dfrac{3+2n}{3}$, $\psi(t)$ *satisfies (H2). Then, Theorem 3.10 implies that equation (32) has the Ulam–Hyers-Rassias stability with respect with* $\psi(t) = \exp(nt)$, $n > 0$. *As example, for* $n = 0.25$ *we get* $\beta_\psi = 2.3333$ *and* $\gamma = 5.1851 \times E_{1/2}(0.44444)$.

Conclusion. In this paper, we studied a class of nonlinear fractional differential equations with a discrete delay. Such problems are real-life models and their stability analysis is one of the most important investigated topics. We used contraction mapping principle to obtain existence and uniqueness result, then we investigated the continuous dependence upon the initial data due to the delay and Ulam's stability which guarantees the existence of an $\varepsilon-$solution.

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