International Journal of Engineering & Applied Sciences (IJEAS)

Vol.6, Issue 3(2014)37-51

FREE VIBRATION OF AXIALLY FUNCTIONALLY GRADED BEAMS IN THERMAL ENVIRONMENT

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Abstract

In this paper, free vibration characteristics of an axially functionally graded (AFG) cantilever beam subject to temperature rising are studied with the effect of material-temperature dependent properties. Material properties of the beam are temperature-dependent and change in the axial direction of the beam according to a power-law function. The considered problem is solved within the Euler-Bernoulli beam theory by using finite element method. The system of equations of motion is derived by using Lagrange's equations. The effects of material distributions and temperature rising on the first three natural frequencies and mode shapes are investigated.

Keywords: Free vibration analysis, Functionally graded materials, Temperature dependent physical properties, Finite element method

1. Introduction

Functionally graded materials (FGMs), a novel class of composites whose composition varies continuously as a function of position along thickness of a structure to achieve a required function. Functionally graded structures have been an area of intensive research overthe last decade. Because of the wide material variations and applications of FGMs, it is important to study the static and dynamic analysis of FG structures, such as beams and plates. Therefore, an intensive study has been conducted recently on vibration of structures made of FGMs (i.e., [1–37]). In recent years, the mechanical behavior of axially functionally graded (AFG) materials has been a topic of active research. Wu et al. [38] used the semi-inverse method to find the solutions to the dynamic equation of axially functionally graded simply supported beams. Aydogdu [39] analyzed the vibration and buckling of axially functionally graded simply-supported beam by using semi-inverse method. Huang and Li [40] presented a new approach for free vibration of axially functionally non-uniform graded beams. Alshorbgy et al. [41] have investigated the dynamic characteristics of non-uniform graded beams with material graduation in axially or transversally thorough the thickness. Shahba et al. [42] studied free vibration and stability analysis of AFG tapered Timoshenko beams by using finite element method. Hein and Feklistova [43] investigated free vibrations of non-uniform and AFG beams using Haar wavelets. Şimşek et al. [44] studied dynamic behavior of an AFG beam under action of a moving harmonic load. Shahba and Rajasekaran [45] studied free vibration and stability of AFG tapered Euler-Bernoulli beams. Huang et al. [46] presented a new approach for investigating the vibration behaviors of AFG Timoshenko beams with non-uniform cross-section. Free vibration of axially inhomogeneous beams is analyzed by Li et al. [47]. Rajasekaran [48,49] investiagted the free bending vibration of AFG tapered and non-uniform beams by using Differential Transformation method and differential quadrature element method. Akgöz and Civalek [50] studied vibration response of AFG tapered microbeams in conjunction with Bernoulli-Euler beam and modified couple stress theory by using Rayleigh-Ritz solution method. Nguyen [51] investigated large displacement analysis of tapered an AFG cantilever beam by using finite element method. Babilio [52] presented the dynamics of an AFG simply supported beam under axial time-dependent load. Rajasekaran and Norouzzadeh Tochaei [53] presented free vibration characteristics of tapered and AFG Timoshenko beams by using differential transformation element method and differential quadrature element method of lowest-order.

In this study, free vibration characteristics of an AFG cantilever beam subject to uniform temperature rising are studied with the effect of material-temperature dependent properties. The considered problem is investigated within the Bernoulli-Euler beam theory by using energy based finite element method. Material properties of the beam change in the axial direction according to a power-law function. In the study, the effects of the material distributions and temperature rising on the first three natural frequencies and mode shapes are presented for AFG beam.

2. Theory and Formulations

A cantilever beam of length *L*, width *b*, thickness *h*, made of AFG elastic material, as shown in Figure 1.

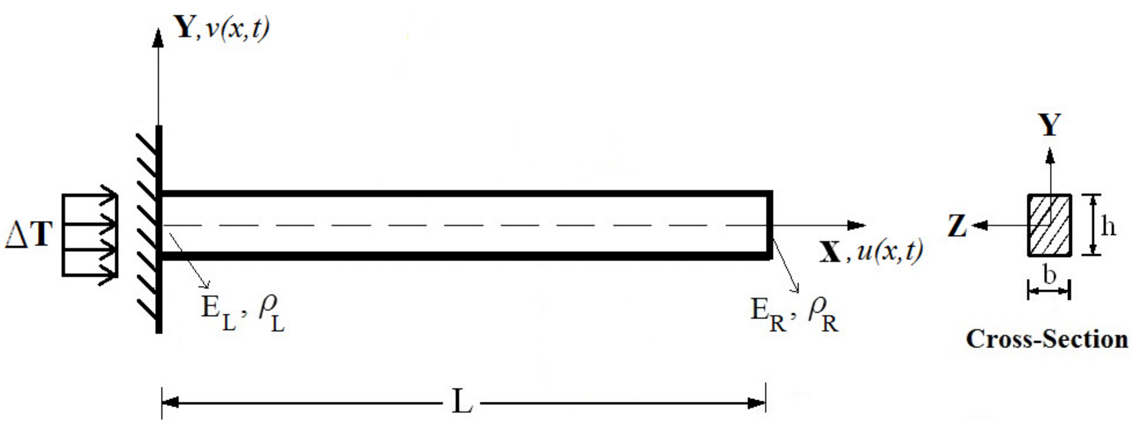


Fig. 1 A cantilever axially functionally graded (AFG) beam under uniform temperature rising DT.

In this study, the material properties are both temperature-dependent and position-dependent. The effective material properties of the functionally graded beam, *P*, i.e., Young's modulus *E,* and mass density r vary continuously in the axial direction (*X* axis) according to a power-law function and a function of temperature *T* (see Touloukian [54]) as follows;

$$P(X,T) = (P_L(T) - P_R(T))\left(1 - \frac{X}{L}\right)^n + P_R(T) \tag{1a}$$

$$P(T) = P_0(P_{-1}T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3) \tag{1b}$$

where P_L and P_R are the material properties of the left and the right surfaces of the beam that depends on temperature (T), $T=T_0+DT$, where T_0 is installation temperature and DT is the uniform temperature rise, It is clear from Eq. (1a) that when $X=-L/2$, $P=P_L$, and when $X=L/2$, $P=P_R$. Where n is the non-negative power-law exponent which dictates the material variation profile through the axial direction of the beam. In Eq. (1b), P_{-1}, P_0, P_1, P_2 and P_3 indicate the coefficients of temperature T and are unique to the constituent materials. In this study, the unit of the temperature is Kelvin (K), the unit of the Young's modulus E is Pascal (Pa) and the unit of the mass density r is kg/m^3. The beams considered in numerical examples are made of Zirconia and Aluminum Oxide. The right surface of the functionally graded beam is Zirconia and the left surface of the functionally graded beam is Aluminum Oxide. The coefficients of temperature T for Zirconia and Aluminum Oxide are listed in Table 1 and 2 (from Reddy and Chin [55]).

Table 1 The coefficients of temperature *T* for Zirconia (from Reddy and Chin [55])

The material properties	P_0	P_{-1}	P_1	P_2	P_3
Thermal expansion coefficient α_X (*1/K*)	12.766×10^{-6}	0	-1.491×10^{-3}	1.0006×10^{-5}	-6.778×10^{-11}
Young's modulus *E* (Pa)	244.27×10^{9}	0	-1.371×10^{-3}	1.214×10^{-6}	-3.681×10^{-10}
Poisson's ratio n	0.2882	0	1.133×10^{-4}	0	0
Mass density r (kg/m^3)	5700	0	0	0	0

Table 2 The coefficients of temperature *T* for Aluminum Oxide (from Reddy and Chin [55])

The material properties	P_0	P_{-1}	P_1	P_2	P_3
Thermal expansion coefficient α_X (*1/K*)	6.8269×10^{-6}	0	1.838×10^{-4}	0	0
Young's modulus *E* (Pa)	349.55×10^{9}	0	-3.853×10^{-4}	4.027×10^{-7}	-1.673×10^{-10}
Poisson's ratio n	0.26	0	0	0	0
Mass density r (kg/m^3)	2700	0	0	0	0

Acoording to the coordinate system *(X,Y,Z)* shown in figure 1, based on Euler-Bernoulli beam theory, the axial and the transverse displacement field are expressed as

$$u(X,Y,t) = u_0(X,t) - Y\frac{\partial v_0(X,t)}{\partial X} \tag{2}$$

$$v(X,Y,t) = v_0(X,t) \tag{3}$$

Where u_0 and v_0 are the axial and the transverse displacements in the mid-plane, t indicates time. Using Eq. (2) and (3), the linear strain- displacement relation can be obtained:

$$\varepsilon_{xx} = \frac{\partial u}{\partial X} = \frac{\partial u_0(X,t)}{\partial X} - Y\frac{\partial^2 v_0(X,t)}{\partial X^2} \tag{4}$$

According to Hooke's law, constitutive equations of the AFG beam are as follows:

$$\begin{aligned}\sigma_{xx} &= E(X,T)\,\varepsilon = E(X,T)[\varepsilon_{xx} - \alpha(X,T)\Delta T] \\ &= E(X,T)[\frac{\partial u_0(X,t)}{\partial X} - Y\frac{\partial^2 v_0(X,t)}{\partial X^2} - \alpha(X,T)\Delta T]\end{aligned} \tag{5}$$

Where s_{xx} and$v$ are normal stresses and normal strains in the X direction, respectively. Based on Euler-Bernoulli beam theory, the elastic strain energy (U_i) of the beam is expressed as

$$U_i = \frac{1}{2}\int_0^L \int_A \sigma_{xx}\,\varepsilon\, dA\, dX \tag{6}$$

By substituting equations (4) and (5) into Eq. (6), elastic strain energy (U_i) can be rewritten as follows:

$$\begin{aligned}U_i &= \frac{1}{2}\int_0^L E(X,T)A(\frac{\partial u_0}{\partial x})^2 dX + \frac{1}{2}\int_0^L E(X,T)I(\frac{\partial^2 v_0}{\partial x^2})^2 dX \\ &- \int_0^L E(X,T)A\,\alpha(X,T)\Delta T(\frac{\partial u_0}{\partial x})\, dX + \frac{1}{2}\int_0^L E(X,T)A(\alpha(X,T)\Delta T)^2\, dX\end{aligned} \tag{7}$$

Where A and I are respectively the cross-sectional area and moment of inertia. Kinetic energy (V) of the FGM beam are expressed as follows:

$$V = \frac{1}{2}\int_0^L \int_A \rho(X,T)[(\frac{\partial u}{\partial t})^2 + (\frac{\partial v}{\partial t})^2]\, dA\, dX \tag{8}$$

By substituting equations (2) and (3) into Eq. (8), Kinetic energy (V) can be rewritten as follows:

$$V = \frac{1}{2}\int_0^L [\rho(X,T)A(\frac{\partial u_0}{\partial t})^2 + \rho(X,T)A(\frac{\partial v_0}{\partial t})^2 + \rho(X,T)I(\frac{\partial^2 v_0}{\partial X \partial t})^2]\, dX \tag{9}$$

Lagrangian functional of the problem is given as follows:

$$L = V - U_i \tag{10}$$

Total nodal displacements q which is written for a two-node beam element, each node has three degrees of freedom, shown in Fig. 2 are defined as follows:

$$\{q(t)\}_{(e)} = [u_i^{(e)}(t),\ v_i^{(e)}(t),\ \theta_i^{(e)}(t),\ u_j^{(e)}(t),\ v_j^{(e)}(t),\ \theta_j^{(e)}(t)]^T \tag{11}$$

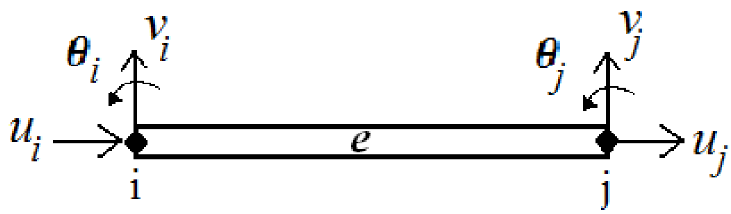


Figure 2. A two-node beam element.

The displacement field of the finite element is expressed in terms of nodal displacements as follows:

$$\begin{aligned} u^{(e)}(X,t) &= \varphi_1^{(U)}(X)\, u_i(t) + \varphi_2^{(U)}(X)\, u_j(t) \\ &= [\varphi^{(U)}]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = [\varphi^{(U)}]\{q\}_U \end{aligned} \tag{12}$$

$$\begin{aligned} v^{(e)}(X,t) &= \varphi_1^{(v)}(X)\, v_i(t) + \varphi_2^{(v)}(X)\, \theta_i(t) + \varphi_3^{(v)}(X)\, v_j(t) + \varphi_4^{(v)}(X)\, \theta_j(t) \\ &= [\varphi^{(V)}]\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = [\varphi^{(V)}]\{q\}_V \end{aligned} \tag{13}$$

where u_i, v_i and θ_i are axial displacements, transverse displacements and slopes at the two end nodes of the beam element, respectively. $\phi_i^{(U)}$ and $\phi_i^{(V)}$ are Hermitian shape functions for axial and transverse degrees of freedom, respectively, which are given in Appendix.

After substituting Equation (11) into Equation (10) and then using the Lagrange's equations gives the following equation;

$$\frac{\partial L}{\partial q_k^{(e)}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k^{(e)}} = 0, \quad k = 1, 2, 3, 4, 5, 6 \tag{14}$$

Where $\dot{q}_k^{(e)}$ indicates the time derivatives of nodal displacements q.

The Lagrange's equations yield the system of equations of motion for the finite element and by use of usual assemblage procedure the following system of equations of motion for the whole system can be obtained as follows:

$$[K]\{q(t)\}+[M]\{\ddot{q}(t)\}=0 \tag{15}$$

where, $[K]$ is the stiffness matrix and $[M]$ is mass matrix. The components of the stiffness matrix $[K]$:

$$[K]=\begin{bmatrix}[K^A] & [0] \\ [0] & [K^D]\end{bmatrix}, \tag{16}$$

Where

$$[K^A]=\int_0^{L_e} E(X,T)A\left[\frac{d\varphi^{(U)}}{dX}\right]^T\left[\frac{d\varphi^{(U)}}{dX}\right]dX\,, \tag{17a}$$

$$[K^D]=\int_0^{L_e} E(X,T)I\left[\frac{d^2\varphi^{(V)}}{dX^2}\right]^T\left[\frac{d^2\varphi^{(V)}}{dX^2}\right]dX\,, \tag{17b}$$

Where L_e indicates the length of the finite beam element. The mass matrix $[M]$ can be expressed as a sum of four sub-matrices as shown below:

$$[M]=[M_U]+[M_V]+[M_\theta] \tag{18}$$

Where

$$[M_U]=\int_0^{L_e} \rho(X,T)A[\varphi^{(U)}]^T[\varphi^{(U)}]dX \tag{19a}$$

$$[M_V]=\int_0^{L_e} \rho(X,T)A[\varphi^{(V)}]^T[\varphi^{(V)}]dX \tag{19b}$$

$$[M_\theta]=\int_0^{L_e} \rho(X,T)A[\frac{d\varphi^{(V)}}{dX}]^T[\frac{d\varphi^{(V)}}{dX}]dX \tag{19c}$$

If the global nodal displacement vector $\{q\}$ is assumed to be harmonic in time with circular frequency w, i.e $\{q\}=\{\hat{q}\}e^{iwt}$ becomes, after imposing the appropriate end conditions, an eigenvalue problem of the form:

$$([K]-\omega^2[M])\{\hat{q}\}=0 \tag{20}$$

Where $\{\hat{q}\}$ is a vector of displacement amplitudes of the vibration.

3. Numerical Results

In the numerical examples, the natural frequencies and the mode shapes of the AFG beams are calculated and presented in figures for different material distributions and temperature rising. The beams considered in numerical examples are made of Zirconia and Aluminum Oxide. The right surface of the AFG beam is Zirconia and the left surface of the AFG beam is Aluminum Oxide. When the power index n=0, the beam material is reduced to full Aluminum Oxide (homogeneous Aluminum Oxide). The coefficients of temperature T for Zirconia and Aluminum Oxide are listed in Table 1 and 2 (from Reddy and Chin [55]). In this study, the unit of the temperature is Kelvin (K). In numerical examples, the initial temperature (installation temperature) of the beam is assumed to be T_0= 300 K. In numerical calculations, the number of finite elements is taken as 100 elements. Unless otherwise stated, it is assumed that the width of the beam is b=0.1m, height of the beam is h=0.1m and length of the beam is L=3m in the numerical results.

In order to establish the accuracy of the present formulation and the computer program developed by the author, the results obtained from the present study are compared with the available results in the literature. For this purpose, the dimensionless fundamental frequency ($\bar{\varpi} = \omega / \sqrt{D_{XX}/I_1}$) of a FG cantilever beam according to the exponential distribution are calculated for different exponential ratios P_R for L/h=20 compared with those of Yang and Chen [2008] and Ke et al. [2009]. As seen from Table 3, the present results are in good agreement with that the results of Yang and Chen [8] and Ke et al. [15].

Table 3. Comparison of the dimensionless fundamental frequency $\bar{w}_1$ of intact FG beams.

P_R	*Present*	*Yang and Chen [2008]*	*Ke et al. [2009]*
0.2	0.8283	0.83	0.8235
1	0.8786	0.88	0.8752
5	0.8283	0.83	0.8235

In figure 3, the effect of the temperature rising on the first three lower natural frequencies w of the AFG beam is shown for different the material distributions n.

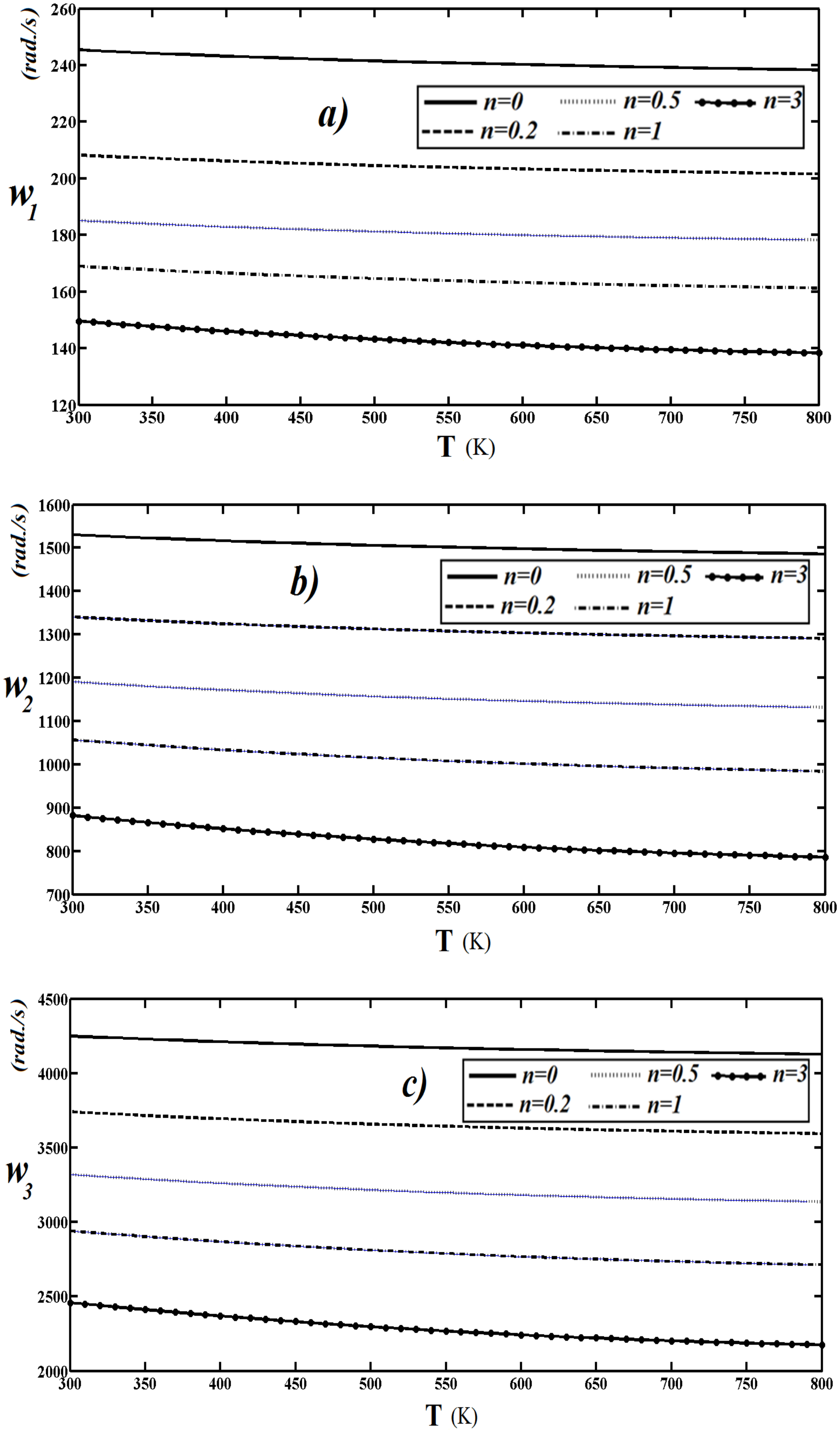
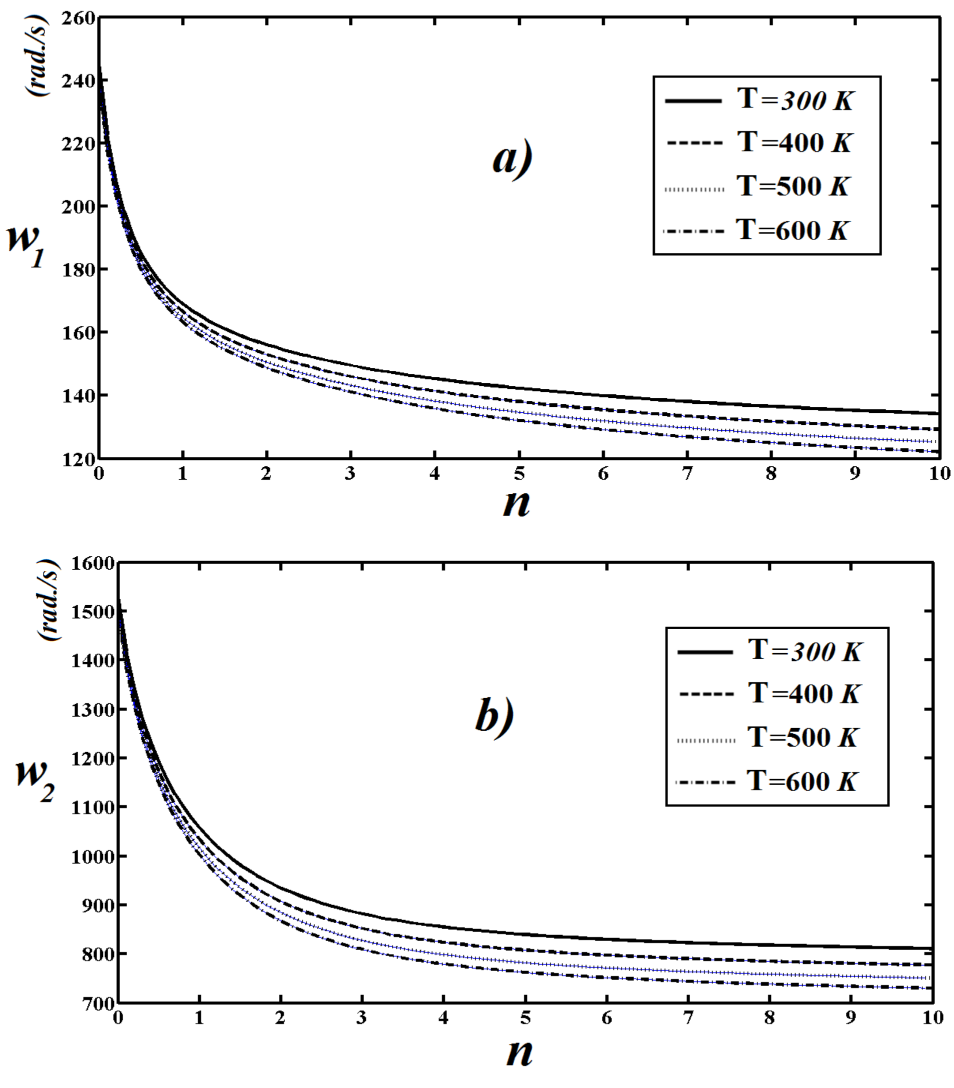


Figure 3. The effect of Temperature (T) on the first three lower natural frequencies w for different the material distributions n. a) First natural frequency (w_1), b) Second natural frequency (w_2) and c) Third natural frequency (w_3).

It is seen from figure 3, with increase in temperature, the fundamental frequency decreases, as expected. This is because; with the temperature increase, the intermolecular distances of the material increase and intermolecular forces decrease according to temperature-dependent physical properties and the beam pile becomes flexible. As a result, the frequencies of the AFG beam decreases. Also it is observed from figure 3 that there are significant different of the material distributions.

In figure 4, the effect of the material distributions n on the first three lower natural frequencies w of the AFG beam is shown for different the temperature (T) .



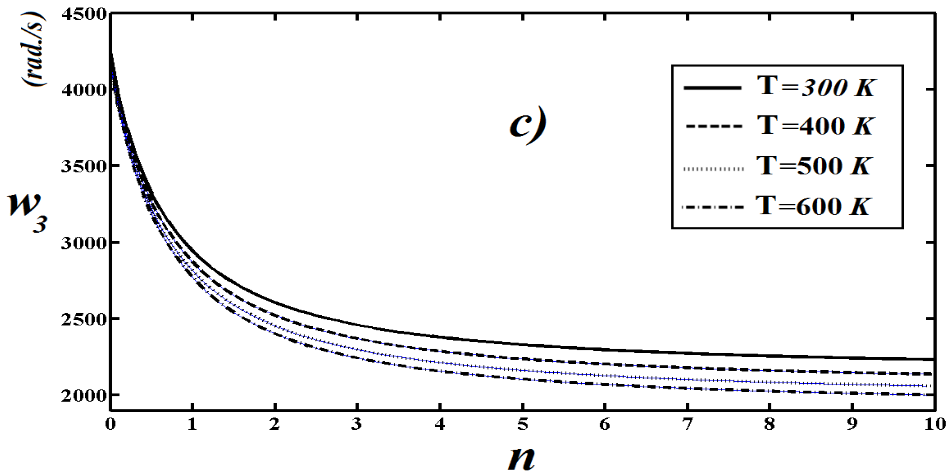
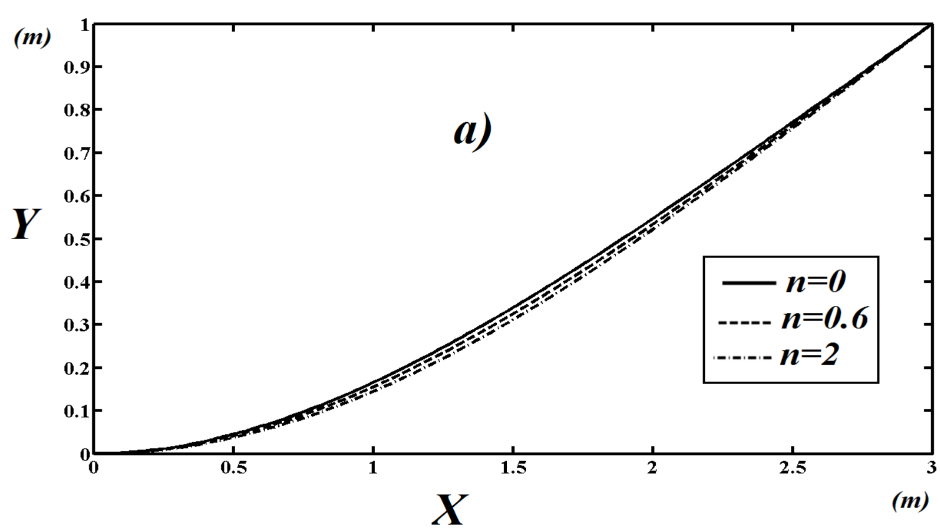


Figure 4. The effect material distributions (n) on the first three lower natural frequencies w for different the Temperature (T). a) First natural frequency (w_1), b) Second natural frequency (w_2) and c) Third natural frequency (w_3).

It is seen from figure 4, with increase in the material power law index n causes decrease in the fundamental frequency for all values of the temoerature (T): Because when the material power law index n increase, the material of the beam get close to Zirconia (right side material) according to Eq. 1 and it is known from the physical properties of the Aluminum Oxide (left side material) and Zirconia (right side material) that the Young modulus of of Aluminum Oxide is greater than that of Zirconia. As a result, the strength of the material increases and the fundamental frequency of the AFG beam decraeses. It is observed from figure 4 that increase in the material power law index n, the curve has an asymptote. In the case of n=∞, the AFG beam is reduced to the homogeneous Zirconia (Right side material) beam according to Eq. 1. Also, it is seen from figure 4 that with increase the material power law index n, the differences of the different temperature (T) increase considerably on the fundamental frequency.

Figure 5 displays the effect of the material distributions n on the first, second and third vibration mode shapes for T=400 K .



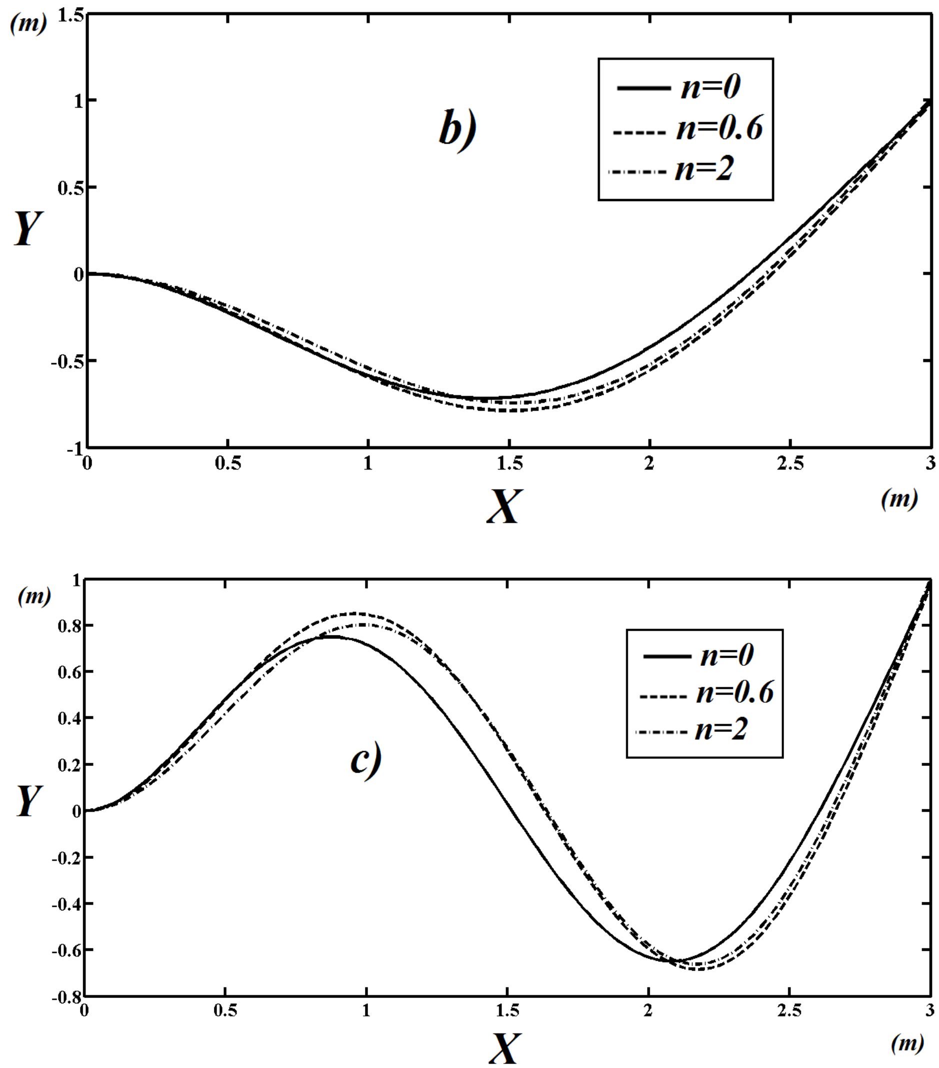


Figure 5. The effect material distributions (n) on the first three lower vibration modes for Temperature T=400 K. a) First mode shape, b) Second mode shape and c) Third mode shape.

It is seen from figure 5 that the material distributions n play important role on the vibration mode shapes.

4. Conclusions

Free vibration analysis of an AFG cantilever beam subjected to temperature rising is investigated under with the effect of material-temperature dependent properties. Material properties of the beam are temperature-dependent and change in the axial direction of the beam according to a power-law function. The considered problem is solved within the Euler-Bernoulli beam theory by using finite element method. The system of equations of motion is derived by using Lagrange's equations. It is observed from the investigations that temperature rising have a great influence on the vibration characteristics of the AFG beam. With increase in the temperature, the vibration characteristics of the AFG beam change considerably. The distribution of the AFG material plays an important role on the vibration frequency and mode of the beam.

Appendix

The interpolation functions for axial degrees of freedom are

$$\varphi^{(U)}(X)=[\varphi_1^{(U)}(X)\quad \varphi_2^{(U)}(X)]^T, \tag{A.1}$$

Where

$$\varphi_1^{(U)}(X)=\left(-\frac{X}{L_e}+1\right), \tag{A.2}$$

$$\varphi_2^{(U)}(X)=\left(\frac{X}{L_e}\right), \tag{A.3}$$

The interpolation functions for transverse degrees of freedom are

$$\varphi^{(V)}(X)=\left[\varphi_1^{(V)}(X)\quad \varphi_2^{(V)}(X)\quad \varphi_3^{(V)}(X)\quad \varphi_4^{(V)}(X)\right]^T, \tag{A.4}$$

Where

$$\varphi_1^{(V)}(X)=\left(1-\frac{3X^2}{L_e^2}+\frac{2X^3}{L_e^3}\right), \tag{A.5}$$

$$\varphi_2^{(V)}(X)=\left(-X+\frac{2X^2}{L_e}-\frac{X^3}{L_e^3}\right), \tag{A.6}$$

$$\varphi_3^{(V)}=\left(\frac{3X^2}{L_e^2}-\frac{2X^3}{L_e^3}\right), \tag{A.7}$$

$$\varphi_4^{(V)}(X)=\left(\frac{X^2}{L_e}-\frac{X^3}{L_e^3}\right), \tag{A.8}$$

Where L_e indicates the length of the finite beam element.

References

[1] Sankar BV, An elasticity solution for functionally graded beams. *Compos Sci Technol,* 61(5):689–96, 2001.

[2] Chakraborty A, Gopalakrishnan S and Reddy JN, A new beam finite element for the analysis of functionally graded materials. *Int J Mech Sci,* 45(3):519–39, 2003.

[3] Chakraborty A and Gopalakrishnan S, A spectrally formulated finite element for wave propagation analysis in functionally graded beams. *Int J Solids Struct,* 40(10):2421–48, 2003.

[4] Aydogdu M and Taskin V, Free vibration analysis of functionally graded beams with simply supported edges. *Mater Des,* 28(5):1651–6, 2007.

[5] Zhong Z and Yu T, Analytical solution of a cantilever functionally graded beam. *Compos Sci Technol,* 67(3–4):481–8, 2007.

[6] Ying J, Lu CF and Chen WQ. Two-dimensional elasticity solutions for functionally graded beams resting on elastic foundations. *Compos Struct*, 84(3):209–19, 2008.

[7] Kapuria S, Bhattacharyya M and Kumar AN, Bending and free vibration response of

layered functionally graded beams: a theoretical model and its experimental validation. *Compos Struct*, 82(3):390–402, 2008.
[8] Yang J and Chen Y, Free vibration and buckling analyses of functionally graded beams with edge cracks. *Compos Struct*, 83(1):48–60, 2008.
[9] Li XF, A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams. *J Sound Vib*, 318(4–5):1210–29, 2008.
[10] Xiang HJ and Yang J, Free and forced vibration of a laminated FGM Timoshenko beam of variable thickness under heat conduction. *Compos Part B: Eng*, 39(2):292–303, 2008.
[11] Kadoli R, Akhtar K and Ganesan N, Static analysis of functionally graded beams using higher order shear deformation theory. *Appl Math Model*, 32(12):2509–25, 2008.
[12] Benatta MA, Mechab I, Tounsi A and Adda Bedia EA. Static analysis of functionally graded short beams including warping and shear deformation effects. *Comput Mater Sci*, 44(2):765–73, 2008.
[13] Sina SA, Navazi HM and Haddadpour H, An analytical method for free vibration analysis of functionally graded beams. Mater Design, 30(3):741–7, 2009.
[14] Pradhan SC and Murmu T, Thermo-mechanical vibration of FGM sandwich beam under variable elastic foundations using differential quadrature method. *J Sound Vib*, 321(1–2):342 62, 2009.
[15] Ke LL, Yang J, Kitipornchai S and Xiang Y, Flexural vibration and elastic buckling of a cracked Timoshenko beam made of functionally graded materials. Mech *Adv Mater Struc*, 16:488–502, 2009.
[16] Kitipornchai S, Ke LL, Yang J and Xiang Y, Nonlinear vibration of edge cracked functionally graded Timoshenko beams. *J Sound Vib*, 324(3–5):962–82, 2009.
[17] Ke LL, Yang J and Kitipornchai S, Postbuckling analysis of edge cracked functionally graded Timoshenko beams under end shortening. *Compos Struct*,90(2):152–60, 2009.
[18] Pradhan SC and Sarkar A, Analyses of tapered FGM beams with nonlocal theory. *Struct Eng Mech*, 32:811–33, 2009.
[19] Şimşek M and Kocatürk T, Free and Forced Vibration of a Functionally Graded Beam Subjected to a Concentrated Moving Harmonic Load. *Compos Struct*, 90(40), 465-473, 2009.
[20] Li, SR., Su, HD. and Cheng, CJ, Free vibration of functionally graded material beams with surface-bonded piezoelectric layers in thermal environment, *Applied Mathematics and Mechanics (English Edition)*, 30 (8), pp. 969-982, 2009.
[21] Anandrao KS, Gupta RK, Ramchandran P and Rao GV, Thermal post-buckling analysis of uniform slender functionally graded material beams. *Struct Eng Mech*, 36(5):545–60, 2010.
[22] Ke LL, Yang J and Kitipornchai S, Nonlinear free vibration of functionally graded carbon nanotube-reinforced composite beams. *Compos Struct*, 92(3):676–83, 2010.
[23] Şimşek M, Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories. *Nucl Eng Des,* 240:697–705, 2010.
[24] Kocatürk T, Şimşek M and Akbaş ŞD, Large displacement static analysis of a cantilever Timoshenko beam composed of functionally graded material. *Sci Eng Compos Mater*, 18(1):21–34, 2011.
[25] Fallah A and Aghdam MM, Nonlinear free vibration and post-buckling analysis of functionally graded beams on nonlinear elastic foundation. *Eur J Mech A –Solids*, 30(4):571 83, 2011.
[26] Kocatürk T and Akbaş ŞD, Post-buckling analysis of Timoshenko beams made of functionally graded material under thermal loading, *Struct Eng Mech*, 41, 775-789, 2012.
[27] Yan T, Yang J and Kitipornchai S, Nonlinear dynamic response of an edge-cracked functionally graded Timoshenko beam under parametric excitation. *Nonlinear Dynamics*, 67(1), 527–540, 2012.

[28] Wei D, Liu Yand Xiang Z, An analytical method for free vibration analysis of functionally graded beams with edge cracks. *Journal of Sound and Vibration,* 331(7), 1686–1700, 2012.
[29] Akbaş ŞD, Geometrically Nonlinear Static Analysis of Edge Cracked Timoshenko Beams Composed of Functionally Graded Material. *Mathematical Problems in Engineering, 2013,* doi: 10.1155/2013/871815, 2013.
[30] Kocatürk T and Akbaş ŞD, Thermal post-buckling analysis of functionally graded beams with temperature-dependent physical properties. *Steel and Composite Structures*, 15, 485-505, 2013.
[31] Akbaş ŞD and Kocatürk T, Post-buckling Analysis of Functionally Graded Three-Dimensional Beams Under the Influence of Temperature. *Journal of Thermal Stresses*, 36, 1233-1254, 2013.
[32] Akbaş ŞD, Free vibration characteristics of edge cracked functionally graded beams by using finite element method. International *Journal of Engineering Trends and Technology*, 4, 4590-4597, 2013.
[33] Alshorbagy AE, Temperature effects on the vibration characteristics of a functionally graded thick beam *Ain Shams Engineering Journal,* 4, 455–464, 2013.
[34] Pradhan KK and Chakraverty S, Free vibration of Euler and Timoshenko functionally graded beams by Rayleigh-Ritz method, Composites Part B: Engineering, 51,175-184, 2013.
[35] Ramesh MNV and Rao NM, Free vibration analysis of rotating functionally-graded cantilever beams. International Journal of Acoustics and Vibrations, 19(1),31-41, 2014.
[36] Hadji L, Daouadji TH, Tounsi A and Bedia EA, A higher order shear deformation theory for static and free vibration of FGM beam. Steel and Composite Structures, 16(5), 507-519, 2014.
[37] Mashat DS, Carrera E, Zenkour AM, Al Khateeb SA and Filippi M,Free vibration of FGM layered beams by various theories and finite elements. Composites Part B: Engineering, 59, 269-278, 2014.
[38] Wu L, Wang Q and Elishakoff I, Semi-inverse method for axially functionally graded beams with an anti-symmetric vibration mode, J Sound Vib, 284:1190–202, 2005.
[39] Aydogdu M, Semi-inverse method for vibration and buckling of axially functionally graded beams. J Reinf Plast Compos, 27:683–91, 2008.
[40] Huang Y and Li XF. A new approach for free vibration of axially functionally graded beams with non-uniform cross-section. *J Sound Vib,* 329:2291–303, 2010.
[41] Alshorbgy AE, Eltaher MA and Mahmoud FF. Free vibration characteristics of a functionally graded beam by finite element method. *App Math Model*, 35:412–25, 2011.
[42] Shahba A, Attarnejad R, Marvi MT and Hajilar S. Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions. *Composites Part B*, 42:801–8, 2011.
[43] Hein H and Feklistova L, Free vibrations of non-uniform and axially functionally graded beams using Haar wavelets. *Engineering Structures*, 33 (12), 3696-3701, 2011.
[44] Şimşek M, Kocatürk T, Akbaş ŞD, Dynamic behavior of an axially functionally graded beam under action of a moving harmonic load. *Composite Structures*, 94 (8), 2358-2364, 2012.
[45] Shahba A.,and Rajasekaran S, Free vibration and stability of tapered Euler-Bernoulli beams made of axially functionally graded materials. *Applied Mathematical Modelling*, 36 (7), 3094-3111, 2012.
[46] Huang Y, Yang LE and Luo QZ, Free vibration of axially functionally graded Timoshenko beams with non-uniform cross-section. *Composites Part B: Engineering*, 45 (1), 1493-1498, 2013.
[47] Li XF, Kang YA and Wu JX, Exact frequency equations of free vibration of

exponentially functionally graded beams. *Applied Acoustics*, 74 (3), 413-420, 2013.
[48] Rajasekaran S, Free vibration of centrifugally stiffened axially functionally graded tapered Timoshenko beams using differential transformation and quadrature. *Applied Mathematical Modelling*, 37 (6), 4440-4463, 2013.
[49] Rajasekaran S, Buckling and vibration of axially functionally graded nonuniform beams using differential transformation based dynamic stiffness approach. *Meccanica*, 48 (5), 1053-1070, 2013.
[50] Akgöz B and Civalek Ö, Free vibration analysis of axially functionally graded tapered Bernoulli-Euler microbeams based on the modified couple stress theory. *Composite Structures*, 98, 314-322, 2013.
[51] Nguyen DK, Large displacement response of tapered cantilever beams made of axially functionally graded material. *Composites Part B: Engineering*, 55, 298-305, 2013.
[52] Babilio E, Dynamics of an axially functionally graded beam under axial load. *European Physical Journal: Special Topics*, 222(7),1519-1539, 2013.
[53] Rajasekaran S and Norouzzadeh Tochaei E, Free vibration analysis of axially functionally graded tapered Timoshenko beams using differential transformation element method and differential quadrature element method of lowest-order. Meccanica**,** 49(4), 995 1009, 2014.
[54] Touloukian YS, *Thermophysical Properties of High Temperature Solid Materials.* Macmillan, New York., 1967.
[55] J.N. Reddy JN and Chin CD, Thermoelastical analysis of functionally graded cylinders and plates. *Journal of Thermal Stresses*, 21(6), 593-626, 1998.