

An Analytical Solution for Free Vibrations of A Cantilever Nanobeam with A Spring Mass System

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Abstract

An analytical solution for the title problem is presented using the nonlocal elasticity theory based on Euler-Bernoulli beam theory. Fourier sine series is used to represent lateral displacement of the nanobeam. Stokes' transformation is applied to derive the coefficient matrix of the corresponding systems of linear equations. This matrix also contains the relationship between spring and mass parameters. A convergence study is provided to show how the first three frequency parameter of the nanobeam would converge by an increase of series terms in the literature. The results are given in a series of figures and tables for various combinations of boundary conditions.

Keywords: Nonlocal elasticity theory, nanobeam, Fourier sine series, Stokes' transformation.

1. Introduction

Nanobeams have attracted a significant attention due to their superior mechanical and electrical properties and their potential applications in electronics, optics and other fields of nanoscience. Nowadays, it is being extensively utilized as nanostructure components for microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS). Since controlled experiments including nanoscale effects are both expensive and difficult, the definition of appropriate mathematical models for micro/nanostructures is an important issue concerning the theoretical framework of micro/nanostructures. In the design of nanobeam-based components such as MEMS and NEMS, it is of great importance to acquire the dynamical behaviors of nanobeams accurately. The application of classical elasticity theory may be questionable in the dynamical analysis of micro/nanostructures. Classical elasticity theory is scale-free theory and this theory lacks the accountability of the small scale effects arising from the small-size. Eringen [1] introduced the nonlocal elasticity theory in order to overcome the shortcomings of classical continuum models. He assumes that the stress field at a point not only depends on the strain field at that point but also on strains at all other points of the body.

The application of classical continuum models may be questionable in the analysis of "smaller" nanostructures. Eringen [1] have pioneered the nonlocal elasticity theory in 1970s. The internal size effect can be considered in the constitutive equations simply as a material scale parameter. Such a nonlocal elasticity theory has been widely accepted and applied to many physical problems of a wide range of interest, including the buckling, bending, and vibration of plate-like structures [2-4] and beam-like structures [5-7] and elements in micro/nano structures. Many research works correlated to

nonlocal elasticity theory have been reported and apply them to analyze the mechanical behavior of structures in nano/microscale, see Refs. [8-17].

In this work, an analytical method, called Stokes' transformation is introduced for the vibration analysis of the cantilever nanobeams with a spring mass system. Fourier sine series is used to represent the displacement function. A coefficient matrix is constructed by applying Stokes' transformation to the nonlocal boundary conditions. As a numerical example, a cantilever nanobeam with a spring mass system is considered. The effects of boundary conditions, mass parameter, and translational spring parameter on the natural frequencies are examined and graphically presented.

2. Formulation of the problem

According to nonlocal elasticity theory, the fourth-order partial differential equation of the nanobeam is given by [18]

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} + \left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2} \right] \rho A \frac{\partial^2 v(x,t)}{\partial t^2} = 0, \quad (1)$$

where EI is the flexural rigidity of the nanobeam, E is Young's modulus, I is the moment of inertia of the cross-sectional area A , ρ the density of the beam material a the internal characteristic length and e_0 is a constant. To analyze the lateral vibration of the nanobeam, the lateral displacement $v(x,t)$ is described by

$$v(x,t) = \Psi(x) \cos(\omega t) \quad (2)$$

where $\Psi(x)$ is the modal displacement function and ω is the natural frequency. The modal displacement function is described as following on identification of relationships between boundary points;

$$v(x) = \left\{ \begin{array}{ll} \Psi_0 & x = 0 \\ \Psi_L & x = L \\ \sum_{j=1}^{\infty} D_j \sin(\beta_j x) & 0 < x < L \end{array} \right\}, \quad (3)$$

where

$$\beta_j = \frac{j\pi}{L}, \quad (4)$$

where L is the length of the nanobeam. The Fourier coefficient (D_j) in Eq. (3) can be conveniently written as;

$$D_j = \frac{2}{L} \int_0^L v(x) \sin(\beta_j x) dx. \quad (5)$$

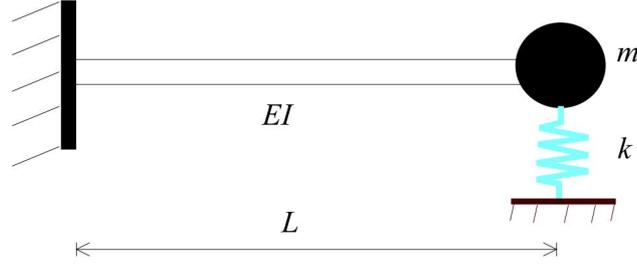


Figure 1: A cantilever nanobeam with a spring mass system

Termwise differentiation of Eq. (3) yields

$$v'(x) = \sum_{k=1}^{\infty} \beta_j D_j \cos(\beta_j x). \quad (6)$$

Eq. (6) can be represented by a Fourier cosine series as follows:

$$v'(x) = \frac{r_0}{L} + \sum_{j=1}^{\infty} r_j \cos(\beta_j x). \quad (7)$$

The Fourier coefficients in Eq. (7) are given by

$$r_0 = \frac{2}{L} \int_0^L v'(x) dx = \frac{2}{L} [v(L) - v(0)], \quad (8)$$

$$r_j = \frac{2}{L} \int_0^L v'(x) \cos(\beta_j x) dx \quad (j=1, 2, \dots). \quad (9)$$

Applying integration by parts, we get;

$$r_j = \frac{2}{L} [v(x) \cos(\beta_j x)]_0^L + \frac{2}{L} \left[\beta_j \int_0^L v(x) \sin(\beta_j x) dx \right], \quad (10)$$

$$r_j = \frac{2}{L} [(-1)^j v(L) - v(0)] + \beta_j D_j. \quad (11)$$

Then, the first derivative of the function is found as follows:

$$\frac{dv(x)}{dx} = \frac{\Psi_L - \Psi_0}{L} + \sum_{j=1}^{\infty} \cos(\beta_j x) \left(\frac{2((-1)^j \Psi_L - \Psi_0)}{L} + \beta_j D_j \right). \quad (12)$$

The above procedure is known as Stokes' transformation. The higher order derivatives of $v(x)$ can be separately determined by employing Stokes' transformation as follows [19]:

$$\frac{d^2 v(x)}{dx^2} = - \sum_{j=1}^{\infty} \beta_j \sin(\beta_j x) \left(\frac{2((-1)^j \Psi_L - \Psi_0)}{L} + \beta_j D_j \right), \quad (13)$$

$$\frac{d^3v(x)}{dx^3} = \frac{\Psi_L - \Psi_0}{L} + \sum_{j=1}^{\infty} \cos(\beta_j x) \left(\frac{2((-1)^j \Psi_L - \Psi_0)}{L} - \beta_j^2 \left(\frac{2((-1)^j \Psi_L - \Psi_0)}{L} + \beta_j D_j \right) \right), \quad (14)$$

$$\frac{d^4v(x)}{dx^4} = - \sum_{j=1}^{\infty} \beta_j \sin(\beta_j x) \left(\frac{2((-1)^j \Psi_L - \Psi_0)}{L} - \beta_j^2 \left(\frac{2((-1)^j \Psi_L - \Psi_0)}{L} + \beta_j D_j \right) \right). \quad (15)$$

Eqs. (13) and (15) are substituted into Eq. (1) to result in

$$\begin{aligned} & \sum_{j=1}^{\infty} \left(\frac{1}{L} \cos(\omega t) \sin(\beta_j x) (-LD_j (A(e_0 a)^2 \rho \omega^2 \beta_j^2 + \rho A \omega^2 - EI \beta_j^4) \right. \\ & - 2\beta_j ((-1)^j (\Psi_L (A(e_0 a)^2 \rho \omega^2 - EI \beta_j^2) + EI \Psi_L) \\ & \left. + \Psi_0 (EI \beta_j^2 - A(e_0 a)^2 \rho \omega^2) + EI \Psi_0) \right) = 0. \end{aligned} \quad (16)$$

Fourier coefficient in Eq. (16) can be written in terms of Ψ_0 , Ψ_L , Ψ_0 and Ψ_L as follows:

$$\begin{aligned} D_j &= \frac{2}{\beta_j^3 L (e_0 a)^2 \beta_j^2 \omega^2 - \omega_j^2 + \omega^2} \left((\Psi_0 - (-1)^j \Psi_L) - \beta_j^2 (\Psi_0 - (-1)^j \Psi_L) + \right. \\ & \left. \frac{\omega^2}{\omega_j^2} \beta_j^4 (e_0 a)^2 (\Psi_0 - (-1)^j \Psi_L) \right), \end{aligned} \quad (17)$$

where

$$\omega_j^2 = \frac{EI}{\rho A} \left(\frac{j^4 \pi^4}{L^4} \right). \quad (18)$$

The function $v(x,t)$ for the free vibration of a nanobeam having free boundary at both ends becomes

$$\begin{aligned} v(x,t) &= \sum_{j=1}^{\infty} \frac{2}{\beta_j^3 L (e_0 a)^2 \beta_j^2 \omega^2 - \omega_j^2 + \omega^2} \left((\Psi_0 - (-1)^j \Psi_L) - \beta_j^2 (\Psi_0 - (-1)^j \Psi_L) + \right. \\ & \left. \frac{\omega^2}{\omega_j^2} \beta_j^4 (e_0 a)^2 (\Psi_0 - (-1)^j \Psi_L) \right) \cos(\omega t) \sin(\beta_j x). \end{aligned} \quad (19)$$

This equation is a more general equation of the present method and can be used with any deformable boundary condition.

3. Boundary conditions

Consider a cantilever nanobeam with translational restraint at the free end with a point mass (see Figure 1). The boundary conditions are mathematically written as,

$$\Psi_0 = 0, \quad \frac{\partial v(x,t)}{\partial x} = 0, \quad x = 0, \quad (20)$$

$$\Psi_L''=0, \quad k\Psi_L - m \frac{\partial^2 v(x,t)}{\partial t^2} = EI \frac{\partial^3 v(x,t)}{\partial x^3}, \quad x=L. \quad (21)$$

The substitution of equations (12), (14) and (17) into Eqs. (20)-(21) leads to the two simultaneous homogeneous equations

$$\left\{ 1 + \sum_{j=1}^{\infty} \frac{2\lambda^4(-1)^j(\pi^2\Delta^2 j^2 + 1)}{\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4} \right\} \Psi_L + \left\{ K - m\pi^4\lambda^4 + \sum_{j=1}^{\infty} \frac{2\lambda^4\pi^2 j^2}{\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4} \right\} \Psi_0'' = 0 \quad (22)$$

$$\left\{ \sum_{j=1}^{\infty} \frac{2j^2}{\pi^2(\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4)} \right\} \Psi_L + \left\{ 1 + \sum_{j=1}^{\infty} \frac{2\lambda^4(-1)^j}{\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4} \right\} \Psi_0'' = 0 \quad (23)$$

where

$$\lambda^4 = \frac{\rho AL^4 \omega^2}{\pi^4 EI}, \quad (24)$$

$$K = \frac{kL^3}{EI}, \quad (25)$$

$$\Delta = \frac{e_0 a}{L}. \quad (26)$$

Eqs. (22)-(23) can be written as a matrix form

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \Psi_L \\ \Psi_0'' \end{bmatrix} = 0 \quad (27)$$

where

$$\Phi_{11} = 1 + \sum_{j=1}^{\infty} \frac{2\lambda^4(-1)^j(\pi^2\Delta^2 j^2 + 1)}{\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4}, \quad (28)$$

$$\Phi_{12} = K - m\pi^4\lambda^4 + \sum_{j=1}^{\infty} \frac{2\lambda^4\pi^2 j^2}{\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4}, \quad (29)$$

$$\Phi_{21} = \sum_{j=1}^{\infty} \frac{2j^2}{\pi^2(\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4)}, \quad (30)$$

$$\Phi_{22} = 1 + \sum_{j=1}^{\infty} \frac{2\lambda^4(-1)^j}{\lambda^4(\pi^2\Delta^2 j^2 + 1) - j^4}. \quad (31)$$

Eq. (27) defines a eigenvalue problem. The eigenvalues could be found by setting the determinant of the coefficient matrix in Eq. (27) to zero

$$\begin{vmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{vmatrix} = 0. \quad (32)$$

The characteristic equation of above determinant can be derived by assigning the different values of (K) and (m) corresponding to the boundary condition.

4. Numerical results

4.1. Validation of the proposed method

In this subsection, it is desired to evaluate the accuracy of the proposed method when applied to a special case of the model described in this work, in which there is no spring mass system at the free end. It can be noted that, by letting ($K=0$), ($\Delta=0$) and ($m=0$), the model will automatically degenerate into the cantilever beam in classical elasticity theory. The λ_i parameters are calculated from Eq. (32) using the first 100 terms of the infinite series. The most important observation from Table 1, is due to the fact that all frequency parameters of the beam are calculated by using the first 100 terms of the series. Improvement in accuracy can be gained by increasing the terms of series. However, as seen from Table 1, the present results seem to be more acceptable.

Table 1. Verification of the proposed method for a beam with clamped-free ends.

	Exact ($\sqrt{\frac{EI}{\rho AL^4}}$)	Present ($\lambda \times \pi$)
Mode-1	1,875	1,882
Mode-2	4,694	4.713
Mode-3	7,855	7.887

4.2. Effects of different parameters

In this subsection, the effects of different parameters will be examined in more detail. A parametric study of the effect of small scale parameter on the first six vibration frequencies has been performed and listed in Table 2.

Table 2. Small-scale (non-local) effect on the vibration frequencies at various non-local parameters with $K=1$, $m=1$.

Δ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.00	0.4282	1.2885	2.2802	3.2784	4.2795	5.2820
0.01	0.4282	1.2881	2.2778	3.2707	4.2619	5.2484
0.02	0.4282	1.2870	2.2705	3.2481	4.2111	5.1539
0.03	0.4282	1.2851	2.2586	3.2120	4.1328	5.0134
0.04	0.4282	1.2825	2.2424	3.1646	4.0343	4.8450
0.05	0.4282	1.2792	2.2224	3.1085	3.9231	4.6643
0.06	0.4282	1.2752	2.1992	3.0462	3.8057	4.4826
0.07	0.4283	1.2705	2.1732	2.9800	3.6871	4.3069
0.08	0.4283	1.2653	2.1450	2.9117	3.5704	4.1408

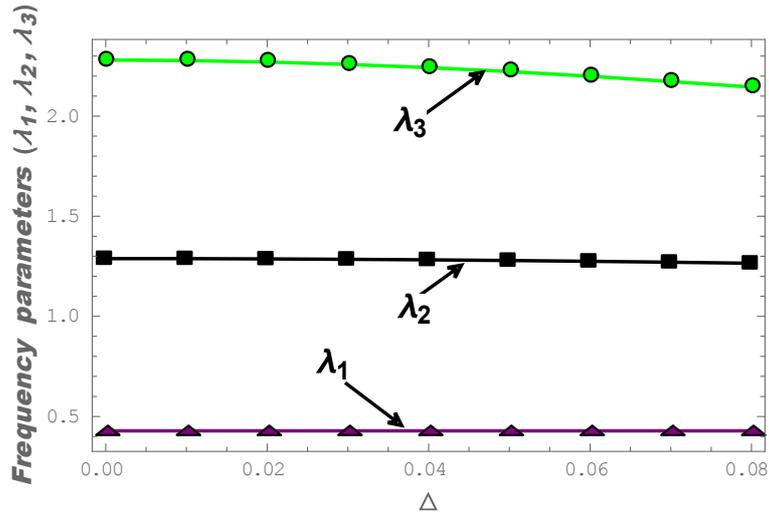


Figure 2: Nonlocal effect on the first three frequencies with $K=1$, $m=1$.

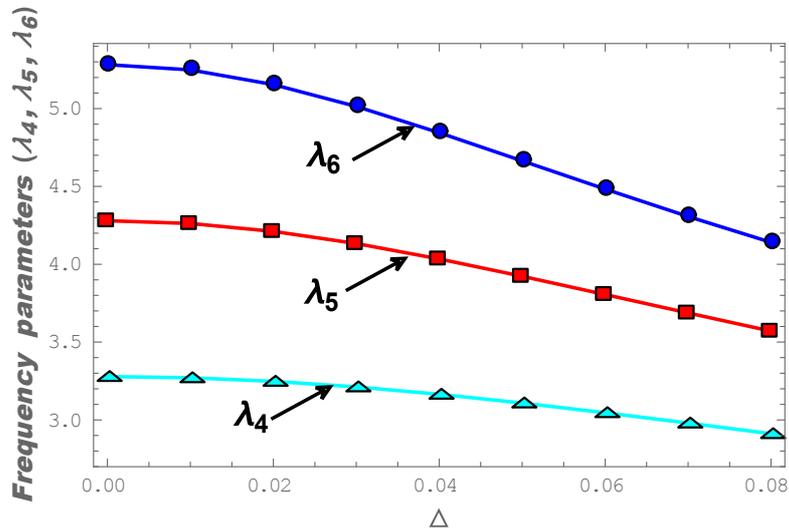


Figure 3: Nonlocal effect on the fourth, fifth and sixth frequency parameters with $K=1$, $m=1$

The simulation results in the Figures 2-3 indicate that the first six vibration frequencies decrease with increasing small scale parameter. Furthermore, for a constant translational spring parameter $K=1$ and attached mass $m=1$, Figure 2 reveals that the nonlocal parameter has little effect on the first three frequency parameter. In contrast, Figure 3 shows that the fourth, fifth and sixth frequency parameters decrease noticeably with increasing the small scale parameter.

In the second numerical example, a dynamical analysis is performed to investigate the effects of mass parameter to examine how it affects the vibration frequencies of the system. The results of analysis are listed in Table 3. Fixing the spring and nonlocal parameters ($K=10$, $\Delta=0.05$) and varying the mass parameter (m) result in a significant change in the vibration frequencies (see Table 3). The first six frequency parameters calculated from the Eq. (32) are plotted in Figure 5. It can be seen from the Figure 5 that, by increasing the mass parameter, the first six vibration frequencies decrease significantly.

It is often a difficult step prediction of a modal displacement function capable of satisfying any restrained boundary conditions. The present procedure for the free vibration analysis of a cantilever

nanobeam with a spring mass system is found to be effective regardless of the boundary conditions. The main advantage of this method is that it is not necessary to select a new set of displacement functions for each change in supporting condition. Only one computer code is required, which can be used any type of cantilever boundary condition by assigning the proper values of (Δ), (K) and (m).

Table 3. Effect of mass parameters on the first six frequency parameter of nanobeam with $K=10$, $\Delta = 0.05$.

m	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.00	0.8441	1.5265	2.4595	3.3395	4.1466	4.8795
0.10	0.7844	1.4146	2.3253	3.1911	3.9912	4.7214
0.20	0.7393	1.3646	2.2817	3.1533	3.9585	4.6932
0.30	0.7041	1.3371	2.2607	3.1366	3.9450	4.6819
0.40	0.6757	1.3197	2.2484	3.1273	3.9376	4.6759
0.50	0.6520	1.3079	2.2403	3.1213	3.9329	4.6721
0.60	0.6319	1.2993	2.2347	3.1172	3.9297	4.6696
0.70	0.6146	1.2928	2.2304	3.1142	3.9274	4.6677
0.80	0.5993	1.2877	2.2272	3.1119	3.9256	4.6663
0.90	0.5858	1.2836	2.2246	3.1100	3.9242	4.6652
1.00	0.5736	1.2802	2.2225	3.1086	3.9231	4.6643

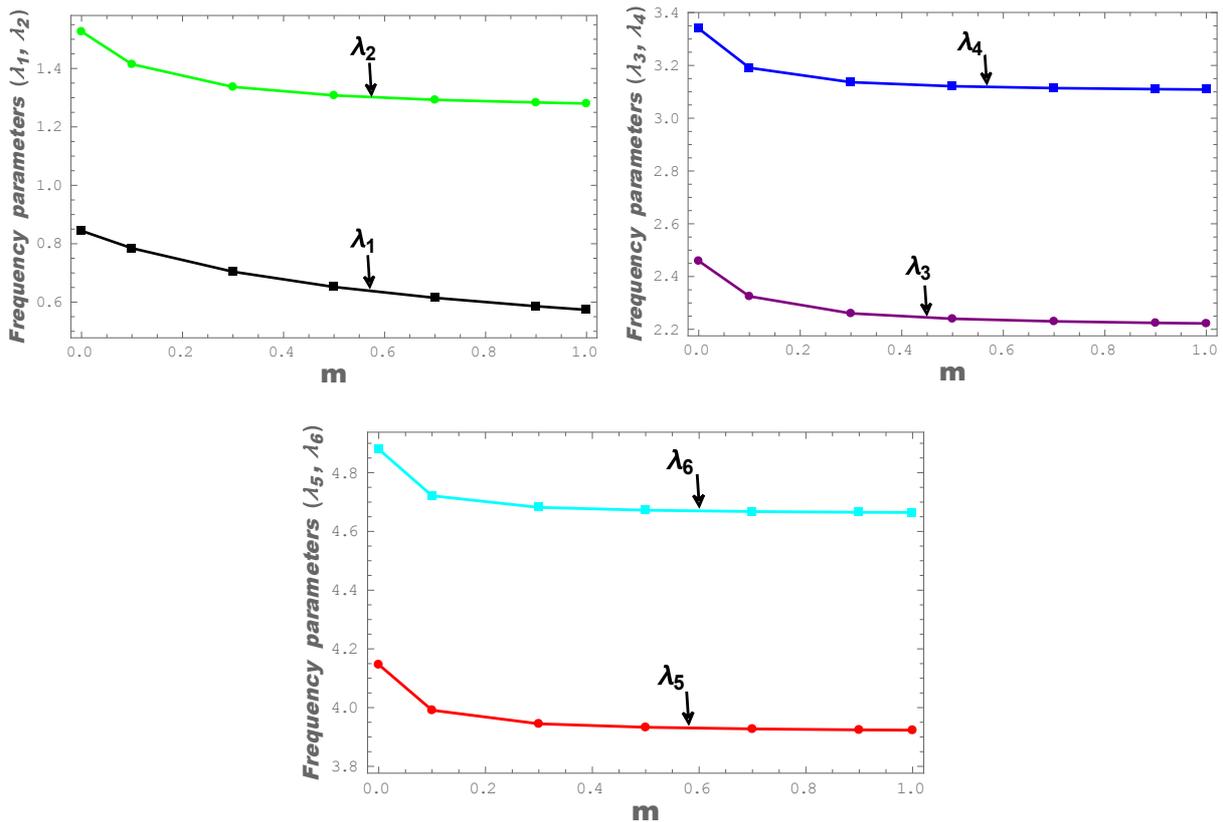


Figure 4: Effect of mass parameters on the first six frequency parameter of nanobeam with $K=10$, $\Delta = 0.05$.

5. Conclusions

In present work, the free vibration of the cantilever nanobeams with a spring mass system is analytically investigated. A Fourier sine series is employed for the modal displacement function and Stokes' transformation is applied for the analysis to obtain a coefficient matrix. The aim of this procedure is to enforce boundary conditions, especially the deformable boundary conditions. The influence of different parameters on the free vibration frequencies is investigated in some numerical examples.

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