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THE MINKOWSKI TYPE INEQUALITIES FOR WEIGHTED FRACTIONAL OPERATORS

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Abstract. In this article, inequalities of reverse Minkowski type involving weighted fractional operators are investigated. In addition, new fractional integral inequalities related to Minkowski type are also established.

1. Introduction

Fractional analysis has drawn attention highly because of its aplications in different areas. Researchers focus on developing different fractional operators in the development of fractional analysis. These different fractional operators are also used in integral inequalities. Hence fractional analysis plays an important role in the development of inequality theory. One of the most useful fractional integral operator is Riemann-liouville fractional integral operator. Scientist who suggested that Riemann-Liouville fractional operator can be used in fractional analysis is Joseph Liouville ([20]). Then, several researchers studied these operators with different inequalities and thus introduced the notion of fractional conformable integrals. In [1], Abdeljawad presented the properties of the conformable fractional operators. Also, in [18], Khan et al. investigated fractional conformable derivatives operators. Similarly, several mathematicians have been interested in and studied conformable fractional operators ([17], [29]). In [15], Katugampola defined a new fractional derivative operator. Also, Katugampola developed a new approach to generalized fractional derivatives. Based on these operators, new theorems were proved by the researchers.

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In [1]- [32], some researchers used new fractional derivative or integral operators such as Riemann-Liouville, Caputo, Hadamard and Katugampola types. Many new results are obtained for functions in $L_p[a,b]$ which is defined as follows:

Definition 1. *For $p \in [1,\infty)$, if the function $\wp$ holds the following inequality*

$$\left(\int_a^b |\wp(\tau)|^p \, d\tau\right)^{\frac{1}{p}} < \infty,$$

then it is said to be in $L_p[a,b]$.

In the mathematical literature the Minkowski's inequality, which is very well known in the literature, has been stated as follows (see [12]):

Theorem 1. *$\int_a^b \wp^p(\tau)d\tau$ and $\int_a^b \hbar^p(\tau)d\tau$ are positive finite reals for $p \geq 1$. Then the inequality*

$$\left(\int_a^b (\wp(\tau)+\hbar(\tau))^p d\tau\right)^{\frac{1}{p}} \leq \left(\int_a^b \wp^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \hbar^p(\tau)d\tau\right)^{\frac{1}{p}}$$

holds.

The reverse Minkowski inequality for classical Riemann integrals is obtained by L. Bougoffa in [5] which is given as the following:

Theorem 2. *Let $\wp, \hbar \in L_p[a,b]$ be two positive functions, with $1 \leq p < \infty$, $0 < \int_a^b \wp^p(\tau)d\tau < \infty$ and $0 < \int_a^b \hbar^p(\tau)d\tau < \infty$. If $0 \leq n \leq \frac{\wp(\tau)}{\hbar(\tau)} \leq N$ for $n, N \in \mathbb{R}^+$ and every $\tau \in [a,b]$, then the inequality*

$$\left(\int_a^b \wp^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \hbar^p(\tau)d\tau\right)^{\frac{1}{p}} \leq c\left(\int_a^b (\wp(\tau)+\hbar(\tau))^p d\tau\right)^{\frac{1}{p}}$$

holds where $c = \frac{N(n+1)+(N+1)}{(n+1)(N+1)}$.

The following theorem is called "Young's inequality" (see [22]):

Theorem 3. *Let $[0,k]$ where $k > 0$ be an interval and h be a function which is increasing and continuous on $[0,k]$. If $b \in [0,\hbar(k)]$, $a \in [0,k]$, $\hbar(0) = 0$ and $\hbar^{-1}$ stands for the inverse function of h, then*

$$\int_0^a \hbar(\tau)d\tau + \int_0^b \hbar^{-1}(\tau)d\tau \geq ab. \tag{1}$$

Example 1. *The function $\hbar : (0,c) \to \mathbb{R}$, $\hbar(\tau) = \tau^{r-1}$ satisfies the conditions mentioned in Theorem 3 for $r > 1$. Applying $\hbar$ to (1) we have*

$$\frac{1}{r}a^r + \frac{1}{s}b^s \geq ab, \qquad a,b \geq 0,\ r \geq 1 \text{ and } \frac{1}{r}+\frac{1}{s} = 1.$$

In other terms, this inequality puts forward the relation between arithmetic mean and geometric mean.

Definition 2. *([4]) Let $\hbar \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}\hbar$ and $J^{\alpha}_{b^-}\hbar$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}\hbar(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^{\tau} (\tau - \theta)^{\alpha-1} \hbar(\theta) d\theta, \quad \tau > a$$

and

$$J^{\alpha}_{b^-}\hbar(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{b} (\theta - \tau)^{\alpha-1} \hbar(\theta) d\theta, \quad \tau < b$$

respectively where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u} u^{\alpha-1} du$. By choosing $\alpha = 0$ in above definitions, we get the function $\hbar$ itself. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 3. *([27])Let (a,b) be an infinite or finite interval on positive real axis and let $\hbar$ is defined on (a,b) with $\hbar \in L_p(a,b)$. Then for $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, definitions of the left-sided and right-sided Hadamard fractional integrals of order α of a real function $\hbar$ are given as*

$$H^{\alpha}_{a^+}\hbar(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^{\tau} \left(\log \frac{\tau}{\theta}\right)^{\alpha-1} \frac{\hbar(\theta)}{\theta} d\theta, \quad a < \tau < b$$

and

$$H^{\alpha}_{b^-}\hbar(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{b} \left(\log \frac{\theta}{\tau}\right)^{\alpha-1} \frac{\hbar(\theta)}{\theta} d\theta, \quad a < \tau < b,$$

respectively.

Definition 4. *([16]) Let $[a,b]$ be a finite interval and $\hbar \in X^p_c(a,b)$ be a real function. Then for $\alpha \in \mathbb{C}$, $\rho > 0$, $Re(\alpha) > 0$, the definitions of left-sided and right sided Katugampola fractional integrals of order α of $\hbar$ are given as*

$${}^{\rho}I^{\alpha}_{a^+}\hbar(\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^{\tau} \frac{\theta^{\rho-1}}{(\tau^{\rho} - \theta^{\rho})^{1-\alpha}} \hbar(\theta) d\theta, \quad \tau > a$$

and

$${}^{\rho}I^{\alpha}_{b^-}\hbar(\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\tau}^{b} \frac{\theta^{\rho-1}}{(\theta^{\rho} - \tau^{\rho})^{1-\alpha}} \hbar(\theta) d\theta, \quad \tau < b,$$

respectively.

As the use of fractional integral operators increased, it became necessary to obtain more general versions of the new results obtaibed. Thus, weighted integral operators began to be presented. While new results are obtained with these operators, general versions of the results in the literature can also be obtained. One of the most effective weighted integral operator presented recently is given in the following:

Definition 5. *([24]) Let $\phi(\tau)$ be a monotonic, positive and increasing function on the finite interval $[a,b]$ and continuously differentiable on (a,b) with $\phi(0)=0$, $0\in[a,b]$. Then for $w(\tau)\neq 0$ and $w^{-1}(\tau)=\frac{1}{w(\tau)}$, the definitions of the weighted fractional integrals of a function (the left-side and right-side respectively) $\hbar$ with respect to ϕ on $[a,b]$ are given as*

$$\left({}_{a+}\Im_w^{\ell:\phi}\hbar\right)(\tau) = \frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(\theta)\left[\phi(\tau)-\phi(\theta)\right]^{\ell-1}\hbar(\theta)w(\theta)d\theta, \tag{2}$$

$$\left({}_{w}\Im_{b-}^{\ell:\phi}\hbar\right)(\tau) = \frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_\tau^b \phi'(\theta)\left[\phi(\theta)-\phi(\tau)\right]^{\ell-1}\hbar(\theta)w(\theta)d\theta, \quad \ell>0. \tag{3}$$

The fractional integral operator given above is studied on in this study because it can give very efficient results in terms of application. Since by choosing $\phi(\tau)=\tau$ and $w(\theta)=1$, the weighted fractional integral operators ((2) and (3)) reduce to the classical Riemann–Liouville fractional integral operators and by choosing other special cases, many forms of fractional integral operators can be obtained.

Obtaining some new general forms of the Minkowski type inequalities using weighted fractional operators is the main aim of this study.

2. Reverse Minkowski Inequalities for Weighted Fractional Operators

Theorem 4. *Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau>a>0$, $\ell>0$, $p\geq 1$. If $0\leq n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ holds for $n,N\in\mathbb{R}^+$ and $t\in[a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)+\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau)\leq c_1\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{1}{p}}(\tau), \tag{4}$$

with $c_1=\frac{N(n+1)+(N+1)}{(n+1)(N+1)}$.

Proof. Under the given condition $\frac{\wp(t)}{\hbar(t)}\leq N$, $t\in[a,\tau]$, it can be written as

$$\wp(t)\leq N(\wp(t)+\hbar(t))-N\wp(t)$$

which implies that

$$(N+1)^p\wp^p(t)\leq N^p(\wp(t)+\hbar(t))^p. \tag{5}$$

Multiplying both sides of (5) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating with respect to t from a to τ, we have

$$\begin{aligned}&\frac{(N+1)^p w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp^p(t)w(t)dt\\ \leq\ &\frac{N^p w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}(\wp+\hbar)^p(t)w(t)dt.\end{aligned}$$

Consequently, we can write

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) \leq \frac{N}{N+1}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{1}{p}}(\tau). \tag{6}$$

On the other hand, as $n\hbar(t) \leq \wp(t)$, it follows

$$\left(1+\frac{1}{n}\right)^p \hbar^p(t) \leq \left(\frac{1}{n}\right)^p (\wp(t)+\hbar(t))^p. \tag{7}$$

Next, multiplying both sides of (7) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1} w(t)$ and then integrating with respect to t from a to τ, we obtain

$$\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq \frac{1}{n+1}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{1}{p}}(\tau). \tag{8}$$

From (6) and (8), the required result follows. □

Remark 1. *Applying Theorem 4 for $\phi(\tau)=\tau$ and $w(\theta)=1$, we obtain Theorem 2.1 in [9].*

Remark 2. *In Theorem 4, if we choose $\phi(\tau)=\tau$, $w(\theta)=1$ and $\ell=1$, we have the reverse Minkowski inequality in [5].*

Inequality (4) is a version of reverse Minkowski inequality obtained with weighted fractional operators.

Theorem 5. *Let $\wp, \hbar \in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau > a > 0$, $\ell > 0$, $p \geq 1$. If $0 \leq n \leq \frac{\wp(t)}{\hbar(t)} \leq N$ holds for $n, N \in \mathbb{R}^+$ and $t \in [a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{2}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{2}{p}}(\tau) \geq c_2\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau),$$

with $c_2 = \frac{(N+1)(n+1)}{N} - 2$.

Proof. Multiplying inequality (6) by inequality (8), we obtain

$$\frac{(N+1)(n+1)}{N}\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq \left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{2}{p}}(\tau). \tag{9}$$

Using the Minkowski inequality, on the right side of (9), we get

$$\begin{aligned} & \frac{(N+1)(n+1)}{N}\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \\ \leq & \left[\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau)\right]^2. \end{aligned}$$

Then, we have

$$\begin{aligned} & \left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{2}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{2}{p}}(\tau) \\ \geq & \left[\frac{(N+1)(n+1)}{N} - 2\right]\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \end{aligned}$$

which is the desired result. □

Remark 3. *Applying Theorem 5 for $\phi(\tau) = \tau$ and $w(\theta) = 1$, we obtain Theorem 2.3 in [9].*

Remark 4. *In Theorem 5, if we choose $\phi(\tau) = \tau$, $w(\theta) = 1$ and $\ell = 1$, we have Theorem 2.2 in [30].*

3. Other Fractional Integral Inequalities

Theorem 6. *Let $\wp, \hbar \in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau > a > 0$, $\ell > 0$, $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $0 \leq n \leq \frac{\wp(t)}{\hbar(t)} \leq N$ holds for $n, N \in \mathbb{R}^+$ and $t \in [a,\tau]$, then the following inequality for weighted fractional operators holds:*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar\right)^{\frac{1}{q}}(\tau) \leq \left(\frac{N}{n}\right)^{\frac{1}{qp}}\left({}_{a+}\Im_w^{\ell:\phi}\wp^{\frac{1}{p}}.\hbar^{\frac{1}{q}}\right)(\tau).$$

Proof. Using the given condition $\frac{\wp(t)}{\hbar(t)} \leq N$, $t \in [a,\tau]$, it can be written

$$\begin{aligned} \wp(t) &\leq N\hbar(t) \\ N^{-\frac{1}{q}}\wp^{\frac{1}{q}}(t) &\leq \hbar^{\frac{1}{q}}(t). \end{aligned} \tag{10}$$

Multiplying both sides of (10) by $\wp^{\frac{1}{p}}(t)$, we can rewrite as follows

$$N^{-\frac{1}{q}}\wp(t) \leq \wp^{\frac{1}{p}}(t)\hbar^{\frac{1}{q}}(t) \tag{11}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Multiplying both sides of (11) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau) - \phi(t)\right]^{\ell-1}w(t)$ and then integrating we have

$$\begin{aligned} &\frac{N^{-\frac{1}{q}}w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp(t)w(t)dt \\ \leq\; &\frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp^{\frac{1}{p}}(t)\hbar^{\frac{1}{q}}(t)w(t)dt. \end{aligned}$$

From weighted fractional operators, we obtain

$$N^{-\frac{1}{pq}}\left({}_{a+}\Im_w^{\ell:\phi}\wp\right)^{\frac{1}{p}}(\tau) \leq \left({}_{a+}\Im_w^{\ell:\phi}\wp^{\frac{1}{p}}.\hbar^{\frac{1}{q}}\right)^{\frac{1}{p}}(\tau). \tag{12}$$

On the contrary, as $n \leq \frac{\wp(t)}{\hbar(t)}$, it follows

$$n^{\frac{1}{p}}\hbar^{\frac{1}{p}}(t) \leq \wp^{\frac{1}{p}}(t). \tag{13}$$

Multiplying both sides of (13) by $\hbar^{\frac{1}{q}}(t)$ and using the relation $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$n^{\frac{1}{p}}\hbar(t) \leq \wp^{\frac{1}{p}}(t)\hbar^{\frac{1}{q}}(t). \tag{14}$$

Multiplying both sides of (14) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we get

$$n^{\frac{1}{pq}}\left({}_{a+}\Im_w^{\ell:\phi}\hbar\right)^{\frac{1}{q}}(\tau)\leq\left({}_{a+}\Im_w^{\ell:\phi}\wp^{\frac{1}{p}}.\hbar^{\frac{1}{q}}\right)^{\frac{1}{q}}(\tau). \tag{15}$$

Conducting the product between (12) and (15), we have

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar\right)^{\frac{1}{q}}(\tau)\leq\left(\frac{N}{n}\right)^{\frac{1}{qp}}\left({}_{a+}\Im_w^{\ell:\phi}\wp^{\frac{1}{p}}\hbar^{\frac{1}{q}}\right)(\tau).$$

where $\frac{1}{p}+\frac{1}{q}=1$. So the proof is completed. □

Theorem 7. *For $\ell>0$, $p,q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau>a>0$. If $0\leq n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ for $n,N\in\mathbb{R}^+$ and for all $t\in[a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau)\leq c_3\left({}_{a+}\Im_w^{\ell:\phi}(\wp^p+\hbar^p)\right)(\tau)+c_4\left({}_{a+}\Im_w^{\ell:\phi}(\wp^q+\hbar^q)\right)(\tau)$$

with $c_3=\frac{2^{p-1}N^p}{p(N+1)^p}$ and $c_4=\frac{2^{q-1}}{q(n+1)^q}$.

Proof. Using the hypothesis, we obtain the following inequality:

$$(N+1)^p\wp^p(t)\leq N^p(\wp+\hbar)^p(t). \tag{16}$$

Multiplying both sides of (16) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we have

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)\leq\frac{N^p}{(N+1)^p}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)(\tau). \tag{17}$$

For $t\in[a,\tau]$, since $0\leq n\leq\frac{\wp(t)}{\hbar(t)}$ holds we get

$$(n+1)^q\hbar^q(t)\leq(\wp+\hbar)^q(t). \tag{18}$$

Similarly, multiplying both sides of (18) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we can write

$$\left({}_{a+}\Im_w^{\ell:\phi}\hbar^q\right)(\tau)\leq\frac{1}{(n+1)^q}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^q\right)(\tau). \tag{19}$$

Using the Young's inequality, we have

$$\wp(t)\hbar(t)\leq\frac{1}{p}\wp^p(t)+\frac{1}{q}\hbar^q(t), \tag{20}$$

again multiplying both sides of (20) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau)\leq\frac{1}{p}\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)+\frac{1}{q}\left({}_{a+}\Im_w^{\ell:\phi}\hbar^q\right)(\tau). \tag{21}$$

Using (17) and (19) in (21), we obtain

$$\begin{aligned}\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau) &\leq \frac{N^p}{p(N+1)^p}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)(\tau) \\ &+\frac{1}{q(n+1)^q}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^q\right)(\tau).\end{aligned} \tag{22}$$

Using the inequality $(x+y)^r \leq 2^{r-1}(x^r+y^r)$, $r>1$, $x,y>0$ in (22), we have

$$\begin{aligned}\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau) &\leq \frac{2^{p-1}N^p}{p(N+1)^p}\left({}_{a+}\Im_w^{\ell:\phi}(\wp^p+\hbar^p)\right)(\tau) \\ &+\frac{2^{q-1}}{q(n+1)^q}\left({}_{a+}\Im_w^{\ell:\phi}(\wp^q+\hbar^q)\right)(\tau).\end{aligned}$$

This is the required result. □

Theorem 8. *For $\ell>0$, $p\geq 1$. Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau>a>0$. If $0<c<n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ for $n,N\in\mathbb{R}^+$ and for all $t\in[a,\tau]$, then*

$$\begin{aligned}\frac{N+1}{N-c}\left({}_{a+}\Im_w^{\ell:\phi}(\wp-c\hbar)^p\right)^{\frac{1}{p}}(\tau) &\leq \left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)+\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \\ &\leq \frac{n+1}{n-c}\left({}_{a+}\Im_w^{\ell:\phi}(\wp-c\hbar)^p\right)^{\frac{1}{p}}(\tau).\end{aligned}$$

Proof. Using the hypothesis $0<c<n\leq N$, we have

$$nc\leq Nc \implies nc+n\leq nc+N\leq Nc+N \implies (N+1)(n-c)\leq(n+1)(N-c).$$

It can be concluded that

$$\frac{N+1}{N-c}\leq\frac{n+1}{n-c}.$$

Also,

$$\begin{aligned}n\leq\frac{\wp(t)}{\hbar(t)}\leq N &\implies n-c\leq\frac{\wp(t)-c\hbar(t)}{\hbar(t)}\leq N-c \\ &\implies \frac{(\wp(t)-c\hbar(t))^p}{(N-c)^p}\leq\hbar^p(t)\leq\frac{(\wp(t)-c\hbar(t))^p}{(n-c)^p}.\end{aligned} \tag{23}$$

Multiplying both sides of (23) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we get

$$\begin{aligned}&\frac{w^{-1}(\tau)}{(N-c)^p\Gamma(\ell)}\int_a^\tau\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}(\wp(t)-c\hbar(t))^p\,w(t)dt \\ \leq\ &\frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\hbar^p(t)w(t)dt \\ \leq\ &\frac{w^{-1}(\tau)}{(n-c)^p\Gamma(\ell)}\int_a^\tau\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}(\wp(t)-c\hbar(t))^p\,w(t)dt\end{aligned}$$

Then, we can write

$$\begin{aligned} \frac{1}{N-c}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp-c\hbar\right)^p\right)^{\frac{1}{p}}(\tau) &\leq \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \\ &\leq \frac{1}{n-c}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp-c\hbar\right)^p\right)^{\frac{1}{p}}(\tau). \end{aligned} \tag{24}$$

Again, we obtain

$$\frac{1}{N}\leq\frac{\hbar(t)}{\wp(t)}\leq\frac{1}{n}\Longrightarrow\frac{n-c}{nc}\leq\frac{\wp(t)-c\hbar(t)}{c\wp(t)}\leq\frac{N-c}{cN}$$

which implies

$$\left(\frac{N}{N-c}\right)^p\left(\wp(t)-c\hbar(t)\right)^p\leq\wp^p(t)\leq\left(\frac{n}{n-c}\right)^p\left(\wp(t)-c\hbar(t)\right)^p. \tag{25}$$

Repeating the same procedure with (25), we have

$$\begin{aligned} \frac{N}{N-c}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp-c\hbar\right)^p\right)^{\frac{1}{p}}(\tau) &\leq \left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) \\ &\leq \frac{n}{n-c}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp-c\hbar\right)^p\right)^{\frac{1}{p}}(\tau). \end{aligned} \tag{26}$$

Adding (24) and (26), the required result is obtained. □

Theorem 9. *For $\ell>0$, $p\geq 1$. Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau>a>0$. If $0\leq a\leq\wp(t)\leq A$ and $0\leq b\leq\hbar(t)\leq B$, $t\in[a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)+\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau)\leq c_5\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp+\hbar\right)^p\right)^{\frac{1}{p}}(\tau) \tag{27}$$

with $c_5=\frac{A(a+B)+B(b+A)}{(a+B)(b+A)}$.

Proof. Under the given conditions, it follows that

$$\frac{1}{B}\leq\frac{1}{\hbar(t)}\leq\frac{1}{b}. \tag{28}$$

Considering the product of (28) and $0\leq a\leq\wp(t)\leq A$, we have

$$\frac{a}{B}\leq\frac{\wp(t)}{\hbar(t)}\leq\frac{A}{b}. \tag{29}$$

From (29), we get

$$\hbar^p(t)\leq\left(\frac{B}{a+B}\right)^p\left(\wp(t)+\hbar(t)\right)^p \tag{30}$$

and

$$\wp^p(t)\leq\left(\frac{A}{b+A}\right)^p\left(\wp(t)+\hbar(t)\right)^p. \tag{31}$$

Multiplying both sides of (30) and (31) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq \frac{B}{a+B}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp+\hbar\right)^p\right)^{\frac{1}{p}}(\tau) \tag{32}$$

and

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) \leq \frac{A}{b+A}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp+\hbar\right)^p\right)^{\frac{1}{p}}(\tau). \tag{33}$$

respectively. The proof of (27) can be concluded by adding (32) and (33). □

Theorem 10. *Let $\wp,\hbar \in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are positive reals for $\tau > a > 0$. If $0 \leq n \leq \frac{\wp(t)}{\hbar(t)} \leq N$ for $n, N \in \mathbb{R}^+$ and for all $t \in [a,\tau]$, then*

$$\frac{1}{N}\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau) \leq \frac{1}{(n+1)(N+1)}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp+\hbar\right)^2\right)(\tau) \leq \frac{1}{n}\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau)$$

for $\ell > 0$.

Proof. Using $0 \leq n \leq \frac{\wp(t)}{\hbar(t)} \leq N$, we obtain

$$\hbar(t)(n+1) \leq \hbar(t)+\wp(t) \leq \hbar(t)(N+1). \tag{34}$$

Also, it follows that $\frac{1}{N} \leq \frac{\hbar(t)}{\wp(t)} \leq \frac{1}{n}$, which yields

$$\wp(t)\left(\frac{N+1}{N}\right) \leq \hbar(t)+\wp(t) \leq \wp(t)\left(\frac{n+1}{n}\right). \tag{35}$$

Evaluating the product between (34) and (35), we get

$$\frac{\wp(t)\hbar(t)}{N} \leq \frac{\left(\hbar(t)+\wp(t)\right)^2}{(n+1)(N+1)} \leq \frac{\wp(t)\hbar(t)}{n}. \tag{36}$$

Multiplying both sides of (36) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\begin{aligned}
&\frac{w^{-1}(\tau)}{N\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp(t)\hbar(t)w(t)dt\\
\leq\ &\frac{w^{-1}(\tau)}{(n+1)(N+1)\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\left(\hbar(t)+\wp(t)\right)^2 w(t)dt\\
\leq\ &\frac{w^{-1}(\tau)}{n\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp(t)\hbar(t)w(t)dt.
\end{aligned}$$

Hence

$$\frac{1}{N}\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau) \leq \frac{1}{(n+1)(N+1)}\left({}_{a+}\Im_w^{\ell:\phi}\left(\wp+\hbar\right)^2\right)(\tau) \leq \frac{1}{n}\left({}_{a+}\Im_w^{\ell:\phi}\wp\hbar\right)(\tau).$$

This completes the proof. □

Theorem 11. *Let $\wp, \hbar \in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau > a > 0$. If $0 < n \leq \frac{\wp(t)}{\hbar(t)} \leq N$ holds for $n, N \in \mathbb{R}^+$ and for all $t \in [a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq 2\left({}_{a+}\Im_w^{\ell:\phi}\Psi^p(\wp,\hbar)\right)^{\frac{1}{p}}(\tau) \tag{37}$$

holds for $\ell > 0$ where $\Psi(\wp(t),\hbar(t)) = \max\left\{N\left[\left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t)\right], \frac{(N+n)\hbar(t)-\wp(t)}{n}\right\}$.

Proof. From the hypothesis $0 < n \leq \frac{\wp(t)}{\hbar(t)} \leq N$, we have

$$0 < n \leq N + n - \frac{\wp(t)}{\hbar(t)} \tag{38}$$

and

$$N + n - \frac{\wp(t)}{\hbar(t)} \leq N. \tag{39}$$

Hence, using (38) and (39), we get

$$\hbar(t) < \frac{(N+n)\hbar(t) - \wp(t)}{n} \leq h(\wp(t),\hbar(t)), \tag{40}$$

where $\Psi(\wp(t),\hbar(t)) = \max\left\{N\left[\left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t)\right], \frac{(N+n)\hbar(t)-\wp(t)}{n}\right\}$.

Using the hypothesis, it follows that $0 < \frac{1}{N} \leq \frac{\hbar(t)}{\wp(t)} \leq \frac{1}{n}$. In this way, we have

$$\frac{1}{N} \leq \frac{1}{N} + \frac{1}{n} - \frac{\hbar(t)}{\wp(t)} \tag{41}$$

and

$$\frac{1}{N} + \frac{1}{n} - \frac{\hbar(t)}{\wp(t)} \leq \frac{1}{n}. \tag{42}$$

From (41) and (42), we obtain

$$\frac{1}{N} \leq \frac{\left(\frac{1}{N}+\frac{1}{n}\right)\wp(t) - \hbar(t)}{\wp(t)} \leq \frac{1}{n},$$

which can be rewritten as

$$\begin{aligned}
\wp(t) &\leq N\left(\frac{1}{N}+\frac{1}{n}\right)\wp(t) - N\hbar(t) \\
&= \left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t) \\
&\leq N\left[\left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t)\right] \\
&\leq \Psi(\wp(t),\hbar(t)).
\end{aligned} \tag{43}$$

We can write from (40) and (43)

$$\wp^p(t) \leq \Psi^p(\wp(t),\hbar(t)) \tag{44}$$

$$\hbar^p(t) \quad \leq \quad \Psi^p(\wp(t), \hbar(t)). \tag{45}$$

Multiplying both sides of (44) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\begin{aligned} & \frac{w^{-1}(\tau)}{\Gamma(\ell)} \int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1} \wp^p(t) w(t) dt \\ \leq \quad & \frac{w^{-1}(\tau)}{\Gamma(\ell)} \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1} \Psi^p(\wp(t), \hbar(t)) w(t) dt. \end{aligned}$$

Accordingly, it can be written as

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) \leq \left({}_{a+}\Im_w^{\ell:\phi}\Psi^p(\wp,\hbar)\right)^{\frac{1}{p}}(\tau). \tag{46}$$

Using the same procedure as above, for (45), we have

$$\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq \left({}_{a+}\Im_w^{\ell:\phi}\Psi^p(\wp,\hbar)\right)^{\frac{1}{p}}(\tau). \tag{47}$$

The required result (37) follows from (46) and (47). □

4. Conclusion

In this paper, first we gave different definitions of fractional integral operators and then we introduced the reverse Minkowski type inequalities using weighted fractional operators. The obtained results are an extension of some known results in the literature. Especially, we would like to emphasize that different types all integral inequalities can be obtained using this operators.

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