Konuralp Journal of Mathematics, *10 (1) (2022) 134-137*

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Iterated Bicrossed Product of Groups

Esra Kırmızı Çetinalp[1]

[1]*Department of Mathematics, Kamil Özdağ Science Faculty, Karamanoğlu Mehmetbey University, Karaman, Turkey*

Abstract

In this paper, we study iterated bicrossed product looking from the viewpoint of Combinatorial Group Theory and describe a new version of iterated bicrossed product of groups. Also, we investigate that the group property of this new product is provided. Then, by considering finite cyclic groups, we give an example for iterated bicrossed product of groups.

Keywords: *Bicrossed product, group, semidirect product.*
2010 Mathematics Subject Classification: *16S15, 20M15, 20L05.*

1. Introduction and Preliminaries

The classification of groups has taken so much interest for ages and the classification problem originates in group theory. In this work, we will pursue to a new classification idea. That is, we will construct a new group structure. This would give have the edge over achieving some new groups in the meaning of products of groups. Also, It is used when new groups are constructed that to come into some properties of initial groups and it is used for some complicated groups are reduced to some simple groups. One turning point in studying the classification for groups was the bicrossed product construction.
The bicrossed product construction was introduced by Zappa [13]. Later on the construction appeared in a paper of Takeuchi [11]. Different names to this product used in literature were knit product [10] and Zappa-Szep product [13]. Bicrossed product constructions were presented and studied for other constructions, like that: algebras, Lie algebras, Hopf algebras, groupoids, lie groups, locally quantum groups. Firstly, in [1], Agore et al. were studied the bicrosssed product for finite groups. The main ingredients in constructing the bicrossed product are the so called matched pairs of groups. The bicrossed product structure generalizes the semidirect product construction for the case when neither factor is required to be normal : a group E is the internal bicrossed product of its subgroups U_1 and U_2 if $HG = E$ and their overlap is trivial. This construction is essential to the quantum double construction.
Iteration of the algebraic structures (direct, semidirect, crossed) has been studied in recent years. Firstly, authors investigated iterated semidirect product of free groups ([3]). Then, in [8], the author conduct a research on iterated crossed product from the point of algebraic constructions. In that paper, the author's objective was the so-called quasi-Hopf two-sided smash product on algebras. After that, in [5], Çetinalp and Karpuz studied iterated crossed product contruction from the point of Combinatorial Group Theory. In this paper, as a continuation of this works, we define an iterative version of bicrossed product. We call this product as *iterated bicrossed product of groups*. This product is more noteworthiness than known group structures since it includes direct product, semidirect products [2], bicrossed product of groups.
Now, we give the basic definitions that will be used throughout the paper. For detailed information on this subject, we can referenced in [1, 4, 5, 6, 7, 9, 12].

Definition 1.1. *Let U_1 and U_2 be two groups and $\alpha : U_2 \times U_1 \to U_1$ and $\beta : U_2 \times U_1 \to U_2$ two maps. We use the notation*

$$\alpha(u_2,u_1) = u_2 \triangleright u_1 \qquad \text{and} \qquad \beta(u_2,u_1) = u_2 \triangleleft u_1$$

for all $u_1 \in U_1$ and $u_2 \in U_2$. If $\alpha : U_2 \times U_1 \to U_1$ is an action of U_2 on U_1 as group automorphisms we represent by $U_1 \rtimes_\alpha U_2$ the semidirect product of U_1 on U_2: $U_1 \rtimes_\alpha U_2 = U_1 \times U_2$ as a set with the multiplication given by

$$(u_1,u_2)(v_1,v_2) = (u_1(u_2 \triangleright v_1), u_2 v_2),$$

Email addresses: **esrakirmizi@kmu.edu.tr (Esra Kırmızı Çetinalp)**

for all $u_1, v_1 \in U_1$ *and* $u_2, v_2 \in U_2$.

Definition 1.2. *A matched pair of groups is a quadruple* (U_1, U_2, α, f) *where* U_1 *and* U_2 *are groups,* $\alpha : U_2 \times U_1 \to U_1$ *is a left action of the group* U_2 *on the set* U_1*,* $\beta : U_2 \times U_1 \to U_2$ *is a right action of the group* U_1 *on the set* U_2 *such that the following compatibility conditions hold:*

$$\left.\begin{array}{c} u_2 \triangleright (u_1 v_1) = (u_2 \triangleright u_1)((u_2 \triangleleft u_1) \triangleright v_1), \\ (u_2 v_2) \triangleright u_1 = u_2 \triangleright (v_2 \triangleright u_1), \\ (u_2 v_2) \triangleleft u_1 = (u_2 \triangleleft (v_2 \triangleright u_1))(v_2 \triangleleft u_1), \\ u_2 \triangleleft u_1 v_1 = u_2 \triangleleft (u_1 \triangleleft v_1), \end{array}\right\} \tag{1.1}$$

for all $u_1, v_1 \in U_1$ *and* $u_2, v_2 \in U_2$ *and . The quadruple* (U_1, U_2, α, f) *is called matched pair if* $u_2 \triangleright 1 = 1$ *and* $1 \triangleleft u_1 = 1$ *for all* $u_1 \in U_1$ *and* $u_2 \in U_2$.

Let U_1 and U_2 be groups and $\alpha : U_2 \times U_1 \to U_1$ and $\beta : U_2 \times U_1 \to U_2$ two maps. The bicrossed product of U_1 and U_2, denoted by $U_1 \bowtie_\beta U_2 = U_1 \bowtie U_2$, is the set $U_1 \times U_2$ with the multiplication

$$(u_1, u_2)(v_1, v_2) = (u_1(u_2 \triangleright v_1), (u_2 \triangleleft v_1)v_2),$$

for all $u_1, v_1 \in U_1$ and $u_2, v_2 \in U_2$. Bicrossed product $U_1 \bowtie U_2$ is a group with the inverse element $(u_2^{-1} \triangleright u_1^{-1},\ (u_2 \triangleleft (u_2^{-1} \triangleright u_1^{-1}))^{-1})$ if and only if (U_1, U_2, α, f) is a matched pair (cf. [1]).

2. Main Results

2.1. A new group construction

In this section, we obtain algebraic structure by defining binary operations on the defined groups. Then, we give a theorem that necessary conditions for this algebraic structures to be group.

Definition 2.1. *Let* $U_1, U_2, \cdots, U_{2n-1}$ *and* U_{2n} *be any groups. A matched pair of these groups is a quadruple* $(U_i, U_{i+1}, \alpha_i, \beta_i)$ $(1 \le i \le 2n-1)$*, where*

$$\alpha_i : U_{i+1} \times U_i \to U_i \qquad \textit{and} \qquad \beta_i : U_{i+1} \times U_i \to U_{i+1}$$

are maps. We use the notation $\alpha_i(u_{i+1}, u_i) = u_{i+1} \triangleright u_i$ *and* $\beta_i(u_{i+1}, u_i) = u_{i+1} \triangleleft u_i$ *for all* $u_i \in U_i$ *and* $u_{i+1} \in U_{i+1}$.
The iterated bicrossed product of groups $U_1, U_2, \cdots, U_{2n}$ *associated to the matched pair with respect to the actions given above is the set* $U_1 \times U_2 \times \cdots \times U_{2n}$ *with the multiplication*

$$(u_1, u_2, \cdots, u_{2n})(v_1, v_2, \cdots, v_{2n}) = (u_1(u_2 \triangleright v_1),\ (u_2 \triangleleft v_1)v_2,\ u_3(u_4 \triangleright v_3),\ (u_4 \triangleleft v_3)v_4, \cdots, u_{2n-1}(u_{2n} \triangleright v_{2n-1}),\ (u_{2n} \triangleleft v_{2n-1})v_{2n})$$

and the following compatability conditions hold:

$$\left.\begin{array}{c} u_{i+1} \triangleright (u_i u_i^{'}) = (u_{i+1} \triangleright u_i)((u_{i+1} \triangleleft u_i) \triangleright u_i^{'}), \\ (u_{i+1} u_{i+1}^{'}) \triangleright u_i = u_{i+1} \triangleright (u_{i+1}^{'} \triangleright u_i), \\ (u_{i+1} u_{i+1}^{'}) \triangleleft u_i = (u_{i+1} \triangleleft (u_{i+1}^{'} \triangleright u_i))(u_{i+1}^{'} \triangleleft u_i), \\ u_{i+1} \triangleleft u_i u_i^{'} = u_{i+1} \triangleleft (u_i \triangleleft u_i^{'}), \end{array}\right\} \tag{2.1}$$

for all $u_i, u_i^{'} \in U_i$ $(1 \le i \le 2n-1)$ *and* $u_{i+1}, u_{i+1}^{'} \in U_{i+1}$ $(1 \le i \le 2n-1)$*. We denote this product by* $U_1 \bowtie U_2 \bowtie \cdots \bowtie U_{2n}$.

The following first main result of this paper unitize when this new product describes a group.

Theorem 2.2. *Let* $U_1, U_2, \cdots, U_{2n}$ *be groups. For all* $u_i \in U_i$ $(1 \le i \le 2n)$*, let us consider the actions given in (2.1). Then the iterated bicrossed product* $U_1 \bowtie U_2 \bowtie \cdots \bowtie U_{2n}$ *defines a group.*

Proof. We verify the group properties for $U_1 \bowtie U_2 \bowtie \cdots \bowtie U_{2n}$. Firstly, we show the associative property. To do that, for any $u_i, v_i, w_i \in U_i$ $(1 \le i \le 2n)$, let $(u_1, u_2, u_3, \cdots, u_{2n})$, $(v_1, v_2, v_3, \cdots, v_{2n})$, $(w_1, w_2, w_3, \cdots, w_{2n}) \in U_1 \bowtie U_2 \bowtie \cdots \bowtie U_{2n}$. So, we have

$$\begin{aligned}
&[(u_1, u_2, u_3, \cdots, u_{2n-1}, u_{2n})(v_1, v_2, v_3, \cdots, v_{2n-1}, v_{2n})](w_1, w_2, w_3, \cdots, w_{2n-1}, w_{2n}) \\
&\quad = (u_1(u_2 \triangleright v_1),\ (u_2 \triangleleft v_1)v_2,\ u_3(u_4 \triangleright v_3), \cdots, u_{2n-1}(u_{2n} \triangleright v_{2n-1}),\ (u_{2n} \triangleleft v_{2n-1})v_{2n})(w_1, w_2, w_3, \cdots, w_{2n-1}, w_{2n}) \\
&\quad = (u_1(u_2 \triangleright v_1)((u_2 \triangleleft v_1)v_2 \triangleright w_1),\ ((u_2 \triangleleft v_1)v_2 \triangleleft w_1)w_2,\ u_3(u_4 \triangleright v_3)((u_4 \triangleleft v_3)v_4 \triangleright w_3), \cdots, \\
&\qquad u_{2n-1}(u_{2n} \triangleright v_{2n-1})((u_{2n} \triangleleft v_{2n-1})v_{2n} \triangleright w_{2n-1}),\ ((u_{2n} \triangleleft v_{2n-1})v_{2n} \triangleleft w_{2n-1})w_{2n}) \\
&\quad = (u_1(u_2 \triangleright v_1)((u_2 \triangleleft v_1) \triangleright v_2 \triangleright w_1),\ u_2 \triangleleft (v_1 \triangleleft (v_2 \triangleleft w_1))(v_2 \triangleleft w_1)w_2,\ u_3(u_4 \triangleright v_3)((u_4 \triangleleft v_3) \triangleright v_4 \triangleright w_3), \cdots, \\
&\qquad u_{2n-1}(u_{2n} \triangleright v_{2n-1})((u_{2n} \triangleleft v_{2n-1}) \triangleright v_{2n} \triangleright w_{2n-1}),\ u_{2n} \triangleleft (v_{2n-1} \triangleleft (v_{2n} \triangleleft w_{2n-1}))(v_{2n} \triangleleft w_{2n-1})w_{2n})
\end{aligned}$$

and

$$
\begin{aligned}
&(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n})[(v_1,v_2,v_3,\cdots,v_{2n-1},v_{2n})(w_1,w_2,w_3,\cdots,w_{2n-1},w_{2n})]\\
&\quad=(u_1,u_2,u_3,\cdots u_{2n-1},u_{2n})(v_1(v_2\triangleright w_1),\,(v_2\triangleleft w_1)w_2,\,v_3(v_4\triangleright w_3),\cdots,v_{2n-1}(v_{2n}\triangleright w_{2n-1}),\,(v_{2n}\triangleleft w_{2n-1})w_{2n})\\
&\quad=(u_1(u_2\triangleright(v_1(v_2\triangleright w_1))),\,(u_2\triangleleft(v_1(v_2\triangleright w_1)))(v_2\triangleleft w_1)w_2,\,u_3(u_4\triangleright(v_3(v_4\triangleright w_3))),\cdots,\\
&\qquad u_{2n-1}(u_{2n}\triangleright(v_{2n-1}(v_{2n}\triangleright w_{2n-1}))),\,(u_{2n}\triangleleft(v_{2n-1}(v_{2n}\triangleright w_{2n-1})))(v_{2n}\triangleleft w_{2n-1})w_{2n})\\
&\quad=(u_1(u_2\triangleright v_1)((u_2\triangleleft v_1)\triangleright v_2\triangleright w_1),\,u_2\triangleleft(v_1\triangleleft(v_2\triangleleft w_1))(v_2\triangleleft w_1)w_2,\,u_3(u_4\triangleright v_3)((u_4\triangleleft v_3)\triangleright v_4\triangleright w_3),\cdots,\\
&\qquad u_{2n-1}(u_{2n}\triangleright v_{2n-1})((u_{2n}\triangleleft v_{2n-1})\triangleright v_{2n}\triangleright w_{2n-1}),\,u_{2n}\triangleleft(v_{2n-1}\triangleleft(v_{2n}\triangleleft w_{2n-1}))(v_{2n}\triangleleft w_{2n-1})w_{2n}).
\end{aligned}
$$

Let $1_{U_1},1_{U_2},\cdots,1_{U_{2n}}$ be the identity elements of groups $U_1,U_2,\cdots,U_{2n}$, respectively. We have

$$
\begin{aligned}
&(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n})(1_{U_1},1_{U_2},1_{U_3},\cdots,1_{U_{n-1}},1_{U_{2n}})\\
&\quad=(u_1(u_2\triangleright 1_{u_1}),\,(u_2\triangleleft 1_{U_1})1_{U_2},\,u_3(u_4\triangleright 1_{U_3}),\cdots,u_{2n-1}(u_{2n}\triangleright 1_{U_{2n-1}}),\,(u_{2n}\triangleleft 1_{U_{2n-1}})1_{U_{2n}})\\
&\quad=(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n})
\end{aligned}
$$

and

$$
\begin{aligned}
&(1_{U_1},1_{U_2},1_{U_3},\cdots,1_{U_{n-1}},1_{U_{2n}})(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n})\\
&\quad=(1_{U_1}(1_{U_2}\triangleright u_1),\,(1_{U_2}\triangleleft u_1)u_2,\,1_{U_3}(1_{U_4}\triangleright u_3),\cdots,1_{U_{2n-1}}(1_{U_{2n}}\triangleright u_{2n-1}),\,(1_{U_{2n}}\triangleleft u_{2n-1})u_{2n})\\
&\quad=(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n}).
\end{aligned}
$$

Finally, let us find inverse element of $(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n})\in U_1\bowtie U_2\bowtie\cdots\bowtie U_{2n}$.

$$
\begin{aligned}
&(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n})(u_1',u_2',u_3',\cdots,u_{2n-1}',u_{2n}')=(1_{U_1},1_{U_2},1_{U_3},\cdots,1_{U_{n-1}},1_{U_{2n}})\\
&\Rightarrow(u_1(u_2\triangleright u_1'),\,(u_2\triangleleft u_1')u_2',\,u_3(u_4\triangleright u_3'),\cdots,u_{2n-1}(u_{2n}\triangleright u_{2n-1}'),\,(u_{2n}\triangleleft u_{2n-1}')u_{2n}')=(1_{U_1},1_{U_2},1_{U_3},\cdots,1_{U_{2n-1}},1_{U_{2n}})
\end{aligned}
$$

and

$$
\begin{aligned}
&(u_1',u_2',u_3',\cdots,u_{2n-1}',u_{2n}')(u_1,u_2,u_3,\cdots,u_{2n-1},u_{2n})=(1_{U_1},1_{U_2},1_{U_3},\cdots,1_{U_{n-1}},1_{U_{2n}})\\
&\Rightarrow(u_1'(u_2'\triangleright u_1),\,(u_2'\triangleleft u_1)u_2,\,u_3'(u_4'\triangleright u_3),\cdots,u_{2n-1}'(u_{2n}'\triangleright u_{2n-1}),\,(u_{2n}'\triangleleft u_{2n-1})u_{2n})=(1_{U_1},1_{U_2},1_{U_3},\cdots,1_{U_{n-1}},1_{U_{2n}}).
\end{aligned}
$$

Therefore, we obtain $u_{2i-1}'=u_{2i}^{-1}\triangleright u_{2i-1}^{-1}\,(1\le i\le n)$ and $u_{2i}'=(u_{2i}\triangleleft(u_{2i}^{-1}\triangleright u_{2i-1}^{-1}))^{-1}(1\le i\le n)$. Then, iterated bicrossed product $U_1\bowtie U_2\bowtie\cdots\bowtie U_{2n}$ is a group with the inverse element $(u_2^{-1}\triangleright u_1^{-1},\,(u_2\triangleleft(u_2^{-1}\triangleright u_1^{-1}))^{-1},\,u_4^{-1}\triangleright u_3^{-1},\cdots,u_{2n}^{-1}\triangleright u_{2n-1}^{-1},\,(u_{2n}\triangleleft(u_{2n}^{-1}\triangleright u_{2n-1}^{-1}))^{-1})$. Hence the result. □

Now, as consequences of Theorem 2.2, we can get a favourable results according to the cases of maps $\alpha_i\,(1\le i\le 2n-1)$ and $\beta_i\,(1\le i\le 2n-1)$.

Corollary 2.3. *Let $(U_i,U_{i+1},\alpha_i,\beta_i)\,(1\le i\le 2n-1)$ be matched pairs.*

1. *Assume $\beta_i\,(1\le i\le 2n-1)$ are trivial maps. Then $U_1\bowtie U_2\bowtie\cdots\bowtie U_{2n}$ is the iterated semidirect product, denoted by $U_1\rtimes U_2\rtimes U_3\cdots\rtimes U_{2n}$ [3].*
2. *Assume $\alpha_i\,(1\le i\le 2n-1)$ and $\beta_i\,(1\le i\le 2n-1)$ are trivial maps. Then $U_1\bowtie U_2\bowtie\cdots\bowtie U_{2n}$ is the direct products of $2n$ groups, denoted by $U_1\times U_2\times U_3\cdots\times U_{2n}$.*

Corollary 2.4. *Let $(U_i,U_{i+1},\alpha_i,\beta_i)\,(1\le i\le 2n-1)$ be matched pairs.*

1. *Let $\alpha_i\,(2\le i\le 2n-1)$ and $\beta_i\,(2\le i\le 2n-1)$ be trivial maps and $U_i\,(3\le i\le 2n)$ be trivial groups. Then $U_1\bowtie U_2\bowtie\cdots\bowtie U_{2n}$ is the bicrossed product $U_1\bowtie U_2$.*
2. *Let $\alpha_i\,(2\le i\le 2n-1)$ and $\beta_i\,(1\le i\le 2n-1)$ be trivial maps and $U_i\,(3\le i\le 2n)$ be trivial groups. Then $U_1\bowtie U_2\bowtie\cdots\bowtie U_{2n}$ is the semi-direct product of U_1 by U_2, denoted by $U_1\rtimes U_2$.*
3. *Let $\alpha_i\,(1\le i\le 2n-1)$ and $\beta_i\,(1\le i\le 2n-1)$ be trivial maps and $U_i\,(3\le i\le 2n)$ be trivial groups. Then $U_1\bowtie U_2\bowtie\cdots\bowtie U_{2n}$ is the direct product $U_1\times U_2$.*

2.2. Example Part:

We take cognizance of finite cyclic groups and give an application of the iterated bicrossed product.

Example 2.5. *Let* $U_1 = U_3 = U_5 = \cdots = U_{2i-1} = \mathbb{Z}_2 = \langle a_{2i-1}; a_{2i-1}^2 = 1 \rangle$ $(1 \leq i \leq n)$ *and* $U_2 = U_4 = U_6 = \cdots = U_{2i} = \mathbb{Z}_3 = \langle a_{2i}; a_{2i}^3 = 1 \rangle$ $(1 \leq i \leq n)$ *be finite cyclic groups. Suppose that, for* $i \in \{1,2,3,\cdots,n\}$,

$$\begin{array}{rcl} \alpha_{2i-1} : \mathbb{Z}_3 \times \mathbb{Z}_2 & \to & \mathbb{Z}_2 \\ (a_{2i}, a_{2i-1}) & \mapsto & \alpha_{2i-1}(a_{2i}, a_{2i-1}) = a_{2i} \triangleright a_{2i-1} \end{array}$$

and

$$\begin{array}{rcl} \beta_{2i-1} : \mathbb{Z}_3 \times \mathbb{Z}_2 & \to & \mathbb{Z}_3 \\ (a_{2i}, a_{2i-1}) & \mapsto & \beta_{2i-1}(a_{2i}, a_{2i-1}) = a_{2i} \triangleleft a_{2i-1} \end{array}$$

are maps, where a_{2i-1} $(1 \leq i \leq n) \in \mathbb{Z}_2$ *and* a_{2i} $(1 \leq i \leq n) \in \mathbb{Z}_3$*. So, we can write*

$$\begin{array}{rl} \alpha_i(1,1) = 1, & \beta_i(1,1) = 1, \\ \alpha_i(1,a_i) = a_i, & \beta_i(1,a_i) = 1, \\ \alpha_i(a_{i+1},1) = 1, & \beta_i(a_{i+1},1) = a_{i+1}, \\ \alpha_i(a_{i+1},a_i) = a_{i+1} \triangleright a_i, & \beta_i(a_{i+1},a_i) = a_{i+1} \triangleleft a_i, \\ \alpha_i(a_{i+1}^2,1) = 1, & \beta_i(a_{i+1}^2,1) = a_{i+1}^2, \\ \alpha_i(a_{i+1}^2,a_i) = a_{i+1}^2 \triangleright a_i, & \beta_i(a_{i+1}^2,a_i) = a_{i+1}^2 \triangleleft a_i. \end{array}$$

Then, iterated bicrossed product $\mathbb{Z}_2 \bowtie \mathbb{Z}_3 \bowtie \mathbb{Z}_2 \cdots \mathbb{Z}_2 \bowtie \mathbb{Z}_3$ *has a generators*

$$a_1,\ a_2,\ a_3,\ \cdots,\ a_{2n-1},\ a_{2n}$$

and relations

$$\begin{array}{c} a_1^2 = 1, a_2^3 = 1, a_3^2 = 1, a_4^3 = 1, \cdots, a_{2n-1}^2 = 1, a_{2n}^3 = 1, \\ a_{i+1}a_i = (a_{i+1} \triangleright a_i)(a_{i+1} \triangleleft a_i)\ (1 \leq i \leq 2n-1), \\ a_{i+1}^2 a_i = (a_{i+1}^2 \triangleright a_i)(a_{i+1}^2 \triangleleft a_i)\ (1 \leq i \leq 2n-1). \end{array}$$

Corollary 2.6. *We note that* U_i $(3 \leq i \leq 2n)$ *are trivial groups given in Example* 2.5*, then the iterated bicrossed product* $\mathbb{Z}_2 \bowtie \mathbb{Z}_3 \bowtie \mathbb{Z}_2 \cdots \mathbb{Z}_2 \bowtie \mathbb{Z}_3$ *reduced to bicrossed product* $\mathbb{Z}_2 \bowtie \mathbb{Z}_3$*. The bicrossed product* $\mathbb{Z}_2 \bowtie \mathbb{Z}_3$ *is also a special case of the structure obtained in study in [[1], Theorem* 3.1*], authors investigated the construction of bicrossed* $\mathbb{Z}_p \bowtie \mathbb{Z}_m$ *of two finite cyclic groups that one of them has prime order.*

References

[1] A. L. Agore, A. Chirvasitu, B. Ion, G. Militaru, Bicrossed products for finite groups, Algebr Represent Theor 12 (2009), 481-488.
[2] F. Ates, Some new monoid and group constructions under semidirect products. Ars Comb., 91 (2009), 203-218.
[3] D.C. Cohen and A.I. Suciu, Homology of iterated semidirect products of free groups, J. Pure Appl. Algebra, 126 (1998), 87–120.
[4] E.K. Çetinalp, E.G. Karpuz, F. Ateş, A.S. Çevik, Two-sided crossed product of groups, Filomat, 30(4) (2016), 1005–1012.
[5] E.K. Çetinalp, E.G. Karpuz, Iterated crossed product of cyclic groups, Bulletin of the Iranian Mathematical Society, 44(6) (2018), 1493–1508.
[6] E.K. Çetinalp, E.G. Karpuz, Crossed Product of Infinite Groups and Complete Rewriting Systems, Turkish Journal Of Mathematics, 45 (1) (2021), 410-422.
[7] A. Emin, F. Ates, S. Ikikardes, I. N. Cangul, A new monoid construction under crossed products, Journal of Inequalities and Applications 244 (2013).
[8] F. Panaite, Iterated crossed products, J. Algebra Appl., 13(7) (2014).
[9] J.M. Howie and N. Ruskuc, Constructions and presentations for monoids, Comm. in Algebra 22 (15) (1994), 6209-6224.
[10] P. W. Michor, Knit products of graded Lie algebras and groups, Suppl. Rendiconti Circolo Matematico di Palermo Ser. II, 22 (1989), 171-175.
[11] M. Takeuchi, Matched pairs of groups and bismah product of Hopf algrebras, Comm. Algebra, 9(8)(1981), 841-882.
[12] S. O. Unal, Green's relations on ternary semihypergroups and crossed hyperproduct of hypergroups, Asian-European Journal of Mathematics, 14(7) (2021).
[13] G. Zappa, Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro, Atti Secondo Congresso Un.Mat Ital. Bologna, (1940), 119-125.