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Dwell Time for the Hurwitz Stability of Switched Linear Differential Equation Systems

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Abstract

In this paper, the problem of dwell time for the Hurwitz stability of switched linear systems is considered. Dwell time is determined based on the solution of Lyapunov matrix equation for the Hurwitz stability of switched linear differential systems. A numerical example illustrating the efficiency of theorem has been given.

1. Introduction

Switched differential systems are used in mathematical modeling of many fields such as automotive engineering, motor engine control, constrained robotics, networked control systems [1]- [5]. Stability analysis of switched systems is a basic issue. Therefore, it is important to study the stability of the switched systems. There are roughly two different types of studies on the stability of the switched systems in the literature. The first of these studies is to examine the stability of the switched systems under the given switching signals and the other is to determine the switching signal that will ensure the stability of the switched systems [6]. In this study, we have focused on the second method for linear switched systems and we have aimed to give a method that determines the dwell for the system to be Hurwitz stable.
Let us consider the linear switched systems described by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \sigma \in S, t \in [0,\infty) \tag{1.1}$$

where $\{A_p \in \mathbb{C}^{m\times m}, p \in \mathscr{P}\}$ is matrix family for $\mathscr{P} = \{1,2,\ldots,N\}$, S is the set of the functions $\sigma : [0,\infty) \to \mathscr{P}$, σ denotes the switching signals. The amount of time passed between the consecutive switching events is called dwell time of system (1.1). If each subsystems are stable then there exists a minimum dwell time that guarantees stability of system (1.1). For system (1.1), the dwell time method refer to allowing certain set of switching signals; namely,

$$S = S_{dwell[\tau]} = \{\sigma(t) | t \in [t_k, t_{k+1}), t_{k+1} - t_k \geq \tau\}$$

and finding the dwell time, the infimum of the numbers τ for which the switched system is Hurwitz stable [7], [8].
This paper is organized as follows. In section 2, preliminary is given. In section 3, the dwell time for Hurwitz stability is determined. Finally, in section 4, a numerical example is given.

2. Preliminaries

Let $A \in \mathbb{C}^{m\times m}$ and $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$, $x_i(t) (i = 1,2,...,m)$ be a differentiable function and consider the following differential equation system:

$$\dot{x}(t) = Ax(t), t \in [0,\infty). \tag{2.1}$$

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Differential equation system (2.1) is stable if for any positive number ε there exists $\delta = \delta(\varepsilon)$ such that $\|x(t)\| \leq \varepsilon$ for $t \in [0,\infty)$ whenever for $\|x(0)\| \leq \delta$. Further, system (2.1) is Hurwitz stable (asymptotically stable) if it is stable and $\|x(t)\| \to 0$ with increase t to infinity [9]- [13]. Lyapunov theorem, a criterion for Hurwitz stability, is as follows.
The matrix A is Hurwitz stable if and only if there is a solution $X = X^* > 0$ of the Lyapunov matrix equation $A^*X + XA = -I$ where I is unit matrix, A^* is adjoint of the matrix A, the matrix $X = \int_0^\infty e^{tA^*}e^{tA}dt$ is positive definite solution of Lyapunov matrix equation [9]- [13].

Theorem 2.1. *Following inequalitiy*

$$\|e^{At}\| \leq \sqrt{\|X^{-1}\|\|X\|}e^{\frac{-t}{2\|X\|}} \tag{2.2}$$

is valid for the Hurwitz stable matrix A where the matrix X is solution of Lyapunov matrix equation [14], [15].

3. Determination of Dwell Time for Hurwitz Stability

Let us give the following theorem, which gives the upper bound of the solution of the system (1.1) to use determining the dwell time for Hurwitz stability.

Theorem 3.1. *If the matrix $A_p(p = 1,2,\dots,N)$ are the Hurwitz stable matrix, then the following equation is provided for the solution of the linear switched system (1.1)*

$$\|x(t)\| \leq (\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}}e^{\frac{-1}{2\|X_{\sigma_{n+1}}\|}(t-t_n)}e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}(t_k-t_{k-1})]}\|x_0\|$$

where $t \in [t_n, t_{n+1})$ and X_{σ_k} is solution of Lyapunov matrix equation $A^*_{\sigma_k}X_{\sigma_k} + X_{\sigma_k}A_{\sigma_k} = -I$.

Proof. Let us consider the switched system (1.1), where $A_p(p = 1,2,\dots,N)$ are the Hurwitz stable matrix. The solution of system (1.1) which is given with the initial value $x(0) = x_0$ is expressed as

$$x(t) = e^{A_{\sigma_{n+1}}(t-t_n)}e^{A_{\sigma_n}(t_n-t_{n-1})}\dots e^{A_{\sigma_1}(t_1-t_0)}x_0, t \in [t_n, t_{n+1})$$

or

$$x(t) = e^{A_{\sigma_{n+1}}(t-t_n)}(\prod_{k=1}^{n}e^{A_{\sigma_k}(t_k-t_{k-1})})x_0, t \in [t_n, t_{n+1}). \tag{3.1}$$

By taking the norm of solution (3.1) and applying the triangle inequality, the following inequality is obtained

$$\|x(t)\| = \|e^{A_{\sigma_{n+1}}(t-t_n)}(\prod_{k=1}^{n}e^{A_{\sigma_k}(t_k-t_{k-1})})x_0\|$$
$$\|x(t)\| \leq \|e^{A_{\sigma_{n+1}}(t-t_n)}\|\prod_{k=1}^{n}\|e^{A_{\sigma_k}(t_k-t_{k-1})}\|\|x_0\|.$$

If we use inequality (2.2) we obtain the upper bound of the solution as:

$$\|x(t)\| \leq \sqrt{\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|}e^{-\frac{(t-t_n)}{2\|X_{\sigma_{n+1}}\|}}\prod_{k=1}^{n}\sqrt{\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|}e^{-\frac{(t_k-t_{k-1})}{2\|X_{\sigma_k}\|}}\|x_0\|$$
$$= (\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}}e^{-\frac{(t-t_n)}{2\|X_{\sigma_{n+1}}\|}}\prod_{k=1}^{n}(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)^{\frac{1}{2}}e^{-\frac{(t_k-t_{k-1})}{2\|X_{\sigma_k}\|}}\|x_0\|$$
$$\|x(t)\| \leq (\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}}e^{\frac{-1}{2\|X_{\sigma_{n+1}}\|}(t-t_n)}e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}(t_k-t_{k-1})]}\|x_0\|.$$

□

Let us denote by $\mathscr{F}$ the set of all transitions between subsystems and give the following theorem, which gives the dwell time for Hurwitz stability.

Theorem 3.2. *Let $A_p(p \in \mathscr{P})$ be Hurwitz stable matrices. Then, the switched system (1.1) is Hurwitz stable for dwell time*

$$\tau > \max_{n \in \mathscr{F}} \frac{\alpha(n)}{\beta(n)} \tag{3.2}$$

where $\alpha(n) = \sum_{k=1}^{n} ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)^{\frac{1}{2}}$ and $\beta(n) = \sum_{k=1}^{n}\frac{1}{2\|X_{\sigma_k}\|}$.

Proof. The upper bound of the solution of the switched differential equation system (1.1) from Theorem 3.1.

$$\|x(t)\| \leq (\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\frac{-1}{2\|X_{\sigma_{n+1}}\|}(t-t_n)} e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}(t_k-t_{k-1})]}\|x_0\|.$$

Consider that $\|X_{\sigma_{n+1}}\| > 1$, $e^{\frac{-(t-t_n)}{2\|X_{\sigma_{n+1}}\|}} < 1$ and $\tau \leq t_k - t_{k-1}$, the above inequality can be written as

$$\|x(t)\| \leq \max_{i\in\mathscr{P}}(\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}\tau]}\|x_0\|$$
$$= \max_{i\in\mathscr{P}}(\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\alpha(n)-\tau\beta(n)}\|x_0\|$$

where $\alpha(n) = \sum_{k=1}^{n} ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)^{\frac{1}{2}}$ and $\beta(n) = \sum_{k=1}^{n} \frac{1}{2\|X_{\sigma_k}\|}$.
Since $\alpha(n) - \tau\beta(n) < 0$, it is seen that $\|x(t)\| \to 0$ as $t \to \infty$. Thus, it is shown that the system (1.1) is Hurwitz stable for dwell time in (3.2). □

4. A Numerical Example

Example 4.1. *Let us consider the following system consisting three Hurwitz stable subsystems:*
$A_1 = \begin{pmatrix} -0.1 & -1 \\ 0.3 & -0.5 \end{pmatrix}, A_2 = \begin{pmatrix} -0.7 & -0.4 \\ 2 & -0.6 \end{pmatrix}, A_3 = \begin{pmatrix} -0.01 & 3 \\ -0.5 & -0.4 \end{pmatrix}$

$$\dot{x}(t) = A_i x(t), x(0) = [10, -12]^T, t \geq 0; i \in \{1,2,3\}. \tag{4.1}$$

Let $\mathscr{D}$ be the switching graph of the system (4.1) given in Figure 4.1.

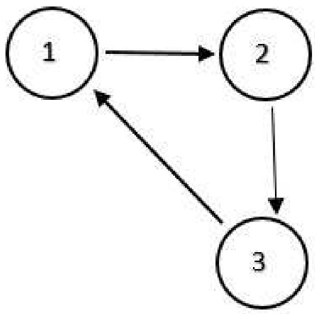

Figure 4.1: Switching graphs of the Cauchy problem consisting of three subsystems with A_1, A_2 and A_3.

For the graph $\mathscr{D}$, the minimum dwell time calculated in Theorem 3.2. is obtained as $\tau > 4.844453$. For this minimum dwell time and switched for graph $\mathscr{D}$, the solution curves given in the graph below are obtained.

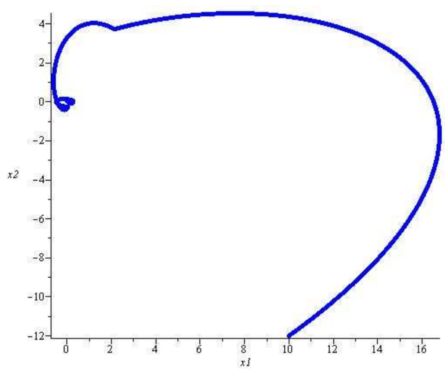

Figure 4.2: State trajectory under constant dwell time ($\tau = 4.85$) condition.

5. Conclusion

In this paper, a new minimum dwell time which make differential equation systems (1.1) Hurwitz stable, is obtained depending on solution of the Lyapunov matrix equation. The effect of the new minimum dwell time is illustrated with an example.

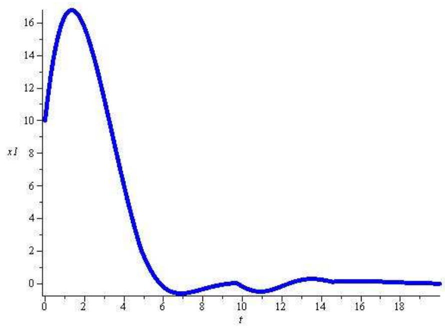

Figure 4.3: State trajectory x_1 of system (4.1).

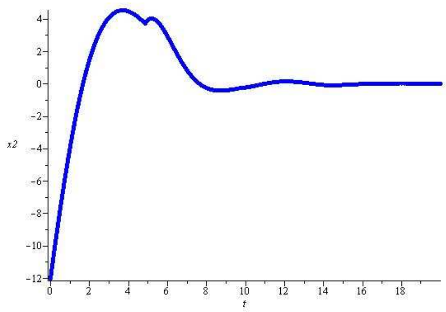

Figure 4.4: State trajectory x_2 of system (4.1).

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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