

Research Article

Oscillation of noncanonical second-order advanced differential equations via canonical transform

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ABSTRACT. In this paper, we develop a new technique to deduce oscillation of a second-order noncanonical advanced differential equation by using established criteria for second-order canonical advanced differential equations. We illustrate our results by presenting two examples.

Keywords: Advanced differential equation, canonical transform, second-order, oscillation.

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1. INTRODUCTION

Consider the second-order noncanonical advanced differential equation

$$(1.1) \quad (\mu_1 \eta')'(t) + f_1(t) \eta(\sigma(t)) = 0, \quad t \geq t_0$$

subject to

$$(P_1) \quad \mu_1, f_1 \in C([t_0, \infty), (0, \infty)),$$

$$(P_2) \quad \sigma \in C^1([t_0, \infty), \mathbb{R}), \sigma'(t) > 0, \sigma(t) \geq t \text{ for all } t \geq t_0,$$

$$(P_3) \quad \text{Equation (1.1) is in noncanonical form, that is,}$$

$$\Omega(t_0) := \int_{t_0}^{\infty} \frac{dt}{\mu_1(t)} < \infty.$$

If (P_3) does not hold, then we say that (1.1) is in canonical form.

In recent years, there are many results dealing with the oscillation of (1.1) and its modifications for the delay case, that is, $\sigma(t) \leq t$, see for example [2, 4, 7, 11, 12, 14], and few results in the case of $\sigma(t) \geq t$, see [1, 3, 5, 6, 8–10, 13–17, 19, 20, 23]. Many authors paid attention to a comparison technique, which is a powerful tool in the theory of oscillation, see, for instance, the papers [11, 19, 21, 24] for more details. Further, many authors used the Riccati transformation method to obtain oscillation criteria for delay equations. For the mixed case, that is, $\sigma(t) \leq t$ and $\sigma(t) \geq t$, the author in [22] discussed the oscillatory and nonoscillatory behavior of systems of differential equations based on the analysis of the corresponding characteristic equations. On the other hand in [10], Jozef Džurina already obtained oscillation criteria for the canonical second-order advanced differential equation

$$(ru')'(t) + p(t)u(\sigma(t)) = 0$$

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from those of a related ordinary differential equation

$$(ru')'(t) + q(t)u(t) = 0.$$

In this paper, we will rewrite (1.1) in noncanonical form equivalently as an equation in canonical form, then apply the results established by Jozef Džurina in [10] to the obtained equation in canonical form, thus establishing new results for our equation (1.1) in noncanonical form.

Section 2 contains some preliminary results, the main results are presented in Section 3, and two illustrative examples are offered in Section 4.

2. PRELIMINARY RESULTS

Throughout, without loss of generality, considering nonoscillatory solutions of (1.1), we restrict our attention to the positive case, since the negative case is similar.

Lemma 2.1. *We have*

$$(2.1) \quad (\mu_1 \eta')' = \frac{1}{\Omega} \left(\mu_1 \Omega^2 \left(\frac{\eta}{\Omega} \right)' \right)'.$$

Proof. A straightforward calculation shows that

$$\begin{aligned} \frac{1}{\Omega} \left(\mu_1 \Omega^2 \left(\frac{\eta}{\Omega} \right)' \right)' &= \frac{1}{\Omega} \left(\mu_1 \Omega^2 \frac{\eta' \Omega - \eta \Omega'}{\Omega^2} \right)' \\ &= \frac{1}{\Omega} \left(\mu_1 \left(\eta' \Omega - \eta \left(-\frac{1}{\mu_1} \right) \right) \right)' \\ &= \frac{1}{\Omega} (\mu_1 \eta' \Omega + \eta)' \\ &= \frac{1}{\Omega} (\Omega (\mu_1 \eta')' + \mu_1 \eta' \Omega' + \eta') \\ &= (\mu_1 \eta')' + \frac{1}{\Omega} \left(\mu_1 \eta' \left(-\frac{1}{\mu_1} \right) + \eta' \right) \\ &= (\mu_1 \eta')', \end{aligned}$$

completing the proof. □

Lemma 2.2. *Equation (1.1) can be written in the equivalent canonical form as*

$$(2.2) \quad (\mu z')'(t) + f(t)z(\sigma(t)) = 0$$

where

$$\mu = \mu_1 \Omega^2, \quad z = \frac{\eta}{\Omega}, \quad \text{and} \quad f = \Omega(\Omega \circ \sigma) f_1.$$

Proof. The equivalence of (1.1) and (2.2) follows from Lemma 2.1. Moreover, since

$$\int_{t_0}^{\infty} \frac{dt}{\mu_1(t)\Omega^2(t)} = \lim_{t \rightarrow \infty} \frac{1}{\Omega(t)} - \frac{1}{\Omega(t_0)} = \infty,$$

(2.2) is in canonical form. □

Corollary 2.1. *The noncanonical differential equation (1.1) has an eventually positive solution if and only if the canonical equation (2.2) has an eventually positive solution.*

From Corollary 2.1, it is clear that the investigation of oscillation of (1.1) is reduced to that of (2.2), and therefore, we deal with only one class of an eventually positive solution, namely,

$$(2.3) \quad z(t) > 0, \quad \mu(t)z'(t) > 0 \quad \text{and} \quad (\mu(t)z'(t))' < 0$$

for $t \geq t_1 \geq t_0$, see [10, Lemma 2.1]. Define

$$w(t) = \int_{t_0}^t \frac{ds}{\mu(s)}.$$

Now, we state a basic oscillation result given in [10, 18], which will be improved in the next section.

Theorem 2.1. *Assume that there exists a constant δ such that*

$$(2.4) \quad w(t) \int_t^\infty f(s)ds \geq \delta > \frac{1}{4}$$

eventually. Then (2.2) is oscillatory.

3. OSCILLATION RESULTS

In this section, we obtain results for (1.1) by applying results from [10] to the equivalent equation (2.2). If the condition (2.4) fails to hold ($\delta \leq 1/4$), then we can derive a new oscillation criterion using the constant δ .

Theorem 3.2. *Let η be a positive solution of (1.1) and suppose*

$$(3.1) \quad w(t) \int_t^\infty f(s)ds \geq \delta > 0$$

eventually. Then

$$\frac{\eta(t)}{\Omega(t)w^\delta(t)}$$

is increasing eventually.

Proof. Let $\eta > 0$ be a solution of (1.1). By Lemma 2.2, $z > 0$ is a solution of (2.2) satisfying (2.3). Hence, the assumption [10, (3.1) of Theorem 3.1] is satisfied, and therefore the conclusion of [10, Theorem 3.1] holds, which says that z/w^δ is strictly increasing, completing the proof. \square

Next, we present a new comparison result.

Theorem 3.3. *Let (3.1) hold. If the differential equation*

$$(3.2) \quad (\mu z')'(t) + \left(\frac{w(\sigma(t))}{w(t)} \right)^\delta f(t)z(t) = 0$$

is oscillatory, then so is (1.1).

Proof. Since [10, assumption (E₂) of Theorem 3.3] is satisfied, (2.2) is oscillatory, and then so is (1.1). \square

Using any criterion for the oscillation of (3.2), we immediately obtain an oscillation result for (1.1).

Theorem 3.4. *Let (3.1) hold. If there exists a constant δ_1 such that*

$$(3.3) \quad w(t) \int_t^\infty \left(\frac{w(\sigma(s))}{w(s)} \right)^\delta f(s)ds \geq \delta_1 > \frac{1}{4}$$

eventually, then (1.1) is oscillatory.

Proof. Use [10, Theorem 3.4] to complete the proof. \square

If the condition (3.3) fails to hold ($\delta_1 \leq 1/4$), then we can derive a new oscillation criterion using the constant δ_1 .

Theorem 3.5. *Let (3.1) hold. Assume that η is a positive solution of (1.1) and*

$$w(t) \int_t^\infty \left(\frac{w(\sigma(s))}{w(s)} \right)^\delta f(s) ds \geq \delta_1 > 0$$

eventually. Then

$$\frac{\eta(t)}{\Omega(t)w^{\delta_1}(t)}$$

is increasing eventually.

Proof. Use [10, Theorem 3.8] to complete the proof. \square

Theorem 3.6. *Let (3.1) and (3.3) hold. If the differential equation*

$$(3.4) \quad (\mu z')'(t) + \left(\frac{w(\sigma(t))}{w(t)} \right)^{\delta_1} f(t)z(t) = 0$$

is oscillatory, then so is (1.1).

Theorem 3.7. *Let (3.1) and (3.3) hold. If there exists a constant δ_2 such that*

$$(3.5) \quad w(t) \int_t^\infty \left(\frac{w(\sigma(s))}{w(s)} \right)^{\delta_1} f(s) ds \geq \delta_2 > \frac{1}{4}$$

eventually, then (1.1) is oscillatory.

The proofs of Theorems 3.6 and 3.7 follow from [10, Theorems 3.9 and 3.10].

For convenience, let us use the additional condition that there is a positive constant β such that

$$(3.6) \quad \frac{w(\sigma(t))}{w(t)} \geq \beta > 1$$

eventually. Thus, in view of (3.1), conditions (3.3) and (3.5) can be written in simpler forms as

$$\delta_1 = \beta^\delta \delta > \frac{1}{4},$$

$$\delta_2 = \beta^{\delta_1} \delta > \frac{1}{4},$$

respectively. Repeating the above process, we have the increasing sequence $\{\delta_n\}$ defined by

$$\delta_0 = \delta,$$

$$\delta_{n+1} = \beta^{\delta_n} \delta.$$

Now as in [10, Theorem 3.12], one can generalize the oscillation criteria obtained in Theorems 3.4 and 3.7.

Theorem 3.8. *Let (3.1) and (3.6) hold. If there exists $n \in \mathbb{N}$ such that $\delta_j \leq 1/4$ for $j = 0, 1, 2, \dots, n-1$ and*

$$\delta_n > \frac{1}{4},$$

then (1.1) is oscillatory.

4. EXAMPLES

We support the obtained results with some examples.

Example 4.1. Consider the second-order advanced differential equation

$$(4.1) \quad (t^2\eta'(t))' + a\lambda\eta(\lambda t) = 0, \quad t \geq 1,$$

where $a > 0$. Here $\mu_1(t) = t^2$, $f_1(t) = a\lambda$, $\sigma(t) = \lambda t$, $\lambda > 1$. A simple calculation shows that

$$\Omega(t) = \frac{1}{t}, \quad \mu(t) = 1, \quad w(t) = t, \quad f(t) = \frac{a}{t^2}.$$

The transformed canonical equation is

$$z''(t) + \frac{a}{t^2}z(\lambda t) = 0.$$

Condition (3.1) clearly holds, and (3.3) becomes

$$a\lambda^\delta > \frac{1}{4}.$$

Now $\delta = a$, and by Theorem 3.4, (4.1) is oscillatory provided

$$a\lambda^a > \frac{1}{4}.$$

For example, if $a = \frac{1}{5}$, then we see that $\lambda \geq 3.052$, and for $\lambda = 1.8$, we need $a \geq 0.22$.

Example 4.2. Consider the second-order advanced differential equation

$$(4.2) \quad (t^2\eta'(t))' + 0.35742\eta(1.61t) = 0.$$

The transformed canonical equation is

$$z''(t) + \frac{0.222}{t^2}z(1.61t) = 0.$$

For (4.2), $\delta_0 = 0.222$ and $\lambda = 1.61$. A simple calculation shows that

$$\delta_1 = 0.2468 \quad \text{and} \quad \delta_2 = 0.24968.$$

Therefore, Theorems 3.4 and 3.7 fail for (4.2). But

$$\delta_3 = 0.25003 > \frac{1}{4}$$

and Theorem 3.8 implies the oscillation of (4.2). However, it is easy to see that [5, Theorems 3, 5, 6], [8, Theorems 3.3, 3.4, and Corollary 4.4] and [4, Theorem 2] do not get oscillation of (4.2). Thus, our result improve these results.

5. CONCLUSION

In this paper, we derive oscillation criteria for the noncanonical equation (1.1) by transforming it to the canonical equation (2.2), and then we use the comparison technique available for the canonical equation (2.2) to get new oscillation criteria for the studied equation (1.1). Our oscillation criteria improve [5, Theorems 3, 5, 6], [8, Theorems 3.3, 3.4 and Corollary 4.4] and [4, Theorem 2] for the special case $\alpha = \beta = 1$. Finally, the results obtained in [10] cannot be applied to (4.1) and (4.2) since they are of noncanonical type.

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