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Research Article

Piecewise polynomial numerical method for Volterra integral equations of the fourth-kind with constant delay

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Abstract

This work studies the fourth-kind integral equation as a mixed system of first and second-kind Volterra integral equations (VIEs) with constant delay. Regularity, smoothing properties and uniqueness of the solution of this mixed system are obtained by using theorems which give the relevant conditions related to first and second-kind VIEs with delays. A numerical collocation algorithm making use of piecewise polynomials is designed and the global convergence of the proposed numerical method is established. Some typical numerical experiments are also performed which confirm our theoretical result.

Mathematics Subject Classification (2020). 65R20, 45D05, 34K06

Keywords. fourth-kind integral equations, Volterra integral equations with constant delay, piecewise polynomial collocation method, convergence analysis

1. Introduction

In this paper, we analyze the numerical solution of mixed system of first and second-kind Volterra integral equations with (constant) delay $\tau > 0$, given by

$$\begin{cases} y(t) = f(t) + (\nu_{11}y)(t) + (\nu_{12}z)(t) + (\nu_{\tau 11}y)(t) + (\nu_{\tau 12}z)(t), & t \in I, \\ \\ 0 = g(t) + (\nu_{21}y)(t) + (\nu_{22}z)(t) + (\nu_{\tau 21}y)(t) + (\nu_{\tau 22}z)(t), & t \in I, \end{cases} \tag{1.1}$$

where $I := [0, T]$ and the Volterra integral operators ν_{kl} and the delay integral operator $\nu_{\tau kl}$ are given by

$$(\nu_{kl}w)(t) = \int_0^t K_{kl}(t,s)w(s)ds,$$

$$(\nu_{\tau kl}w)(t) = \int_0^{t-\tau} \hat{K}_{kl}(t,s)w(s)ds, \qquad k,l = 1,2, \quad t \in I,$$

with

$$y(t) = \phi(t), \quad z(t) = \varphi(t), \quad t \in [-\tau, 0).$$

Also, $y, f : I \to \mathbb{R}^{d_1}$, $z, g : I \to \mathbb{R}^{d_2}$, $\phi : [-\tau, 0] \to \mathbb{R}^{d_1}, \varphi : [-\tau, 0] \to \mathbb{R}^{d_2}$, $K_{kk}(.,.)$, $\hat{K}_{kk}(.,.) \in L(\mathbb{R}^{d_k})$, $K_{12}(.,.)$, $\hat{K}_{12}(.,.) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$, $K_{21}(.,.)$, $\hat{K}_{21}(.,.) \in L(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ are

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continuous functions and $L(.,.)$ is the linear transformation space.

The Volterra equations with delays are encountered in physical and biological modeling processes [2, 12]. A historical survey of mathematical models in biology, which can be described by Volterra integral equations with constant delays has been presented in the monograph [6].

The numerical solutions of delay integral equations have been investigated by many authors (see, for example, [1,3–7,10,11,16,17,19]). To the lest of our knowledge, numerical analysis of mixed system (1.1) is new in the literature and there are a few available results which investigate these systems numerically. Bulatov et al. [8, 9] considered the integral equation

$$A(t)x(t) = \int_{t-\tau}^{t} K(t,s)x(s)ds = f(t), \quad t \in I,$$

with initial value

$$x(t) = x^0(t), \quad t \in [-\tau, 0).$$

Here, $A(t)$ and $K(t,s)$ are sufficiently smooth $n \times n$-matrices, $\tau > 0$ is a known constant and $det A(t) \equiv 0$. Sufficient conditions for the existence and uniqueness of a continuous solution of this system were given. For more details see [8, 9] and reference therein.

Here we propose the numerical solution of the mixed system (1.1) based on piecewise polynomial collocation methods that construct collocation solutions in a certain polynomial spline space $S_{m-1}^{-1}(\Omega_N)$, where Ω_N represent a uniform partition of I. This is a linear space of discontinuous polynomial spline functions of degree $m-1$ whose dimension is Nm [6]. The succeeding sections of this paper are organized as follows. In section 2, we investigate regularity, smoothing properties and uniqueness of the solution of system (1.1). In section 3, the collocation method based on piecewise polynomials is applied for solving system (1.1) numerically and the global convergence of this method is established in section 4. The paper concludes in section 5 by illustrating the efficiency of the method on some numerical examples.

2. Regularity and smoothing properties of solution

Consider the semi-explicit system (1.1) and let $|\det(K_{22}(t,t))| \geq k_0 > 0, \ \forall t \in I$. By differentiating the second equation of (1.1) with respect to t, substituting for y in the resulting equation using the first equation and applying some elementary manipulations, we get

$$\begin{aligned} z(t) = &\tilde{g}(t) + (\tilde{\nu}_{21}y)(t) + (\tilde{\nu}_{22}z)(t) + (\tilde{\nu}_{\tau 21}y)(t) + (\tilde{\nu}_{\tau 22}z)(t) \\ &+\tilde{K}_{21}(t,t-\tau)y(t-\tau) + \tilde{K}_{22}(t,t-\tau)z(t-\tau), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \tilde{g}(t) =& \quad K_{22}^{-1}(t,t)\Big(g'(t) + K_{21}(t,t)f(t)\Big), \\ (\tilde{\nu}_{21}y)(t) =& \quad K_{22}^{-1}(t,t)\Big(\int_0^t \Big(\frac{\partial K_{21}(t,s)}{\partial t} + K_{21}(t,t)K_{11}(t,s)\Big)y(s)ds\Big), \\ (\tilde{\nu}_{22}z)(t) =& \quad K_{22}^{-1}(t,t)\Big(\int_0^t \Big(\frac{\partial K_{22}(t,s)}{\partial t} + K_{21}(t,t)K_{12}(t,s)\Big)z(s)ds\Big), \\ (\tilde{\nu}_{\tau 21}y)(t) =& \quad K_{22}^{-1}(t,t)\Big(\int_0^{t-\tau} \Big(\frac{\partial \hat{K}_{21}(t,s)}{\partial t} + K_{21}(t,t)\hat{K}_{11}(t,s)\Big)y(s)ds\Big), \\ (\tilde{\nu}_{\tau 22}z)(t) =& \quad K_{22}^{-1}(t,t)\Big(\int_0^{t-\tau} \Big(\frac{\partial \hat{K}_{22}(t,s)}{\partial t} + K_{21}(t,t)\hat{K}_{12}(t,s)\Big)z(s)ds\Big), \\ \tilde{K}_{21}(t,t-\tau) =& \quad K_{22}^{-1}(t,t)\hat{K}_{21}(t,t-\tau), \\ \tilde{K}_{22}(t,t-\tau) =& \quad K_{22}^{-1}(t,t)\hat{K}_{22}(t,t-\tau). \end{aligned} \tag{2.2}$$

Equation (2.1) together with the first equation of (1.1) are the same as the second kind Volterra integral equations with constant delay given by

$$\begin{cases} y(t) = f(t) + (\nu_{11}y)(t) + (\nu_{12}z)(t) + (\nu_{\tau 11}y)(t) + (\nu_{\tau 12}z)(t), \\ \\ z(t) = \tilde{g}(t) + (\tilde{\nu}_{21}y)(t) + (\tilde{\nu}_{22}z)(t) + (\tilde{\nu}_{\tau 21}y)(t) + (\tilde{\nu}_{\tau 22}z)(t) \\ \qquad + \tilde{K}_{21}(t, t-\tau)y(t-\tau) + \tilde{K}_{22}(t, t-\tau)z(t-\tau). \end{cases} \tag{2.3}$$

The solution of system (2.3) is continuous at $t = 0$ only if the initial functions are such that

$$\begin{cases} \phi(0) = f(0) - \int_{-\tau}^{0} \hat{K}_{11}(0, s)\phi(s)ds - \int_{-\tau}^{0} \hat{K}_{12}(0, s)\varphi(s)ds, \\ \\ \varphi(0) = \tilde{g}(0) - \int_{-\tau}^{0} \bar{K}_{21}(0, s)\phi(s)ds - \int_{-\tau}^{0} \bar{K}_{22}(0, s)\varphi(s)ds \\ \qquad + \tilde{K}_{21}(0, -\tau)\phi(-\tau) + \tilde{K}_{22}(0, -\tau)\varphi(-\tau) \end{cases} \tag{2.4}$$

where $\bar{K}_{21}(0, s) = K_{22}^{-1}(0, 0)K_{21}(0, 0)\hat{K}_{11}(0, s) + \frac{\partial \hat{K}_{21}}{\partial t}(0, s)$ and $\bar{K}_{22}(0, s) = K_{22}^{-1}(0, 0)$ $K_{21}(0, 0)\hat{K}_{12}(0, s) + \frac{\partial \hat{K}_{22}}{\partial t}(0, s)$. The conditions of existence and uniqueness of solutions related to the mixed system (1.1) can be investigated by considering the system (2.3) and theorems about existence and uniqueness of the solution of second-kind Volterra integral equations with non-vanishing delay (see [6, Theorem 4.1.1]). Note that using differentiation, we reduce system (1.1) to a regular system of integral equations of the second-kind. However, this reduction to a second-kind Volterra system is not practical from a numerical point of view.

3. Numerical method

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $I := [0, T]$, such that $t_n = nh, n = 0, ..., N$ and $\Omega_N := \{t_0, t_1, \cdots, t_N = T\}$, $\sigma_0 := [t_0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$ $(1 \leq n \leq N-1)$. The mesh Ω_N is assumed to be constrained(i.e, $h = \frac{\tau}{r}$ for some $r \in \mathbb{N}$). Consider the set of collocation parameters $\{c_j\}_{j=1}^{m}$, where $0 < c_1 < \cdots < c_m \leq 1$, and define the set $X_N = \{t_{n,j} = t_n + c_j h\}$ of the collocation points.

Definition 3.1. For a given mesh Ω_N the piecewise polynomial space $S_{\mu}^{(d)}(\Omega_N)$, with $\mu \geq 0$, $-1 \leq d < \mu$ is given by

$$S_{\mu}^{(d)}(\Omega_N) := \{w \in C^d(I) : w|_{\sigma_n} \in \pi_{\mu}, 0 \leq n \leq N-1\}.$$

Here, π_{μ} denotes the space of (real) polynomials of degree not exceeding μ and $C^d(I)$ as the set of all the functions on I, which together with their derivatives of orders up to d. It is readily verified that $S_{\mu}^{(d)}(\Omega_N)$ is a (real) linear vector space whose dimension is given by $\dim S_{\mu}^{(d)}(\Omega_N) = N(\mu - d) + d + 1$.

The collocation solution $u, v \in S_{m-1}^{(-1)}(\Omega_N)$, $(\mu = m-1, d = -1)$ to equation (1.1) is then given by

$$\begin{cases} u(t) = & f(t) + \int_0^t K_{11}(t, s)u(s)ds + \int_0^t K_{12}(t, s)v(s)ds \\ & + \int_0^{t-\tau} \hat{K}_{11}(t, s)u(s)ds + \int_0^{t-\tau} \hat{K}_{12}(t, s)v(s)ds, \\ \\ 0 = & g(t) + \int_0^t K_{21}(t, s)u(s)ds + \int_0^t K_{22}(t, s)v(s)ds \\ & + \int_0^{t-\tau} \hat{K}_{21}(t, s)u(s)ds + \int_0^{t-\tau} \hat{K}_{22}(t, s)v(s)ds, \quad t \in X_N, \end{cases} \tag{3.1}$$

with

$$u(t) = \phi(t), \quad v(t) = \varphi(t), \quad t \in [-\tau, 0).$$

If $t = t_{n,j}$ is such that $t_{n,j} - \tau = t_{n-r,j} < 0$, the values of u, v are determined by the given initial functions. On each subinterval σ_n, the approximations u, v are the polynomials of degree $m-1$ and can be expressed in the form

$$u(t_n + \rho h) = \sum_{j=1}^{m} U_{n,j} L_j(\rho), \tag{3.2}$$

$$v(t_n + \rho h) = \sum_{j=1}^{m} V_{n,j} L_j(\rho), \tag{3.3}$$

where $U_{n,j} = u(t_n + c_j h), V_{n,j} = v(t_n + c_j h)$ and $L_j(\rho)$ represents the Lagrange canonical polynomials for the collocation parameters $\{c_j\}$. Let us apply the change of variable $\rho = (s - t_i)/h, \ \ (i = 0, ..., n)$, and insert (3.2), (3.3) into system (3.1). There are two cases that we deal with separately.

Case I: If $n - r < 0$, then

$$\begin{aligned} U_{n,j} = \quad & f(t_{n,j}) + h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \Big(\int_0^1 K_{11}(t_{n,j}, t_i + \rho h) U_{i,k} L_k(\rho) d\rho \Big) \\ & + h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \Big(\int_0^1 K_{12}(t_{n,j}, t_i + \rho h) V_{i,k} L_k(\rho) d\rho \Big) \\ & + h \sum_{k=1}^{m} \Big(\int_0^{c_j} K_{11}(t_{n,j}, t_n + \rho h) U_{n,k} L_k(\rho) d\rho \Big) \\ & + h \sum_{k=1}^{m} \Big(\int_0^{c_j} K_{12}(t_{n,j}, t_n + \rho h) V_{n,k} L_k(\rho) d\rho \Big) \\ & - h \sum_{i=n-r+1}^{-1} \Big(\int_0^1 \hat{K}_{11}(t_{n,j}, t_i + \rho h) \phi(t_i + \rho h) d\rho \Big) \\ & - h \sum_{i=n-r+1}^{-1} \Big(\int_0^1 \hat{K}_{12}(t_{n,j}, t_i + \rho h) \varphi(t_i + \rho h) d\rho \Big) \\ & - h \int_{c_j}^1 \hat{K}_{11}(t_{n,j}, t_{n-r} + \rho h) \phi(t_{n-r} + \rho h) d\rho \\ & - h \int_{c_j}^1 \hat{K}_{12}(t_{n,j}, t_{n-r} + \rho h) \varphi(t_{n-r} + \rho h) d\rho, \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
0 = \quad & g(t_{n,j}) + h\sum_{i=0}^{n-1}\sum_{k=1}^{m}\Big(\int_0^1 K_{21}(t_{n,j},t_i+\rho h)U_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{i=0}^{n-1}\sum_{k=1}^{m}\Big(\int_0^1 K_{22}(t_{n,j},t_i+\rho h)V_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} K_{21}(t_{n,j},t_n+\rho h)U_{n,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} K_{22}(t_{n,j},t_n+\rho h)V_{n,k}L_k(\rho)d\rho\Big) \\
& - h\sum_{i=n-r+1}^{-1}\Big(\int_0^1 \hat{K}_{21}(t_{n,j},t_i+\rho h)\phi(t_i+\rho h)d\rho\Big) \\
& - h\sum_{i=n-r+1}^{-1}\Big(\int_0^1 \hat{K}_{22}(t_{n,j},t_i+\rho h)\varphi(t_i+\rho h)d\rho\Big) \\
& - h\int_{c_j}^1 \hat{K}_{21}(t_{n,j},t_{n-r}+\rho h)\phi(t_{n-r}+\rho h)d\rho \\
& - h\int_{c_j}^1 \hat{K}_{22}(t_{n,j},t_{n-r}+\rho h)\varphi(t_{n-r}+\rho h)d\rho.
\end{aligned}
\tag{3.5}
$$

Case II: If $n-r \geq 0$, then

$$
\begin{aligned}
U_{n,j} = \quad & f(t_{n,j}) + h\sum_{i=0}^{n-1}\sum_{k=1}^{m}\Big(\int_0^1 K_{11}(t_{n,j},t_i+\rho h)U_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{i=0}^{n-1}\sum_{k=1}^{m}\Big(\int_0^1 K_{12}(t_{n,j},t_i+\rho h)V_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} K_{11}(t_{n,j},t_n+\rho h)U_{n,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} K_{12}(t_{n,j},t_n+\rho h)V_{n,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{i=0}^{n-r-1}\sum_{k=1}^{m}\Big(\int_0^1 \hat{K}_{11}(t_{n,j},t_i+\rho h)U_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{i=0}^{n-r-1}\sum_{k=1}^{m}\Big(\int_0^1 \hat{K}_{12}(t_{n,j},t_i+\rho h)V_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} \hat{K}_{11}(t_{n,j},t_{n-r}+\rho h)U_{n-r,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} \hat{K}_{12}(t_{nj},t_{n-r}+\rho h)V_{n-r,k}L_k(\rho)d\rho\Big),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
0 = \quad & g(t_{n,j}) + h\sum_{i=0}^{n-1}\sum_{k=1}^{m}\Big(\int_0^1 K_{21}(t_{n,j}, t_i+\rho h)U_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{i=0}^{n-1}\sum_{k=1}^{m}\Big(\int_0^1 K_{22}(t_{n,j}, t_i+\rho h)V_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} K_{21}(t_{n,j}, t_n+\rho h)U_{n,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} K_{22}(t_{n,j}, t_n+\rho h)V_{n,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{i=0}^{n-r-1}\sum_{k=1}^{m}\Big(\int_0^1 \hat{K}_{21}(t_{n,j}, t_i+\rho h)U_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{i=0}^{n-r-1}\sum_{k=1}^{m}\Big(\int_0^1 \hat{K}_{22}(t_{n,j}, t_i+\rho h)V_{i,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} \hat{K}_{21}(t_{n,j}, t_{n-r}+\rho h)U_{n-r,k}L_k(\rho)d\rho\Big) \\
& + h\sum_{k=1}^{m}\Big(\int_0^{c_j} \hat{K}_{22}(t_{n,j}, t_{n-r}+\rho h)V_{n-r,k}L_k(\rho)d\rho\Big).
\end{aligned} \tag{3.7}
$$

Approximating the integrals in the obtained system by using appropriate quadrature rules

$$
\int_0^{c_i} w(s) \approx \sum_{j=1}^{m} a_{ij}w(c_j), \quad \int_0^1 w(s) \approx \sum_{j=1}^{m} b_j w(c_j), \quad \int_{c_i}^1 w(s) \approx \sum_{j=1}^{m} \tilde{a}_{ij}w(c_j)
$$

where $\tilde{a}_{ij} = \int_{c_i}^1 L_j(\rho)d\rho,\ \ a_{ij} = \int_0^{c_i} L_j(\rho)d\rho,\ \ b_j = \int_0^1 L_j(\rho)d\rho$. For n such that $n-r<0$, we get

$$
\left\{
\begin{aligned}
\overline{\mathbf{U}}_n = \quad & \overline{\mathbf{f}}_n + \overline{\mathbf{K}}_{11}\overline{\mathbf{U}}_n + \overline{\mathbf{K}}_{12}\overline{\mathbf{V}}_n + \sum_{i=0}^{n-1}\left(\overline{\mathbf{K}}_{11i}\overline{\mathbf{U}}_i + \overline{\mathbf{K}}_{12i}\overline{\mathbf{V}}_i\right) \\
& - \sum_{i=n-r+1}^{-1}\left(\hat{\mathbf{K}}_{11i}\overline{\phi}_i + \hat{\mathbf{K}}_{12i}\overline{\varphi}_i\right) - \tilde{\mathbf{K}}_{11}\overline{\phi}_{n-r} - \tilde{\mathbf{K}}_{12}\overline{\varphi}_{n-r}, \\
0 = \quad & \overline{\mathbf{g}}_n + \overline{\mathbf{K}}_{21}\overline{\mathbf{U}}_n + \overline{\mathbf{K}}_{22}\overline{\mathbf{V}}_n + \sum_{i=0}^{n-1}\left(\overline{\mathbf{K}}_{21i}\overline{\mathbf{U}}_i + \overline{\mathbf{K}}_{22i}\overline{\mathbf{V}}_i\right) \\
& - \sum_{i=n-r+1}^{-1}\left(\hat{\mathbf{K}}_{21i}\overline{\phi}_i + \hat{\mathbf{K}}_{22i}\overline{\varphi}_i\right) - \tilde{\mathbf{K}}_{21}\overline{\phi}_{n-r} - \tilde{\mathbf{K}}_{22}\overline{\varphi}_{n-r},
\end{aligned}
\right. \tag{3.8}
$$

and for n such that $n-r \geq 0$, we have

$$
\left\{
\begin{aligned}
\overline{\mathbf{U}}_n = \quad & \overline{\mathbf{f}}_n + \overline{\mathbf{K}}_{11}\overline{\mathbf{U}}_n + \overline{\mathbf{K}}_{12}\overline{\mathbf{V}}_n + \sum_{i=0}^{n-1}\left(\overline{\mathbf{K}}_{11i}\overline{\mathbf{U}}_i + \overline{\mathbf{K}}_{12i}\overline{\mathbf{V}}_i\right) \\
& + \sum_{i=0}^{n-r-1}\left(\hat{\mathbf{K}}_{11i}\overline{\mathbf{U}}_i + \hat{\mathbf{K}}_{12i}\overline{\mathbf{V}}_i\right) + \hat{\mathbf{K}}_{11}\overline{\mathbf{U}}_{n-r} + \hat{\mathbf{K}}_{12}\overline{\mathbf{V}}_{n-r}, \\
0 = \quad & \overline{\mathbf{g}}_n + \overline{\mathbf{K}}_{21}\overline{\mathbf{U}}_n + \overline{\mathbf{K}}_{22}\overline{\mathbf{V}}_n + \sum_{i=0}^{n-1}\left(\overline{\mathbf{K}}_{21i}\overline{\mathbf{U}}_i + \overline{\mathbf{K}}_{22i}\overline{\mathbf{V}}_i\right) \\
& + \sum_{i=0}^{n-r-1}\left(\hat{\mathbf{K}}_{21i}\overline{\mathbf{U}}_i + \hat{\mathbf{K}}_{22i}\overline{\mathbf{V}}_i\right) + \hat{\mathbf{K}}_{21}\overline{\mathbf{U}}_{n-r} + \hat{\mathbf{K}}_{22}\overline{\mathbf{V}}_{n-r},
\end{aligned}
\right. \tag{3.9}
$$

where

$$\overline{\phi}_i = \Big(\phi(t_i + c_1 h), \ldots, \phi(t_i + c_m h)\Big)^T, \overline{\varphi}_i = \Big(\varphi(t_i + c_1 h), \ldots, \varphi(t_i + c_m h)\Big)^T,$$

$$\overline{\mathbf{U}}_i = \Big(U_{i,1}, \ldots, U_{i,m}\Big)^T, \overline{\mathbf{V}}_i = \Big(V_{i,1}, \ldots, V_{i,m}\Big)^T,$$

and

$$\begin{array}{llll}
\overline{\mathbf{f}}_n = & \Big(f(t_{n,1}), \ldots, f(t_{n,m})\Big)^T, & \overline{\mathbf{g}}_n = & \Big(g(t_{n,1}), \ldots, g(t_{n,m})\Big)^T, \\
\overline{\mathbf{K}}_{pq} = & \{hK_{pq}(t_{n,i}, t_{n,j})a_{ij}\}_{i,j=1}^m, & \overline{\mathbf{K}}_{pqi} = & \{hb_l K_{pq}(t_{n,k}, t_{i,l})\}_{k,l=1}^m, \\
\hat{\mathbf{K}}_{pq} = & \{h\hat{K}_{pq}(t_{n,i}, t_{n-r,j})a_{ij}\}_{i,j=1}^m, & \hat{\mathbf{K}}_{pqi} = & \{hb_l \hat{K}_{pq}(t_{n,k}, t_{i,l})\}_{k,l=1}^m, \\
\tilde{\mathbf{K}}_{pq} = & \{h\hat{K}_{pq}(t_{n,i}, t_{n-r,j})\tilde{a}_{ij}\}_{i,j=1}^m, & p, q = 1, 2, &
\end{array}$$

The collocation approximations (3.2) and (3.3) are obtained by solving the linear systems (3.8) and (3.9) on each subinterval $\sigma_n,\ n = 0, \cdots, N-1$.

4. Convergence

In this section, based on the interpolation error, we analyze the collocation error, and deduce the global convergence result below.

Theorem 4.1. *Assume that the given functions in* (1.1) *for* $D = \{(t,s) : 0 \le s \le t \le T\}$, *and* $D_\tau = I \times [-\tau, T-\tau]$ *satisfy* $f \in C^m(I), \phi, \varphi \in C^m([-\tau, 0]), K_{11}, K_{12} \in C^m(D), \hat{K}_{11}, \hat{K}_{12} \in C^m(D_\tau)$, $g \in C^{m+1}(I), K_{21}, K_{22} \in C^{m+1}(D), \hat{K}_{21}, \hat{K}_{22} \in C^{m+1}(D_\tau)$ *and* $|K_{22}(t,t)| \ge k_0 > 0$, $\forall t \in I$. *Let* $(u, v) \in S_{m-1}^{-1}(\Omega_N)$ *be the collocation approximation of the solution* (y, z) *in equation* (1.1) *which is defined by* (3.2) *and* (3.3). *If* $(0 < c_m \le 1)$, *the collocation approximation* u *converges to the solution* y *for* $-1 \le \lambda \le 1$ *and the following order of convergence holds:*

$$||y - u||_\infty = \mathcal{O}(h^m).$$

If $c_m = 1$, *the collocation approximation* v *converges to the solution* z, *and if* $c_m < 1$, *the collocation approximation* v *converges to the solution* z *for any* $m \ge 1$ *if and only if*

$$-1 \le \lambda = (-1)^m \prod_{i=1}^m \frac{1 - c_i}{c_i} \le 1.$$

Furthermore, the following order of convergence holds:

$$||z - v||_\infty = \left\{ \begin{array}{ll} \mathcal{O}(h^m), & \textit{if} \quad \lambda \in [-1, 1), \\ \mathcal{O}(h^{m-1}), & \textit{if} \quad \lambda = 1, \end{array} \right. \tag{4.1}$$

as $h \to 0$ *with* $Nh \le const$.

Proof. The exact solutions y and z satisfy

$$\begin{aligned}
y(t_n + \rho h) = & \sum_{j=1}^m L_j(\rho) Y_{n,j} + r_n(\rho), \quad r_n(\rho) = h^m \tfrac{y^{(m)}\eta_n(\rho)}{m!} \prod_{i=1}^m (\rho - c_i), \\
z(t_n + \rho h) = & \sum_{j=1}^m L_j(\rho) Z_{n,j} + s_n(\rho), \quad s_n(\rho) = h^m \tfrac{z^{(m)}\varsigma_n(\rho)}{m!} \prod_{i=1}^m (\rho - c_i),
\end{aligned} \tag{4.2}$$

where $Y_{n,j} = y(t_n + c_j h)$ and $Z_{n,j} = z(t_n + c_j h)$. It follows that the errors $e = y - u$ and $\epsilon = z - v$ have the representation

$$e_n(t_n + \rho h) = \sum_{j=1}^m L_j(\rho) e_n(t_{n,j}) + \mathcal{O}(h^m), \tag{4.3}$$

$$\epsilon_n(t_n + \rho h) = \sum_{j=1}^{m} L_j(\rho)\epsilon_n(t_{n,j}) + \mathcal{O}(h^m). \tag{4.4}$$

where $e_n = e|_{\sigma_n}$ and $\epsilon_n = \epsilon|_{\sigma_n}$. The errors also satisfy

$$\begin{cases} e_n(t_{n,j}) = & \int_0^{t_{n,j}} K_{11}(t_{n,j}, s)e_n(s)ds + \int_0^{t_{n,j}} K_{12}(t_{n,j}, s)\epsilon_n(s)ds \\ & + \int_0^{t_{n,j}-\tau} \hat{K}_{11}(t_{n,j}, s)e_n(s)ds + \int_0^{t_{n,j}-\tau} \hat{K}_{12}(t_{n,j}, s)\epsilon_n(s)ds, \\ 0 = & \int_0^{t_{n,j}} K_{21}(t_{n,j}, s)e_n(s)ds + \int_0^{t_{n,j}} K_{22}(t_{n,j}, s)\epsilon_n(s)ds, \\ & + \int_0^{t_{n,j}-\tau} \hat{K}_{21}(t_{n,j}, s)e_n(s)ds + \int_0^{t_{n,j}-\tau} \hat{K}_{22}(t_{n,j}, s)\epsilon_n(s)ds. \end{cases} \tag{4.5}$$

If $t_{n,j} - \tau < 0$, then

$$\begin{cases} e_n(t_{n,j}) = & h\sum_{l=0}^{n-1} \int_0^1 (K_{11}(t_{n,j}, t_l + sh)e_l(t_l + sh) + K_{12}(t_{n,j}, t_l + sh)\epsilon_l(t_l + sh))ds \\ & +h\int_0^{c_j} (K_{11}(t_{n,j}, t_n + sh)e_n(t_n + sh) + K_{12}(t_{n,j}, t_n + sh)\epsilon_n(t_n + sh))ds, \\ 0 = & h\sum_{l=0}^{n-1} \int_0^1 (K_{21}(t_{n,j}, t_l + sh)e_l(t_l + sh) + K_{22}(t_{n,j}, t_l + sh)\epsilon_l(t_l + sh))ds \\ & +h\int_0^{c_j} (K_{21}(t_{n,j}, t_n + sh)e_n(t_n + sh) + K_{22}(t_{n,j}, t_n + sh)\epsilon_n(t_n + sh))ds. \end{cases} \tag{4.6}$$

Considering (4.6) and using a similar procedure as outlined in [18] (see section 3 of [18]), we can obtain the estimates of the error stated in the theorem. Now let $t_{n,j} - \tau \geq 0$. Then from (4.5), we have

$$\begin{aligned} e_n(t_{n,j}) = & h\sum_{l=0}^{n-1} \int_0^1 (K_{11}(t_{n,j}, t_l + sh)e_l(t_l + sh) + K_{12}(t_{n,j}, t_l + sh)\epsilon_l(t_l + sh))ds \\ & +h\int_0^{c_j} (K_{11}(t_{n,j}, t_n + sh)e_n(t_n + sh) + K_{12}(t_{n,j}, t_n + sh)\epsilon_n(t_n + sh))ds \\ & +h\sum_{l=0}^{n-r-1} \int_0^1 (\hat{K}_{11}(t_{n,j}, t_l + sh)e_l(t_l + sh) + \hat{K}_{12}(t_{n,j}, t_l + sh)\epsilon_l(t_l + sh))ds \\ & +h\int_0^{c_j} (\hat{K}_{11}(t_{n,j}, t_{n-r} + sh)e_{n-r}(t_{n-r} + sh) + \hat{K}_{12}(t_{n,j}, t_{n-r} + sh)\epsilon_{n-r}(t_{n-r} + sh))ds, \end{aligned} \tag{4.7}$$

$$\begin{aligned} 0 = & h\sum_{l=0}^{n-1} \int_0^1 (K_{21}(t_{n,j}, t_l + sh)e_l(t_l + sh) + K_{22}(t_{n,j}, t_l + sh)\epsilon_l(t_l + sh))ds \\ & +h\int_0^{c_j} (K_{21}(t_{n,j}, t_n + sh)e_n(t_n + sh) + K_{22}(t_{n,j}, t_n + sh)\epsilon_n(t_n + sh))ds, \\ & +h\sum_{l=0}^{n-r-1} \int_0^1 (\hat{K}_{21}(t_{n,j}, t_l + sh)e_l(t_l + sh) + \hat{K}_{22}(t_{n,j}, t_l + sh)\epsilon_l(t_l + sh))ds \\ & +h\int_0^{c_j} (\hat{K}_{21}(t_{n,j}, t_{n-r} + sh)e_{n-r}(t_{n-r} + sh) + \hat{K}_{22}(t_{n,j}, t_{n-r} + sh)\epsilon_{n-r}(t_{n-r} + sh))ds. \end{aligned} \tag{4.8}$$

We now rewrite (4.8) with n replaced by $n-1$ and $j=m$, subtract this equation from (4.8) and divide by h, to obtain

$$\begin{aligned}
&\int_0^{c_j} (K_{21}(t_{n,j}, t_n+sh)e_n(t_n+sh) + K_{22}(t_{n,j}, t_n+sh)\epsilon_n(t_n+sh))ds = \\
&\int_0^{c_m} (K_{21}(t_{n-1,m}, t_{n-1}+sh)e_{n-1}(t_{n-1}+sh) + K_{22}(t_{n-1,m}, t_{n-1}+sh)\epsilon_{n-1}(t_{n-1}+sh))ds \\
&-\sum_{l=0}^{n-1}\int_0^1 (K_{21}(t_{n,j}, t_l+sh)e_l(t_l+sh) + K_{22}(t_{n,j}, t_l+sh)\epsilon_l(t_l+sh))ds \\
&+\sum_{l=0}^{n-2}\int_0^1 (K_{21}(t_{n-1,m}, t_l+sh)e_l(t_l+sh) + K_{22}(t_{n-1,m}, t_l+sh)\epsilon_l(t_l+sh))ds \\
&-\sum_{l=0}^{n-r-1}\int_0^1 (\hat{K}_{21}(t_{n,j}, t_l+sh)e_l(t_l+sh) + \hat{K}_{22}(t_{n,j}, t_l+sh)\epsilon_l(t_l+sh))ds \\
&+\sum_{l=0}^{n-r-2}\int_0^1 (\hat{K}_{21}(t_{n-1,m}, t_l+sh)e_l(t_l+sh) + \hat{K}_{22}(t_{n-1,m}, t_l+sh)\epsilon_l(t_l+sh))ds \\
&-\int_0^{c_j} (\hat{K}_{21}(t_{n,j}, t_{n-r}+sh)e_{n-r}(t_{n-r}+sh) + \hat{K}_{22}(t_{n,j}, t_{n-r}+sh)\epsilon_{n-r}(t_{n-r}+sh))ds \\
&+\int_0^{c_m} (\hat{K}_{21}(t_{n-1,m}, t_{n-1-r}+sh)e_{n-1-r}(t_{n-1-r}+sh) \\
&+\hat{K}_{22}(t_{n-1,m}, t_{n-1-r}+sh)\epsilon_{n-1-r}(t_{n-1-r}+sh))ds.
\end{aligned} \tag{4.9}$$

Now, without loss of generality, we consider the following two cases:

Case I. If $c_m = 1$, then for $j = 1, \cdots, m$

$$\begin{aligned}
&K_{pq}(t_{n,j}, t_l+sh) - K_{pq}(t_{n-1,m}, t_l+sh) \\
&= c_j h K_{pq,t}(t_n, t_l+sh) + (1-c_m)hK_{pq,t}(t_n, t_l+sh) + \mathcal{O}(h), \\
&\hat{K}_{pq}(t_{n,j}, t_l+sh) - \hat{K}_{pq}(t_{n-1,m}, t_l+sh) \\
&= c_j h \hat{K}_{pq,t}(t_n, t_l+sh) + (1-c_m)h\hat{K}_{pq,t}(t_n, t_l+sh) + \mathcal{O}(h), \quad p,q=1,2,
\end{aligned} \tag{4.10}$$

where $K_{pq,t}(,) = \dfrac{\partial K_{pq}}{\partial t}$ and the unspecified first arguments in the partial derivatives of $K_{pq}, p,q = 1,2$, are those arising in the Taylor's remainder terms. Using (4.10) and inserting (4.3), (4.4) into equations (4.7), (4.9), the following linear system can be derived

$$\mathbf{A}^{(n,n)}\mathbf{E}_n = h\sum_{l=0}^{n-1}\mathbf{B}^{(n,l)}\mathbf{E}_l + h\sum_{l=0}^{n-r-1}\mathbf{C}^{(n,l)}\mathbf{E}_l + \mathbf{D}^{(n,n-r)}\mathbf{E}_{n-r} + \mathcal{O}(h^m), \tag{4.11}$$

where $\mathbf{E}_n = \begin{pmatrix} e_n \\ \epsilon_n \end{pmatrix}$, $e_n = (e(t_{n1}), \ldots, e(t_{nm}))^T$, and $\epsilon_n = (\epsilon(t_{n1}), \ldots, \epsilon(t_{nm}))^T$, and

$$\mathbf{A}^{(n,n)} = \begin{pmatrix} I - hK_{11}^{(n,n)} & -hK_{12}^{(n,n)} \\ K_{21}^{(n,n)} & K_{22}^{(n,n)} \end{pmatrix},$$

$$\mathbf{B}^{(n,l)} = \begin{pmatrix} B_{11}^{(n,l)} & B_{12}^{(n,l)} \\ \tilde{B}_{21}^{(n,l)} & \tilde{B}_{22}^{(n,l)} \end{pmatrix}, \quad \mathbf{C}^{(n,l)} = \begin{pmatrix} C_{11}^{(n,l)} & C_{12}^{(n,l)} \\ \tilde{C}_{21}^{(n,l)} & \tilde{C}_{22}^{(n,l)} \end{pmatrix},$$

$$\mathbf{D}^{(n,n-r)} = \begin{pmatrix} h\hat{K}_{11}^{(n,n-r)} & h\hat{K}_{12}^{(n,n-r)} \\ -\hat{K}_{21}^{(n,n-r)} & -\hat{K}_{22}^{(n,n-r)} \end{pmatrix},$$

such that for $p,q = 1,2$

$$K_{pq}^{(n,n)} = \begin{pmatrix} \displaystyle\int_0^{c_j} K_{pq}(t_{nj}, t_n+\rho h)L_k(\rho)d\rho \\ j,k = 1,\cdots,m \end{pmatrix},$$

$$\hat{K}_{pq}^{(n,n-r)} = \begin{pmatrix} \int_0^{c_j} \hat{K}_{pq}(t_{nj}, t_{n-r} + \rho h) L_k(\rho) d\rho \\ j,k = 1, \cdots, m \end{pmatrix},$$

$$B_{pq}^{(n,l)} = \begin{pmatrix} \int_0^{1} K_{pq}(t_{nj}, t_l + \rho h) L_k(\rho) d\rho \\ j,k = 1, \cdots, m \end{pmatrix},$$

$$\tilde{B}_{pq}^{(n,l)} = \begin{pmatrix} \int_0^{1} c_j \frac{\partial K_{pq}}{\partial t}(t_n, t_l + \rho h) L_k(\rho) d\rho \\ j,k = 1, \cdots, m \end{pmatrix},$$

$$C_{pq}^{(n,l)} = \begin{pmatrix} \int_0^{1} \hat{K}_{pq}(t_{nj}, t_l + \rho h) L_k(\rho) d\rho \\ j,k = 1, \cdots, m \end{pmatrix},$$

$$\tilde{C}_{pq}^{(n,l)} = \begin{pmatrix} \int_0^{1} c_j \frac{\partial \hat{K}_{pq}}{\partial t}(t_n, t_l + \rho h) L_k(\rho) d\rho \\ j,k = 1, \cdots, m \end{pmatrix}.$$

Since $|(K_{22}(t,t))| \geq k_0 > 0$, the inverse of the matrix $\mathbf{A}^{(n,n)}$ exists and is bounded if h is sufficiently small. It then follows from (4.3), (4.4) and Gronwall's inequality that

$$\|e_n\|_\infty = \mathcal{O}(h^m), \qquad \|\epsilon_n\|_\infty = \mathcal{O}(h^m).$$

Case II. $c_m < 1$

In order to describe the key ideas without having to resort to complex notation, we can assume that $K_{22}(t,s) = 1$ or we can employ the Taylor series expansion K_{22} as:

$$K_{22}(t_{n,i}, t_l + sh) = K_{22}(t_n, t_l) + O(h), \ (l = 0, \cdots, n).$$

Using (4.10) and inserting (4.3), (4.4) into the equations (4.7), (4.9), we have

$$\begin{aligned} \hat{\mathbf{A}}^{(n,n)} \mathbf{E}_n &= \hat{\mathbf{B}}^{(n,n-1)} \mathbf{E}_{n-1} + h \sum_{l=0}^{n-2} \tilde{\mathbf{B}}^{(n,l)} \mathbf{E}_l + h \sum_{l=0}^{n-r-2} \tilde{\mathbf{C}}^{(n,l)} \mathbf{E}_l \\ &+ \hat{\mathbf{C}}^{(n,n-r-1)} \mathbf{E}_{n-r-1} + \mathbf{D}^{(n,n-r)} \mathbf{E}_{n-r} + O(h^m), \end{aligned} \tag{4.12}$$

where

$$\hat{\mathbf{A}}^{(n,n)} = \begin{pmatrix} I - hK_{11}^{(n,n)} & -hK_{12}^{(n,n)} \\ K_{21}^{(n,n)} & P \end{pmatrix},$$

$$\hat{\mathbf{B}}^{(n,n-1)} = \begin{pmatrix} hK_{11}^{(n,n-1)} & hK_{12}^{(n,n-1)} \\ K_{21}(t_{n-1}, t_{n-1}) S + O(h) & Q \end{pmatrix},$$

$$\tilde{\mathbf{B}}^{(n,l)} = \begin{pmatrix} B_{11}^{(n,l)} & B_{12}^{(n,l)} \\ \breve{B}_{21}^{(n,l)} & \breve{B}_{22}^{(n,l)} \end{pmatrix}, \ \tilde{\mathbf{C}}^{(n,l)} = \begin{pmatrix} C_{11}^{(n,l)} & C_{12}^{(n,l)} \\ \breve{C}_{21}^{(n,l)} & \breve{C}_{22}^{(n,l)} \end{pmatrix},$$

$$\hat{\mathbf{C}}^{(n,n-r-1)} = \begin{pmatrix} C_{11}^{(n,n-r-1)} & C_{12}^{(n,n-r-1)} \\ \hat{K}_{21}(t_{n-1}, t_{n-1}) S + O(h) & \hat{K}_{22}(t_{n-1}, t_{n-1}) S + O(h) \end{pmatrix},$$

with $S = \Gamma_m \Upsilon_m^T P - \Gamma_m \mathbf{b}$,

$$\Gamma_m = (1, 1, \cdots, 1)^T, \ \ \Upsilon_m = (0, 0, \cdots, 1)^T, \ \mathbf{b} = (b_1, b_2, \cdots, b_m)^T$$

$$P = \begin{pmatrix} \int_0^{c_i} L_j(s) ds \\ i,j = 1, \cdots, m \end{pmatrix}, \qquad Q = \begin{pmatrix} -\int_{c_m}^{1} L_j(s) ds \\ i,j = 1, \cdots, m \end{pmatrix},$$

$$\breve{B}_{pq}^{(n,l)} = \begin{pmatrix} \int_0^{1} (c_j + (1 - c_m)) \frac{\partial K_{pq}}{\partial t}(t_n, t_l + \rho h) L_k(\rho) d\rho \\ j,k = 1, \cdots, m \end{pmatrix},$$

$$\breve{C}_{pq}^{(n,l)} = \left(\begin{array}{c} \int_0^1 (c_j + (1-c_m))\frac{\partial \hat{K}_{pq}}{\partial t}(t_n, t_l + \rho h) L_k(\rho) d\rho \\ j,k = 1,\cdots,m \end{array} \right).$$

It can be verified that the inverse of the matrix $\hat{\mathbf{A}}^{(n,n)}$ has the form

$$(\hat{\mathbf{A}}^{(n,n)})^{-1} = \left(\begin{array}{cc} I + \mathcal{O}(h) & \mathcal{O}(h) \\ \breve{K}_{21}^{(n,n)} & P^{-1} \end{array} \right),$$

provided h is sufficiently small in which case we also have

$$(\hat{\mathbf{A}}^{(n,n)})^{-1}\hat{\mathbf{B}}^{(n,n-1)} = \left(\begin{array}{cc} \mathcal{O}(h) & \mathcal{O}(h) \\ P^{-1}S + \mathcal{O}(h) & P^{-1}Q + \mathcal{O}(h) \end{array} \right).$$

According to Lemma 2.4.3 of Brunner [6], we know that $P^{-1}Q$ has a nontrivial eigenvalue as

$$\lambda = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i}. \tag{4.13}$$

Multiplying (4.12) by $(\hat{\mathbf{A}}^{(n,n)})^{-1}$, using the elementary theory of the difference equations [13] and considering the nontrivial eigenvalue of $P^{-1}Q$ by (4.13), we can conclude with the assertion (4.1) following the steps in [14, 18] with the help of Lemma 6 of [15]. □

5. Numerical Results

In this section, we illustrate the theoretical results obtained in the previous section by the following two examples with $\tau = \frac{1}{2}$. All computations are performed with the Mathematica® software.

Example 5.1. Consider the mixed system of first and second-kind Volterra integral equations with constant delay given by

$$\begin{cases} y(t) = & f(t) + \int_0^t e^{s-t} y(s) ds + \int_0^t (t+s) z(s) ds \\ & + \int_0^{t-\frac{1}{2}} t \sin s y(s) ds + \int_0^{t-\frac{1}{2}} ts z(s) ds, \\ 0 = & g(t) + \int_0^t e^{s+t} y(s) ds + \int_0^t (s + t^2 + 1) z(s) ds \\ & + \int_0^{t-\frac{1}{2}} \sin s y(s) ds + \int_0^{t-\frac{1}{2}} (ts+3) z(s) ds, \qquad t \in [0,1], \\ y(t) = & \sin t + 1, \qquad z(t) = \cos t, \qquad t \in [-\frac{1}{2}, 0), \end{cases} \tag{5.1}$$

where $f(t)$ and $g(t)$ such that the exact solution is:

$$y(t) = \sin t + 1, \quad z(t) = \cos t.$$

Let $(u,v) \in S_{m-1}^{-1}(\Omega_N)$ be the collocation approximation of the solution (y,z) for the equation in (5.1) which is defined by (3.2) and (3.3). Gauss points (i.e., the zeros of $P_m(2s-1)$ in which P_m denotes the Legendre polynomial of degree m) are chosen as collocation parameters. Orders of convergence from the maximum errors at the grid points have been reported in Table 1 which confirm the theoretical results of Theorem 4.1. The error behaviors related to the spline collocation method for the different values of m and N in Examples 1 are shown in Figures 1 and 2.

Remark 5.2. Note that for the Gauss points as collocation parameters, we have $c_m < 1$ and $\lambda = 1$ (if m is even), $\lambda = -1$ (if m is odd). Also, the order of convergence p defined as follows

$$p := \log_2\Big(\frac{\|e_N\|_\infty}{\|e_{2N}\|_\infty}\Big).$$

Example 5.3. Consider the mixed system with constant delay given by

$$\begin{cases} AX(t) = & F(t) + \displaystyle\int_0^t K(t,s)X(s)ds + \int_0^{t-\frac{1}{2}} \hat{K}(t,s)X(s)ds, \quad t \in [0,1], \\ X(t) = & (e^{-t}(t+1), \cos(t+1), e^t)^T, \qquad\qquad\qquad\qquad t \in [-\frac{1}{2}, 0), \end{cases} \tag{5.2}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K(t,s) = \begin{pmatrix} s-t & t+s+2 & s+t^2 \\ s+t & \sin(t+1) & s+t \\ 1+t^2 & \cos t & e^{s+t} \end{pmatrix},$$

$$\hat{K}(t,s) = \begin{pmatrix} s^2t & t^3+1 & 1+t \\ t^2s & (t+1) & s+t+1 \\ te^s & \cos s & e^{s+t+2} \end{pmatrix}, \quad X(t) = \big(\ x(t), y(t), z(t)\ \big)^T,$$

$F(t) = (f(t), g(t), h(t))^T$ such that the exact solution is:

$$x(t) = e^{-t}(t+1), \quad y(t) = \cos(t+1), \quad z(t) = e^t.$$

Let u_1, u_2, v be the approximation of the exact solutions x, y, z, respectively. The spline collocation method has been implemented for system (5.2)and the orders of convergence have been reported in Table 2.

5.1. Figures and Tables

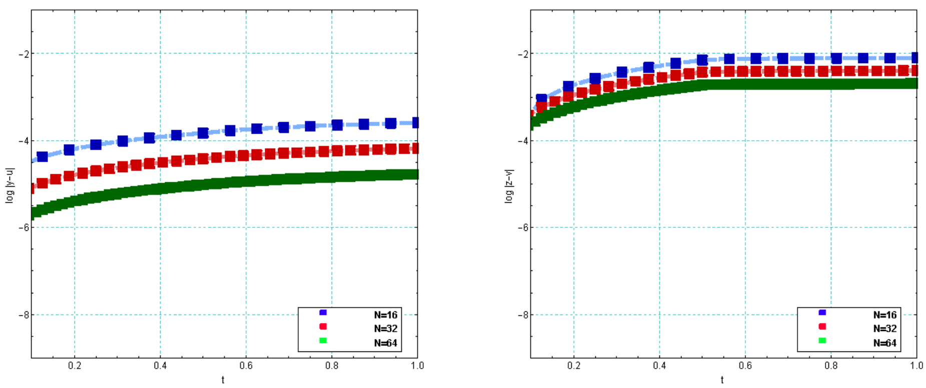

Figure 1. Point-wise absolute errors of y with $m = 2$ in Example 1 (left). Point-wise absolute errors of z with $m = 2$ in Example 1 (right).

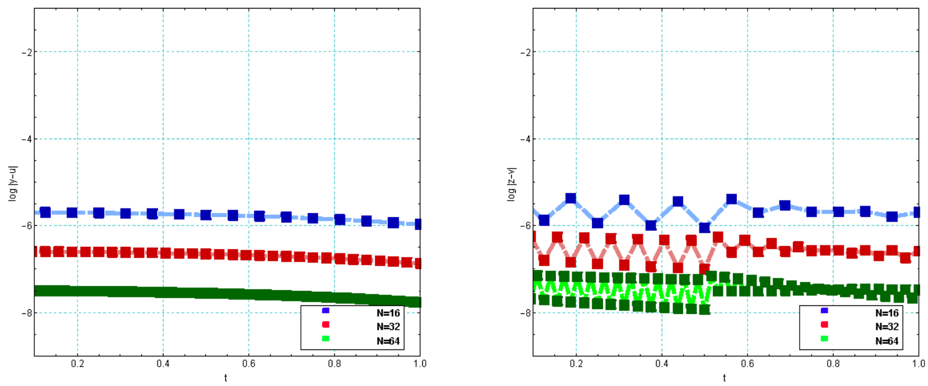

Figure 2. Point-wise absolute errors of y with $m = 3$ in Example 1 (left). Point-wise absolute errors of z with $m = 3$ in Example 1 (right).

	u			v		
m	N=16	N=32	N=64	N=16	N=32	N=64
2	1.91	1.95	1.97	0.94	0.97	0.98
3	2.97	2.98	2.99	2.94	2.97	2.98

Table 1. Orders of convergence of u and v in Example 1.

	u_1			u_2			v		
m	N=16	N=32	N=64	N=16	N=32	N=64	N=16	N=32	N =64
2	2.68	2.53	2.03	1.95	1.98	1.99	0.86	0.93	0.97
3	3.76	3.46	3.03	3.15	3.09	3.04	2.97	2.98	2.99

Table 2. Order of convergence of u_1, u_2 and v in Example 2.

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References

[1] I. Ali, H. Brunner and T. Tang, *Spectral methods for pantograph-type differential and integral equations with multiple delays* Front. Math. China **4**, 49-61, 2009.

[2] J. Belair, *Population models with state-dependent delays.* In: Arino, O., Axelrod, D.E., Kimmel, M. (eds.) Mathematical Population Dynamics, 165-176. Marcel Dekker, New York, 1991.

[3] H. Brunner, *Iterated collocation methods for Volterra integral equations with delay arguments.* Math. Comput. **62**, 581-599, 1994.

[4] H. Brunner, *Collocation and continuous implicit Runge-Kutta methods for a class of delay Volterra integral equations.* J. Comput. Appl. Math. **53**, 61-72, 1994.

[5] H. Brunner, *The discretization of neutral functional integro-differential equations by collocation methods.* J. Anal. Appl. **18**, 393-406, 1999.

[6] H. Brunner, *Collocation methods for Volterra integral and related functional differential equations.* Cambridge university press, Cambridge, 2004.

[7] H. Brunner and Y. Yatsenko, *Spline collocation methods for nonlinear Volterra integral equations with unknown delay.* J. Comput. Appl. Math. **71**, 67-81, 1996.

[8] M.V. Bulatov and M. N. Machkhina, *On a class of integro-algebraic equations with variable integration limits* Zh. Sredn. Mat. Obshch. **12** (2), 40-45, 2010.

[9] M.V. Bulatov, M. N. Machkhina and V.N. Phat, *Existence and uniqueness of solutions to nonlinear integral-algebraic equations with variable limits of integrations* Commun. Appl. Nonlinear Anal. **21** (1), 65-76, 2014

[10] F. Calio, E. Marchetti and R. Pavani, *About the deficient spline collocation method for particular differential and integral equations with delay.* Rend. Sem. Mat. Univ. Pol. Torino, **61**, 287-300, 2003.

[11] F. Calio, E. Marchetti, R. Pavani and G. Micula, *About some Volterra problems solved by a particular spline collocation.* Studia Univ. Babes Bolyai. **48**, 45-52, 2003.

[12] K.L. Cooke,*An epidemic equation with immigration.* Math. Biosci. **29**, 135-158, 1976.

[13] S. N. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, 1999.

[14] F. Ghoreishi, M. Hadizadeh and S. Pishbin , *On the convergence analysis of the spline collocation method for system of integral algebraic equations of index-2*, Int. J. Comput. Methods. **9** (4) , 1250048, 2012.

[15] E. Hairer, Ch. Lubich and S.P. Nrset, *Order of convergence of one-step methods for Volterra integral equations of the second kind*, SIAM. J. Numer. Anal. **20**, 569-579, 1983.

[16] V. Horvat, *On collocation methods for Volterra integral equations with delay arguments.* Math. Commun. **4**, 93-109, 1999.

[17] Q. Hu, *Multilevel correction for discrete collocation solutions of Volterra integral equations with delay arguments.* Appl. Numer. Math. **31**, 159-171, 1999.

[18] J. P. Kauthen, *The numerical solution of integral-algebriac equations of index-1 by polynomial spline collocation methods*, Math. Comp. **236**, 1503-1514, 2000.

[19] M. Khasi, F. Ghoreishi and M. Hadizadeh, *Numerical analysis of a high order method for state-dependent delay integral equations.* Numerical Algorithms, **66**, 177-201, 2013.