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Well-posedness of the 3D Stochastic Generalized Rotating Magnetohydrodynamics Equations

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Abstract

In this paper we treat the 3D stochastic incompressible generalized rotating magnetohydrodynamics equations. By using littlewood-Paley decomposition and Itô integral, we establish the global well-posedness result for small initial data (u_0, b_0) belonging in the critical Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}(\mathbb{R}^3)$. In addition, the proof of local existence is also founded on a priori estimates of the stochastic parabolic equation and the iterative contraction method.

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1. Introduction

We introduce the Stochastic generalized magnetohydrodynamic equations (SGMDC) with the Coriolis force in the whole space $\mathbb{R}^3$

$$\begin{cases} u_t + u\cdot\nabla u + \mu(-\Delta)^\alpha u + Se_3\times u - b\cdot\nabla b + \nabla\pi = f\dot{W} & \text{in } \Omega\times(0,+\infty)\times\mathbb{R}^3,\\ b_t + u\cdot\nabla b + \nu(-\Delta)^\alpha b - b\cdot\nabla u = g\dot{W} & \text{in } \Omega\times(0,+\infty)\times\mathbb{R}^3,\\ \nabla\cdot u = 0, \qquad \nabla\cdot b = 0, & \\ (u,b)|_{t=0} = (u_0, b_0), & \end{cases} \tag{1}$$

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where the unknowns are the random vector fields $u = u(\omega, t, x)$, $b = b(\omega, t, x)$ and the random scalar field $\pi = P + \frac{1}{2}|b|^2$ which respectively denote the fluid velocity, the magnetic field and the total fluid pressure, $\nu > 0$ is the magnetic diffusivity, $\mu > 0$ is the viscosity, $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$ are the incompressible conditions, $S \in \mathbb{R}$ denotes twice the speed of rotation around the vertical vector $e_3 = (0, 0, 1)$, and $f\dot{W}$ and $g\dot{W}$ are random external forces; W is a Wiener process of infinite dimension. The equation (1) is a system of equations which controls the motion of electrically conducting fluids, such as plasma. The equation (1) is understood in the Itô sense. It is constructed by combining the Maxwell and Navier–Stokes equations and serves a real interest in the fields of geophysics, astrophysics, cosmology, plasma physics and several other branches of applied science.

First, we will notice that in the deterministic case, i.e. when $f = g = 0$, the equation (1) becomes the generalized magnetohydrodynamic equations with Coriolis force which is employed to show why the earth has a non-zero large-scale magnetic field whose polarity reverses over several hundred centuries when $\alpha = 1$. For this topic, we invite the reader to consult [2] and the references therein. Let us take the time to briefly mention some recent results in this direction; Wang and Wu [10] realized in the first the global well-posedness and Gevrey class regularity of the solution of the generalized rotating magnetohydrodynamics equations if the initial data are in the Lei-Lin space $\mathcal{X}^{1-2\alpha}$ with $\alpha \in [\frac{1}{2}, 1]$.

For the stochastic case, we refer to [8] for results associated to the existence and uniqueness of the global solution to the stochastic magnetohydrodynamic equation in the framework of Besov spaces when $\alpha = 1$ and $S = 0$. The papers [1] offer several results related to the identification of the regularity of the driving noises and conditions on α for which the existence of a martingale solution of the fractional stochastic magnetohydrodynamic system with $(-\Delta)^\alpha, \alpha > 0$, in $\mathbb{R}^d, d = 2, 3$ when $S = 0$, is proven.

When $b = 0$, $S \neq 0$ and $g = 0$, the system (1) is thus reconciled to the stochastic rotating Navier–Stokes equation ruled by an additive white noise. About this topic, Wang [9] studied the spatial analyticity and uniqueness of the global mild solution, including when a stochastic external force is high and as well as when initial data is essentially arbitrarily large, since the speed of the rotation is fast enough. Wang and Wu [11] investigated Itô integral and Littlewood–Paley theory to guarantee the well-posedness of stochastic Navier–Stokes equations with Coriolis force in Fourier–Besov spaces $\mathrm{F}\dot{\mathrm{B}}^{s}_{p,q}$, and their corresponding results in the deterministic case.

The purpose of this work is to establish the uniform global and local existence of the solution to the 3D Stochastic generalized rotating magnetohydrodynamic equations (1) in the FBM–space (Fourier-Besov-Morrey space) $\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$ with sufficiently small initial data. In fact, this space covers many classical spaces, like some special cases, e.g. the Lei-Lin's space $\mathcal{X}^s = \mathcal{F}\dot{\mathcal{N}}^{s}_{1,0,1}$, the Fourier-Herz space $\dot{\mathcal{B}}^s_q = \mathcal{F}\dot{\mathcal{N}}^{s}_{1,0,q}$ and the Fourier-Besov-Lebesgue space $\mathrm{F}\dot{\mathrm{B}}^{s}_{p,q} = \mathcal{F}\dot{\mathcal{N}}^{s}_{p,0,q}$. Inspired by the results [11, 9], we show the uniform global existence by meaning that the initial data is independent of the speed of rotation S. More precisely, this paper extends the results of existence of the solution of the stochastic rotating Navier–Stokes equation given in [11] in the Fourier-Besov-Lebesgue space $\mathrm{F}\dot{\mathrm{B}}^{1-2\alpha+\frac{3}{p'}}_{p,q}$ to the results of existence of the solution of 3D stochastic generalized rotating magnetohydrodynamic in the Fourier Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$. Throughout this paper, we use FBM–space to designate Fourier-Besov-Morrey space, C is the constant that can be different depending on the place.

Before putting forward our result, we first introduce the corresponding generalized Stokes-Coriolis semi-group. In particular, we study the following linear Stokes problem with the Coriolis force

$$\begin{cases} u_t + \mu(-\Delta)^\alpha u + Se_3 \times u + \nabla\pi = 0, & \text{in } [0, +\infty) \times \mathbb{R}^3, \\ \nabla . u = 0, & \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^3. & \end{cases} \tag{2}$$

The solution of the equation (2) is obtainable from the generalized Stokes-Coriolis semigroup $G_{S,\alpha}$ [7, 12, 10], which is expressed as

$$G_{S,\alpha}(t)u = \mathcal{F}^{-1}[\cos(S\frac{\xi_3}{|\xi|}t)I + \sin(S\frac{\xi_3}{|\xi|}t)R(\xi)] * (e^{-\mu(-\Delta)^\alpha t}u)\,,$$

where u is the divergence-free vector fields, $t \geq 0$, I stands for the identity matrix in $\mathbb{R}^3$ and $R(\xi)$ represents the skew-symmetric matrix symbol of the Riesz transform, as follows

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

In order to present our main result, we give the sense of a mild solution of (1).

Definition 1.1. *Let $\left(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in[0,T]}, \mathbf{P}\right)$ be a filtered probability space with the expectation $\mathbf{E}$ and $T > 0$. We designate by $\mathcal{M}_T$ the smallest σ -algebra of $\mathbf{F}_t$ -adapted distribution processes f defined on $\Omega \times [0,T] \times \mathbb{R}^n$ which are progressively measurable; more precisely $f(\omega, t, \cdot) \in \mathbf{F}_t \times \mathcal{B}([0,t])$, for all $t \in [0,T]$.*

Definition 1.2. *Let $\left(\Omega, \mathbf{F}, \mathbf{P}, (\mathbf{F}_t)_{t\geq 0}, W\right)$ be a fixed probability basis. the divergence free process $\begin{pmatrix} u \\ b \end{pmatrix}$ is a global mild solution of problem (1), if $\begin{pmatrix} u(\omega,\cdot) \\ b(\omega,\cdot) \end{pmatrix} \in \tilde{L}^4_t\big(\mathcal{F}\dot{\mathcal{N}}^{4-\frac{3}{2}\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\big) \cap \mathcal{M}_t$ for all $t \geq 0$ such that*

$$\begin{aligned} \begin{pmatrix} u \\ b \end{pmatrix} =& \mathcal{A}_{S,\alpha}(t)\begin{pmatrix} u_0 \\ b_0 \end{pmatrix} + \int_0^t \mathcal{A}_{S,\alpha}(t-s')\mathbb{P}\begin{pmatrix} f \\ g \end{pmatrix} dW \\ & - \int_0^t \mathcal{A}_{S,\alpha}(t-s')\mathbb{P}\begin{pmatrix} \nabla\cdot(u\otimes u - b\otimes b) \\ \nabla\cdot(u\otimes b - b\otimes u) \end{pmatrix}(s',\cdot)\,ds', \end{aligned} \tag{3}$$

where $\mathcal{A}_{S,\alpha}(t) = \begin{pmatrix} G_{S,\alpha}(t) & 0 \\ 0 & Q_{\nu,\alpha}(t) \end{pmatrix}$ and $Q_{\nu,\alpha}(t) := e^{-\nu(-\Delta)^\alpha t} = \mathcal{F}^{-1}(e^{-\nu|\xi|^{2\alpha}t})$.

Definition 1.3. *For a fixed probability basis $\left(\Omega, \mathbf{F}, \mathbf{P}, \{\mathbf{F}_t\}_{t\in[0,T]}, W\right)$, a divergence free process $\begin{pmatrix} u \\ b \end{pmatrix}$ is a local solution of (1), if there exists a positive random time τ, such that $\begin{pmatrix} u \\ b \end{pmatrix}(\omega) \in \tilde{L}^4_\tau\big(\mathcal{F}\dot{\mathcal{N}}^{4-\frac{3}{2}\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\big) \cap \mathcal{M}_\tau$ and satisfies the relation*

$$\begin{aligned} \begin{pmatrix} u \\ b \end{pmatrix} =& \mathcal{A}_{S,\alpha}(t)\begin{pmatrix} u_0 \\ b_0 \end{pmatrix} + \int_0^t \mathcal{A}_{S,\alpha}(t-s')\mathbb{P}\begin{pmatrix} f \\ g \end{pmatrix} dW \\ & - \int_0^t \mathcal{A}_{S,\alpha}(t-s')\mathbb{P}\begin{pmatrix} \nabla\cdot(u\otimes u - b\otimes b) \\ \nabla\cdot(u\otimes b - b\otimes u) \end{pmatrix}(s',\cdot)\,ds', \\ & P\text{-a.s., for any } t \in [0,\tau]. \end{aligned}$$

We are now able to state our two main results of this work.

Theorem 1.4. *Let $0 \leq \lambda < 3$, $1 \leq q \leq +\infty$ and $2/3 < \alpha \leq (7+\lambda)/6$. Let $\left(\Omega, \boldsymbol{F}, \boldsymbol{P}, \{\boldsymbol{F}_t\}_{0\leq t\leq T}, W\right)$ be a probability basis. Assume that u_0, b_0 are $\boldsymbol{F}_0$ measurable, small enough with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$, and $f, g \in \mathcal{M}_T$. Suppose that for any nonnegative S and for any positive T,*

$$(1+T)\,\Big\| \begin{pmatrix} f \\ g \end{pmatrix} \Big\|_{\tilde{L}^4_\Omega \tilde{L}^4_T\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} + \Big\| \begin{pmatrix} u_0 \\ b_0 \end{pmatrix} \Big\|_{\tilde{L}^4_\Omega\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} < +\infty.$$

Then there exist a random set $\tilde{\Omega}$ with positive probability and a unique global mild solution of equation (1) in $\tilde{L}^4\big(0,T;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\big)$ for all $\omega \in \tilde{\Omega}$.

Theorem 1.5. *Under the assumptions of Theorem 1.4. If there is a positive constant K such that*

$$(1+T)\left\|\begin{pmatrix} f\\ g\end{pmatrix}\right\|_{\tilde{L}^4_\Omega \tilde{L}^4_T\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} + \left\|\begin{pmatrix} u_0\\ b_0\end{pmatrix}\right\|_{\tilde{L}^4_\Omega\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} < K,$$

then there exist a random set $\tilde{\Omega}$ with positive probability, a random time $\tau(\omega) > 0$, and a process $(u,b) \in \tilde{L}^4_\tau\big(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\big) \cap \mathcal{M}_\tau$ for all $\omega \in \tilde{\Omega}$, and (u,b) is a local solution of equation (1).

Remark 1.6. *When $b=0$ and $\lambda = 0$, Wang and Wu [11] obtained the same result in the space $\mathrm{F}\dot{\mathrm{B}}^s_{2,q}(\mathbb{R}^3) = \mathcal{F}\dot{\mathcal{N}}^s_{2,0,q}(\mathbb{R}^3)$. In fact, Theorem 1.4 extends the global existence results of the solution of the stochastic rotating Navier–Stokes equation in the Fourier Besov-Lebesgue space $\mathrm{F}\dot{\mathrm{B}}^{\frac{5}{2}-\frac{3}{2}\alpha}_{2,q}$ to the results of existence of the solution of 3D stochastic generalized rotating magnetohydrodynamic in the Fourier Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{p,\lambda,q}(\mathbb{R}^3)$. The proof of Theorem 1.4 is based on two methods, that is, the use of the dissipative equation and the classical semi-group approach.*

In order to prove Theorem 1.5, we apply an iterative contraction method, so that the solution is exactly the limit in the space $\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{p,\lambda,q}$ of the considered Picard sequence.

2. Preliminaries

This section briefly gives some notations and specifies the fundamental properties of FBM-spaces that are constructed using a Fourier variable localization method on known Morrey spaces. For $1 \le p < \infty$, $0 \le \lambda < n$, the Morrey spaces $\mathrm{M}^\lambda_p = \mathrm{M}^\lambda_p(\mathbb{R}^n)$ are the set of functions $f \in L^p_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{M}^\lambda_p} = \sup_{x\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,r))} < \infty, \tag{4}$$

where $B(x,r)$ is the ball in $\mathbb{R}^n$ with center x and radius r.

Next, let us recall briefly the definition of the Littlewood-Paley decomposition. More precisely, we take χ and φ two nonnegative smooth radial functions such that

$$\operatorname{supp}\varphi \subset \{\xi\in\mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}, \quad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n\backslash\{0\},$$
$$\operatorname{supp}\chi \subset \{\xi\in\mathbb{R}^n : |\xi| \le \frac{4}{3}\}, \quad \chi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1, \quad \xi\in\mathbb{R}^n.$$

We denote the set of all polynomials on $\mathbb{R}^n$ by $\mathcal{P}$ and we designate $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. S' is the space of tempered distributions.

Let us insert some standard localization operators, such as,

$$\dot{\Delta}_j u = \big(\mathcal{F}^{-1}\varphi_j\big) * u = \varphi(2^{-j}D)u = 2^{jn}\int h(2^j y)u(x-y)\,dy,$$
$$\dot{S}_j u = \sum_{k\le j-1}\dot{\Delta}_k u = \chi(2^{-j}D)u = 2^{jn}\int \tilde{h}(2^j y)u(x-y)\,dy,$$

where $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$.

Let $s\in\mathbb{R}$, $1 \le p < +\infty$, $1 \le q \le +\infty$, and $0 \le \lambda < n$. The FBM-space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)$ is the set of all distributions $u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, such that $\varphi_j\hat{u} \in \mathrm{M}^\lambda_p$, for all $j\in\mathbb{Z}$, and

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)} = \begin{cases} \Big\{\sum_{j\in\mathbb{Z}} 2^{jqs}\|\varphi_j\hat{u}\|^q_{\mathrm{M}^\lambda_p}\Big\}^{1/q} & for\ \ q<\infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\varphi_j\hat{u}\|_{\mathrm{M}^\lambda_p} & for\ \ q=\infty. \end{cases}$$

Besides, for $1 \leq \rho \leq \infty$ and $T \in (0, \infty]$, the space $\tilde{L}^{\rho}_{T}(\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})$ is defined as the set of tempered distributions in $S'(\mathbb{R} \times \mathbb{R}^n)/\mathcal{P}$ with respect to the norm

$$\|u(t,x)\|_{\tilde{L}^{\rho}_{T}(\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})} = \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^{q}_{L^{\rho}([0,T],\mathrm{M}^{\lambda}_{p})} \Big\}^{1/q} < +\infty,$$

with the usual modification for $q = +\infty$.

The following lemma is consecrated to give Young's inequality and Bernstein's type inequality associated to Morrey spaces.

Lemma 2.1. *[3, Lemma 2.1] Let $1 \leq p_1, p_2 < \infty$ and $0 \leq \lambda_1,\ \lambda_2 < n$.*

1. *Assume that $h \in L^1$ and $g \in \mathrm{M}^{\lambda_1}_{p_1}$, then*

$$\|h * g\|_{\mathrm{M}^{\lambda_1}_{p_1}} \leq \|h\|_{L^1} \|g\|_{\mathrm{M}^{\lambda_1}_{p_1}}, \tag{5}$$

2. *Let $1 \leq p_2 \leq p_1 < \infty$ such that*

$$\frac{n-\lambda_1}{p_1} \leq \frac{n-\lambda_2}{p_2}.$$

If $\mathrm{supp}(\widehat{h}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq A2^m\}$ then there is a constant $C > 0$ independent of h and m such that

$$\|(i\xi)^{\beta}\widehat{h}\|_{\mathrm{M}^{\lambda_2}_{p_2}} \leq C 2^{m|\beta| + m(\frac{n-\lambda_2}{p_2} - \frac{n-\lambda_1}{p_1})} \|\widehat{h}\|_{\mathrm{M}^{\lambda_1}_{p_1}}, \tag{6}$$

where β is a multi-index and $m \in \mathbb{Z}$.

Now, let us define the Chemin-Lerner type space of Bochner.

Definition 2.2. *Let $1 \leq p, \sigma < \infty, 1 \leq q, \rho \leq \infty, s \in \mathbb{R}$ and $T > 0$. Chemin-Lerner type space of Bochner $\tilde{L}^{\sigma}_{\Omega}\tilde{L}^{\rho}_{T}F\dot{N}^{s}_{p,\lambda,q}(\mathbb{R}^n)$ is defined as the space of distribution process $u \in \mathcal{M}_T$ such that $u(\omega,t) \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}, P$ -a.s. and the quasinorm*

$$\|u\|_{\tilde{L}^{\sigma}_{\Omega}\tilde{L}^{\rho}_{T}F\dot{N}^{s}_{p,\lambda,q}} = \begin{cases} \big\| \{2^{js}[\mathrm{E}(\ \| \varphi_j \hat{u} \|_{L^{\rho}_{T}\mathrm{M}^{\lambda}_{p}})^{\sigma}]^{\frac{1}{\sigma}}\}\big\|_{l^q(j\in\mathbb{Z})} & for\ \ q < \infty, \\ \sup\limits_{j\in\mathbb{Z}} 2^{js}[\mathrm{E}(\|\varphi_j \hat{u}\|_{L^{\rho}_{T}\mathrm{M}^{\lambda}_{p}})^{\sigma}]^{\frac{1}{\sigma}} & for\ \ q = \infty \end{cases}$$

is finite.

3. Linear and Bilinear Estimates

In this section, we start by giving the linear nonhomogeneous dissipative equation

$$\begin{cases} u_t + \nu(-\triangle)^{\alpha}u = h(t,x) \quad (t,x) \in \mathbb{R}^{+} \times \mathbb{R}^3 \\ \quad u(0,x) = u_0(x) \ \ x \in \mathbb{R}^3 \ , \end{cases} \tag{7}$$

and we give the following lemma.

Lemma 3.1. *Let $s \in \mathbb{R}, 0 < T \leq \infty, 0 \leq \lambda < 3, 1 \leq p < \infty,$ and $1 \leq q, \rho \leq \infty$. If $u_0 \in \mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}$ and $h \in \tilde{L}^{\rho}_{T}(\mathcal{F}\dot{\mathcal{N}}^{s-2\alpha+\frac{2\alpha}{\rho}}_{p,\lambda,q})$, then the Cauchy problem (7) admits a unique solution $u(t,x)$ such that for all $\rho_1 \in [\rho, +\infty]$*

$$\|u\|_{\tilde{L}^{\rho_1}_{T}(\mathcal{F}\dot{\mathcal{N}}^{s+\frac{2\alpha}{\rho_1}}_{p,\lambda,q})} \leq C\big(\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q}} + \|h\|_{\tilde{L}^{\rho}_{T}(\mathcal{F}\dot{\mathcal{N}}^{s+\frac{2\alpha}{\rho}-2\alpha}_{p,\lambda,q})}\big).$$

Besides, if $q < +\infty$, then $u \in \mathcal{C}([0,T); \mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})$.

Proof. The proof of the lemma 3.1 is very similar to the one shown in [13, Lemma 2.3] with minor modifications.

Now, we recall some useful lemmas which we will use in the sequel.

Lemma 3.2. *[4, Lemma 10] Assume that* $0 < T \leq \infty$, $0 \leq \lambda < 3$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ *and* $u_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$. *Then there is a constant* $C > 0$ *such that*

$$\|G_{S,\alpha}(t)u_0\|_{\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{4-\frac{3}{2}\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} \leq C\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}. \tag{8}$$

Lemma 3.3. *[4, Proposition 11] Let* $0 \leq \lambda < 3$, $2 \leq p < \infty$, $1 \leq q \leq +\infty$, $0 < T \leq +\infty$ *and* $\frac{2}{3} < \alpha \leq \frac{5}{3} + \frac{\lambda-3}{3p}$, *and put*

$$X = \tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{4-\frac{3}{2}\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}),$$

there is a constant $C = C(p,q) > 0$ *depending on* p, q *such that*

$$\|hg\|_{\tilde{L}^2(\mathcal{F}\dot{\mathcal{N}}^{5-3\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} \leq C\|h\|_X\|g\|_X. \tag{9}$$

Lemma 3.4. *[5, Lemma 3.1] Let* $s \in \mathbb{R}$, $0 < T \leq \infty$, $0 \leq \lambda < 3, 1 \leq p < \infty$, $1 \leq q, \rho, r \leq \infty$ *and* $h \in \tilde{L}^r_T(\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$. *There exists a constant* $C > 0$ *such that*

$$\|\int_0^t G_{S,\alpha}(t-s')h(s')ds'\|_{\tilde{L}^\rho_T(\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} \leq C\|h\|_{\tilde{L}^r_T(\mathcal{F}\dot{\mathcal{N}}^{s-2\alpha-\frac{2\alpha}{\rho}+\frac{2\alpha}{r}}_{p,\lambda,q})}.$$

4. A Priori Estimates

To find the solution of the equation (1), it is also necessary to treat the random term in (3), by superposition principle, we need to consider the following auxiliary problem

$$\begin{cases} dv + \mu(-\Delta)^\alpha v dt + \mathcal{S}e_3 \times v dt = f dW & \text{in } \Omega \times (0, +\infty) \times \mathbb{R}^3, \\ v|_{t=0} = u_0 & \text{on } \Omega \times \mathbb{R}^3. \end{cases} \tag{10}$$

Remark 4.1. *The deterministic version of the equation* (10) *is given in [5, p.6].*

The Fourier transform of (10) with respect to the spatial variable gives

$$\begin{cases} d\hat{v} + \mu|\xi|^{2\alpha}\hat{v}dt + \mathcal{S}e_3 \times \hat{v}dt = \hat{f}dW & \text{in } \Omega \times (0, +\infty) \times \mathbb{R}^3, \\ \hat{v}|_{t=0} = \hat{u}_0 & \text{on } \Omega \times \mathbb{R}^3. \end{cases} \tag{11}$$

Now, we can conclude that this linear stochastic ODE has a unique solution. Consequently, we can obtain the solution of the original equation (10) by inverse Fourier transform. To obtain the solution of the equation (1) by using the fixed point argument, we should also estimate the solution of the equation (10).

Lemma 4.2. *Let* u_0 *be* F_0 *measurable and* f *progressively measurable on* $\Omega \times [0, T] \times \mathbb{R}^3$, *and for any* $q \in [2, +\infty)$, $u_0 \in \tilde{L}^4_\Omega \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$, $f \in \tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$, *the solution* v *of* (10) *belongs to the space*

$$\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$$

and

$$\|v\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} \leq C(1+T)\|f\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + \|u_0\|_{\tilde{L}^4_\Omega \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}.$$

Proof. To prove this estimate, we should first multiply φ_j on both sides of the first equation of (11), we get

$$d(\varphi_j\hat{v}) = -[\mu|\xi|^{2\alpha}(\varphi_j\hat{v}) + \mathcal{S}e_3\times(\varphi_j\hat{v})]dt + (\varphi_j\hat{f})dW.$$

Applying Itô's formula to $\|\varphi_j\hat{v}\|_{\mathrm{M}_2^\lambda}$, we obtain

$$\begin{aligned}
d\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} &= d(\sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\frac{\lambda}{2}}\|\varphi_j\hat{v}\|_{L^2(B(x,r))})^2\\
&= \sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda} d\|\varphi_j\hat{v}\|^2_{L^2(B(x,r))}\\
&= \sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda}\big[2\langle\varphi_j\hat{v},\ -[\mu|\xi|^{2\alpha}(\varphi_j\hat{v}) + \mathcal{S}e_3\times(\varphi_j\hat{v})]dt + (\varphi_j\hat{f})dW\rangle\\
&\quad + \|\varphi_j\hat{f}\|^2_{L^2(B(x,r))}dt\big]\\
&= \sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda}[(-2\mu\||\cdot|^\alpha\varphi_j\hat{v}\|^2_{L^2(B(x,r))} + \|\varphi_j\hat{f}\|^2_{L^2(B(x,r))})dt\\
&\quad + 2\langle\varphi_j\hat{v},\ \varphi_j\hat{f}\rangle dW]\\
&\lesssim (-2\||\cdot|^\alpha\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \|\varphi_j\hat{f}\|^2_{\mathrm{M}_2^\lambda})dt + 2\sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda}\langle\varphi_j\hat{v},\ \varphi_j\hat{f}\rangle dW
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(B(x,r))$. Reapplying Itô's formula to $(\|\varphi_j\hat{v}\|^2 + \varepsilon)^2$ for $\varepsilon > 0$, it results from the above equality that

$$\begin{aligned}
d(\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \varepsilon)^2 \lesssim\ & 2(\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \varepsilon)[(-2\||\cdot|^\alpha\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \|\varphi_j\hat{f}\|^2_{\mathrm{M}_2^\lambda})dt\\
& + 2\sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda}\langle\varphi_j\hat{v},\ \varphi_j\hat{f}\rangle dW]\\
& + 4\sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda}\langle\varphi_j\hat{f},\ \varphi_j\hat{v}\rangle^2 dt. \qquad (12)
\end{aligned}$$

Now, we consider the sequence of stopping times

$$\tau_N = \begin{cases} \inf\{t \geq 0 : \|\varphi_j\hat{v}\| > N\}, & \text{if } \{t\ : \|\varphi_j\hat{v}\| > N\} \neq \emptyset,\\ T, & \text{if } \{t\ :\ \|\varphi_j\hat{v}\| > N\} = \emptyset \end{cases}$$

for $N = 1, 2, \ldots$. Integrating (12) on $[0,\ t]$ for $t \leq \min\{T,\ \tau_N\}$ and taking the expectation of the resulting estimate, we obtain

$$\begin{aligned}
&\mathrm{E}(\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \varepsilon)^2 - \mathrm{E}(\|\varphi_j\hat{u}_0\|^2_{\mathrm{M}_2^\lambda} + \varepsilon)^2\\
&\leq 4\mathrm{E}\int_0^t (\sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda}\langle\varphi_j\hat{f},\varphi_j\hat{v}\rangle)^2 ds - 4\mathrm{E}\int_0^t (\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \varepsilon)\||\cdot|^\alpha\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} ds\\
&\quad + 2\mathrm{E}\int_0^t (\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \varepsilon)\|\varphi_j\hat{f}\|^2_{\mathrm{M}_2^\lambda} ds + 4\mathrm{E}\int_0^t (\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} + \varepsilon)\sup_{x\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda}\langle\varphi_j\hat{v},\ \varphi_j\hat{f}\rangle dW\\
&=: L_1 + L_2 + L_3 + L_4.
\end{aligned}$$

Since L_2 is already in the desired form, it is enough to estimate L_1, L_3 and L_4 by using Young and Hölder's inequalities.

$$\begin{aligned}
L_1 &\lesssim \mathrm{E}\int_0^t \|\varphi_j\hat{f}\|^2_{\mathrm{M}_2^\lambda}\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda} ds\\
&\lesssim \mathrm{E}\sup_{s\in[0,t]}\|\varphi_j\hat{v}\|^2_{\mathrm{M}_2^\lambda}\int_0^t \|\varphi_j\hat{f}\|^2_{\mathrm{M}_2^\lambda} ds\\
&\lesssim \epsilon\mathrm{E}\sup_{s\in[0,t]}\|\varphi_j\hat{v}\|^4_{\mathrm{M}_2^\lambda} + C_\epsilon t\mathrm{E}\int_0^t \|\varphi_j\hat{f}\|^4_{\mathrm{M}_2^\lambda} ds.
\end{aligned}$$

According to Young's inequality, we obtain

$$L_3 \lesssim \epsilon \mathrm{E} \sup_{s\in[0,t]} (\|\varphi_j \hat{v}\|^2_{\mathrm{M}^{\lambda}_2} + \varepsilon)^2 + C_\epsilon t \mathrm{E} \int_0^t \|\varphi_j \hat{f}\|^4_{\mathrm{M}^{\lambda}_2} ds.$$

In order to estimate the stochastic integral L_4, we first apply the Burkholde-Davis-Gundy inequality and then Young's inequality to obtain

$$\begin{aligned} L_4 &\lesssim \mathrm{E} \sup_{s\in[0,t]} \Big| \int_0^{s'} (\|\varphi_j v\|^2_{\mathrm{M}^{\lambda}_2} + \varepsilon)\langle \varphi_j v,\ \varphi_j \hat{f}\rangle dW \Big| \\ &\lesssim \mathrm{E} \sup_{s\in[0,t]} (\|\varphi_j v\|^2_{\mathrm{M}^{\lambda}_2} + \varepsilon)\|\varphi_j v\|_{\mathrm{M}^{\lambda}_2} \Big(\int_0^t \|\varphi_j \hat{f}\|^2_{\mathrm{M}^{\lambda}_2} ds \Big)^{\frac{1}{2}} \\ &\lesssim \epsilon \mathrm{E} \sup_{s\in[0,t]} [(\|\varphi_j v\|^2_{\mathrm{M}^{\lambda}_2} + \varepsilon)\|\varphi_j \hat{v}\|_{\mathrm{M}^{\lambda}_2}]^{\frac{4}{3}} + C_\epsilon \mathrm{E} t \int_0^t \|\varphi_j \hat{f}\|^4_{\mathrm{M}^{\lambda}_2} ds. \end{aligned}$$

Based on the estimates of L_1, L_2, L_3 and L_4, choosing $\epsilon > 0$ sufficiently small and passing to the limit as $\varepsilon \to 0$, we obtain

$$\begin{aligned} &\mathrm{E} \sup_{t\in[0,T\wedge\tau_N]} \|\varphi_j \hat{v}\|^4_{\mathrm{M}^{\lambda}_2} + \mathrm{E} \int_0^{T\wedge\tau_N} \|\varphi_j \hat{v}\|^2_{\mathrm{M}^{\lambda}_2} \||\cdot|^\alpha \varphi_j \hat{v}\|^2_{\mathrm{M}^{\lambda}_2} ds \\ &\lesssim \mathrm{E}\|\varphi_j \hat{u}_0\|^4_{\mathrm{M}^{\lambda}_2} + [1 + (T\wedge\tau_N)]\mathrm{E} \int_0^{T\wedge\tau_N} \|\varphi_j \hat{f}\|^4_{\mathrm{M}^{\lambda}_2} ds. \end{aligned} \tag{13}$$

Given the conditions on f and u_0, we assert that $\mathrm{E} \sup_{t\in[0,T\wedge\tau_N]} \|\varphi_j \hat{v}\|^4_{\mathrm{M}^{\lambda}_2}$ is bounded by a constant without N. Consequently, let $N \to \infty$ and applying the fact that $\lim_{N\to\infty} \tau_N = T$, P-*a.s.*, we obtain

$$\mathrm{E} \sup_{t\in[0,T]} \|\varphi_j \hat{v}\|^4_{\mathrm{M}^{\lambda}_2} + 2^{2\alpha j} \mathrm{E} \int_0^T \|\varphi_j \hat{v}\|^4_{\mathrm{M}^{\lambda}_2} ds \lesssim \mathrm{E}\|\varphi_j \hat{u}_0\|^4_{\mathrm{M}^{\lambda}_2} + (1+T)\mathrm{E} \int_0^T \|\varphi_j \hat{f}\|^4_{\mathrm{M}^{\lambda}_2} ds. \tag{14}$$

Consequently

$$2^{2\alpha j} \mathrm{E} \int_0^T \|\varphi_j \hat{v}\|^4_{\mathrm{M}^{\lambda}_2} ds \lesssim \mathrm{E}\|\varphi_j \hat{u}_0\|^4_{\mathrm{M}^{\lambda}_2} + (1+T)\mathrm{E} \int_0^T \|\varphi_j \hat{f}\|^4_{\mathrm{M}^{\lambda}_2} ds.$$

Multiplying the above estimate by $2^{(\frac{5}{2}-2\alpha+\frac{\lambda}{2})j}$ and taking l^q norm, we obtain

$$\|v\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} \lesssim (1+T)\|f\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}, + \|u_0\|_{\tilde{L}^4_\Omega \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}.$$

Remark 4.3. *According to the inequality* (14)*, we can deduce that*

$$\mathrm{E} \sup_{t\in[0,T]} \|\varphi_j \hat{v}\|^4_{\mathrm{M}^{\lambda}_2} \lesssim \mathrm{E}\,\|\varphi_j \hat{u}_0\|^4_{\mathrm{M}^{\lambda}_2} + (1+T)\mathrm{E} \int_0^T \|\varphi_j \hat{f}\|^4_{\mathrm{M}^{\lambda}_2} ds.$$

Under the same condition given in Lemma 4.2, we can obtain the solution v of (10) *in the space $\tilde{L}^4_\Omega \tilde{L}^\infty_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$ with*

$$\|v\|_{\tilde{L}^4_\Omega \tilde{L}^\infty_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} \lesssim (1+T)\|f\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + \|u_0\|_{\tilde{L}^4_\Omega \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}.$$

Lemma 4.4. *Supposing that the condition of Lemma 4.2 is verified, the solution v of the equation* (10) *satisfies the following statement: There is a set $\tilde{\Omega}$ with positive probability such that $v(\omega,\cdot,\cdot) \in \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$ and there is a positive constant $\widetilde{C}$ such that*

$$\|v(\omega,\cdot,\cdot)\|_{\tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} \leq \widetilde{C}\big[\,\|u_0\|_{\tilde{L}^4_\Omega \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + (1+T)\|f\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}\big]$$

for all $\omega \in \tilde{\Omega}$.

Proof. The proof is given in the Fourier-Besov space [11, p.11]. First, we consider the following set

$$\Omega^* :=\Big\{ \omega : \|v(\omega,\cdot,\cdot)\|_{\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} > \widetilde{C}\big[\,\|u_0\|_{\tilde{L}^4_\Omega\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + (1+T)\|f\|_{\tilde{L}^4_\Omega\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}\big]\Big\} \tag{15}$$

for some positive constant $\widetilde{C}$ chosen later.

Estimate (15), Lemma 4.2 and Chebychev's inequality give

$$\begin{aligned}\mathbf{P}(\Omega^*) &\le \frac{\|v(\omega,\cdot,\cdot)\|^4_{\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}}{\big[\widetilde{C}\,\|u_0\|_{\tilde{L}^4_\Omega\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + (1+T)\|f\|_{\tilde{L}^4_\Omega\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}\big]^4}\\ &\le \big(\frac{C}{\widetilde{C}}\big)^4\end{aligned}$$

where C is a constant given in Lemma 4.2. Put $\tilde{\Omega} = \Omega\backslash\Omega^*$ and $\widetilde{C} > C$, then

$$\mathbf{P}(\tilde{\Omega}) = 1 - \mathbf{P}(\Omega^*) \ge 1 - \big(\frac{C}{\widetilde{C}}\big)^4 > 0.$$

Clearly, $\tilde{\Omega}$ is to fulfill the needs.

Proof of Theorem 1.4. Usually, the mild solution (u, b) for the equation (1) can be reformulated as follows

$$\begin{aligned} u &= G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}f\,dW - \int_0^t G_{S,\alpha}(t-s')\mathbb{P}\nabla\cdot(u\otimes u - b\otimes b)(s',\cdot)\,ds',\\ b &= Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}g\,dW - \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}\nabla\cdot(u\otimes b - b\otimes u)(s',\cdot)\,ds', \end{aligned} \tag{16}$$

with $Q_{\nu,\alpha}(t) := e^{-\nu(-\Delta)^\alpha t} = \mathcal{F}^{-1}(e^{-\nu|\xi|^{2\alpha}t})$.

The system (16) can easily be rewritten as follows

$$\begin{pmatrix} u\\ b\end{pmatrix} = \begin{pmatrix} G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}f\,dW\\ Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}g\,dW\end{pmatrix} + \begin{pmatrix} L(u,u)-L(b,b)\\ K(u,b)-K(b,u)\end{pmatrix} := \psi(u,b),$$

where

$$L(u,v) = -\int_0^t G_{S,\alpha}(t-s')\mathbb{P}\nabla\cdot(u\otimes v)(s',\cdot)\,ds',$$

and

$$K(u,v) = -\int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}\nabla\cdot(u\otimes v)(s',\cdot)\,ds'.$$

Put

$$X = \tilde{L}^4_T\,\big(\,\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\,\big)\,.$$

Lemma 3.4 and Lemma 3.3 give

$$\begin{aligned} &\|L(u,u)-L(b,b)\|_{\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q})}\\ &= \Big\|\int_0^t G_{S,\alpha}(t-s')\mathbb{P}\nabla.(u\otimes u - b\otimes b)(s')ds'\Big\|_{\tilde{L}^2_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q})}\\ &\le C\|\nabla.(u\otimes u - b\otimes b)\|_{\tilde{L}^2_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{7}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q})}\\ &\le C(\|u\|^2_X + \|b\|^2_X)\,. \end{aligned} \tag{17}$$

To prove Theorem 1.4 we require that

$$\| \big(G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}fdW_{s'} \,, Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}gdW_{s'}\big) \|_X \leq CC_0.$$

Lemma 4.4 leads to

$$\begin{aligned}
&\| G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}fdW_{s'} \|_{\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} \\
&\leq C\big(\|u_0\|_{\tilde{L}^4_\Omega(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} + (1+T)\|f\|_{\tilde{L}^4_\Omega\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q})}\big) \\
&\leq \frac{CC_0}{2}.
\end{aligned}$$

Concerning the second equation, we note that $K(u,b)$ can be viewed as the solution to the heat equation (7) with $u_0 = 0, h = \mathbb{P}\nabla.(u\otimes v)$. From Lemma 3.1 with $s = \frac{5}{2} - 2\alpha + \frac{\lambda}{2}$, $\rho_1 = 4$, $\rho = 2$, and applying Lemma 3.3 we get

$$\begin{aligned}
&\|K(u,b) - K(b,u)\|_{\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} \\
=& \ \Big\|\int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}\nabla.(u\otimes b - b\otimes u)(s')ds'\Big\|_{\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} \\
\leq& \ 2C\|\nabla.(u\otimes b)\|_{\tilde{L}^2_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} \\
\leq& \ 2C\|u\otimes b\|_{\tilde{L}^2_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{7}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} \\
\leq& \ 2C\|u\|_X\|b\|_X\,.
\end{aligned} \tag{18}$$

In a similar way, Lemma 4.4 with $S = 0$ yields

$$\begin{aligned}
&\| Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}gdW_{s'} \|_{\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} \\
&\leq C\big(\|b_0\|_{\tilde{L}^4_\Omega(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} + (1+T)\|g\|_{\tilde{L}^4_\Omega\tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q})}\big) \\
&\leq \frac{CC_0}{2},
\end{aligned}$$

Since

$$\begin{aligned}
&\| \big(G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}fdW_{s'} \,, Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}gdW_{s'}\big) \|_X \\
&\leq CC_0,
\end{aligned} \tag{19}$$

we put

$$H = \Big\{(u,b)|(u,b)\in X, \|(u,b)\|_X \leq 2CC_0\Big\},$$

where C_0 is a constant that can be determined later. The combination of (17), (18) and (19) gives

$$\begin{aligned}
&\|\psi(u,b)\|_X \\
&\leq \| \big(G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}fdW_{s'} \,, Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}gdW_{s'}\big) \|_X \\
&\quad + C(\|u\|_X^2 + \|b\|_X^2 + 2\|u\|_X\|b\|_X) \\
&\leq CC_0 + C\|(u,b)\|_X^2 \\
&\leq CC_0 + 4C^3C_0^2.
\end{aligned} \tag{20}$$

It follows that for all $(u,b) \in H, \psi(u,b) \in H$ if a choice of $C_0 < \frac{1}{16C^2}$ is realized.

However, for any (u,v), $(a,b) \in H$, we get

$$\begin{aligned}
&\|\psi(u,v) - \psi(a,b)\|_X \\
&\leq \|L(u,u) - L(a,a)\|_X + \|L(v,v) - L(b,b)\|_X \\
&\quad + \|K(a,b) - K(u,v)\|_X + \|K(b,a) - K(v,u)\|_X \\
&\leq \|L(u,u-a) + L(u-a,a)\|_X + \|L(v-b,b) + L(v,v-b)\|_X \\
&\quad + \|K(a,b-v) + K(a-u,v)\|_X + \|K(b,a-u) + K(b-v,u)\|_X \\
&\leq C((\|u\|_X + \|a\|_X)\|u-a\|_X + (\|v\|_X + \|b\|_X)\|v-b\|_X) \\
&\quad + C((\|v\|_X + \|b\|_X)\|u-a\|_X + (\|u\|_X + \|a\|_X)\|v-b\|_X) \\
&\leq 8C^2C_0(\|u-a\|_X + \|v-b\|_X) \\
&\leq \frac{1}{2}\|(u,v) - (a,b)\|_X.
\end{aligned}$$

Finally, we find that ψ is a contraction mapping from H to H. Hence, the Banach fixed point theorem leads to the fact that ψ has a unique fixed point $(u,b) \in H$ which is the solution of equation (1). □

Proof of Theorem 1.5. In order to obtain a sequence of approximate solutions (u^k, b^k) on $\mathbb{R}\times\mathbb{R}^3$, we employ the following iterative scheme,

$$\begin{cases} (u^0, b^0) = (u_0, b_0) \\ u^{k+1} = u^0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}f dW - \int_0^t G_{S,\alpha}(t-s')\mathbb{P}\nabla\cdot(u^k\otimes u^k - b^k\otimes b^k)(s',\cdot)\,ds', \\ b^{k+1} = b^0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}g dW - \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}\nabla\cdot(u^k\otimes b^k - b^k\otimes u^k)(s',\cdot)\,ds'. \end{cases} \tag{21}$$

Let us now show the uniform boundedness of $\{u^k, b^k\}_{k\in\mathbb{N}}$ in $X\times X$ for some $T>0$, with $X = \tilde{L}_T^4\big(\mathcal{F}\dot{\mathcal{N}}_{2,\lambda,q}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}\big)$.

Using (20) and (21), we obtain the following inequality.

$$\begin{aligned}
&\|(u^{k+1}, b^{k+1})\|_X \\
&\leq \Big\| \Big(G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}f dW_{s'}\,, Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}g dW_{s'}\Big) \Big\|_X \\
&\quad + C(\| u^k \|_X^2 + \| b^k \|_X^2 + 2\| u^k \|_X \| b^k \|_X).
\end{aligned}$$

Then

$$\| (u^{k+1}, b^{k+1}) \|_X \leq C\big(\| (\theta, \vartheta) \|_X + \| (u^k, b^k) \|_X^2 \big), \tag{22}$$

where $\theta = G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}f dW_{s'}$ and $\vartheta = Q_{\nu,\alpha}(t)b_0 + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}g dW_{s'}$.

Let $\tau(\omega, C_0)$ be the stopping time given by

$$\tau(\omega, C_0) = \begin{cases} \inf D_\omega\,, & \text{if } D_\omega \neq \emptyset \\ T, & \text{if } D_\omega = \emptyset \end{cases} \tag{23}$$

where C_0 is a positive number and

$$D_\omega = \Big\{ t\in[0,T] : \Big\|\begin{pmatrix} \theta \\ \vartheta \end{pmatrix}(\omega,\cdot)\Big\|_{\tilde{L}_t^4\big(\mathcal{F}\dot{\mathcal{N}}_{2,\lambda,q}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}\big)} \geq C_0,\ P-\text{a.s} \Big\}.$$

Since θ solves (10) and ϑ is also a solution of (10) when $S=0$, Lemma 4.4 gives

$$\begin{aligned}
&C \,\Big\| \Big(G_{S,\alpha}(t)u_0 + \int_0^t G_{S,\alpha}(t-s')\mathbb{P}f dW_{s'}\,,\ Q_{\nu,\alpha}(t)b_0 \\
&\quad + \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}g dW_{s'}\Big) \Big\|_{\tilde{L}_t^4\big(\mathcal{F}\dot{\mathcal{N}}_{2,\lambda,q}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}\big)} \\
&\leq C_0 P - \text{a.s.}
\end{aligned}$$

Then

$$\| (\theta,\vartheta)(\omega,.) \|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq CC_0, \forall\omega\in\bar{\Omega}. \tag{24}$$

It should be noted that $\|.\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}$ is non decreasing and continuous as a function of t. Therefore, $\tau(\omega, C_0)$ exists and is positive for all $\omega\in\bar{\Omega}$.

Given a $C_0 > 0$ small enough, such that $C_0 < \frac{1}{16C^2}$ where C is constant from Lemma 4.4. Let τ the corresponding $\tau(\omega, C_0)$ for which we prove by induction that

$$\|(u^k,b^k)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq C_0, \text{ for any } k\in\mathbb{N}, \text{ and } \omega\in\bar{\Omega}.$$

Suppose that $\|(u^k,b^k)(\omega,.)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq R, \forall\omega\in\bar{\Omega}$.

Then, according to (22) and (24), it follows that

$$\|(u^{k+1},b^{k+1})(\omega,.)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq CC_0 + C\|(u^k,b^k)(\omega,.)\|^2_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}, \forall\omega\in\bar{\Omega}.$$

Consequently

$$\|(u^{k+1},b^{k+1})(\omega,.)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq CC_0 + CR^2, \forall\omega\in\bar{\Omega}.$$

Let $C_0 > 0$ be sufficiently small that if $\|(u_0,b_0)(\omega,.)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq C_0$, then $1-4CC_0 > 0$, and we take the smallest root R of $CC_0 + CR^2 = R$, that is,

$$R = \frac{1-\sqrt{1-4C_0C^2}}{2C}. \tag{25}$$

As follows, if $\|(u^k,b^k)(\omega,.)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq R$, then

$$\|(u^{k+1},b^{k+1})(\omega,.)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \leq R$$

involves that $\|(u^k,b^k)\|_{\tilde{L}^4_t\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}, k\in\mathbb{N}$ is uniformly bounded in the space $\tilde{L}^4_t\big(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\big)$.

Now, we bound the difference vector $(u^{k+1}-u^k, b^{k+1}-b^k)$.

Firstly, note that

$$\begin{aligned} u^{k+1}-u^k &= \int_0^t G_{S,\alpha}(t-s')\mathbb{P}\nabla\cdot[u^{k-1}\otimes u^{k-1} - u^k\otimes u^k \\ &\quad - (b^{k-1}\otimes b^{k-1} - b^k\otimes b^k)](s',\cdot)\,ds' \\ &= \int_0^t G_{S,\alpha}(t-s')\mathbb{P}\nabla\cdot\big[u^{k-1}\otimes(u^{k-1}-u^k) + (u^{k-1}-u^k)\otimes u^k \\ &\quad - \big(b^{k-1}\otimes(b^{k-1}-b^k) - (b^{k-1}-b^k)\otimes b^k\big)\big](s',\cdot)\,ds'. \end{aligned}$$

Then

$$
\begin{aligned}
&\|u^{k+1}-u^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\leq \big\|\nabla\cdot\big[u^{k-1}\otimes(u^{k-1}-u^k)+(u^{k-1}-u^k)\otimes u^k\\
&\quad-\big(b^{k-1}\otimes(b^{k-1}-b^k)-(b^{k-1}-b^k)\otimes b^k\big)\big]\big\|_{\tilde{L}^2_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\leq \big\|u^{k-1}\otimes(u^{k-1}-u^k)+(u^{k-1}-u^k)\otimes u^k\\
&\quad-\big(b^{k-1}\otimes(b^{k-1}-b^k)-(b^{k-1}-b^k)\otimes b^k\big)\big\|_{\tilde{L}^2_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{7}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\leq C\Big[\big(\|u^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}+\|u^{k-1}\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\big)\\
&\quad\times\|u^{k-1}-u^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\quad+\big(\|b^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}+\|b^{k-1}\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\big)\\
&\quad\times\|b^{k-1}-b^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\Big]\\
&\leq 2CR\|(u^{k-1}-u^k,b^{k-1}-b^k)\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}.
\end{aligned}\tag{26}
$$

For the second equation we have

$$
\begin{aligned}
b^{k+1}-b^k=\int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}\nabla\cdot[&u^{k-1}\otimes b^{k-1}-u^k\otimes b^k\\
&-(b^{k-1}\otimes u^{k-1}-b^k\otimes u^k)](s',\cdot)\,ds'
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|b^{k+1}-b^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\quad\leq C\Big(\ \big\|\nabla\cdot\big[u^{k-1}\otimes b^{k-1}-u^k\otimes b^k\big]\big\|_{\tilde{L}^2_T\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\qquad+\big\|\nabla\cdot\big[b^{k-1}\otimes u^{k-1}-b^k\otimes u^k\big]\big\|_{\tilde{L}^2_T\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\ \Big)\\
&\leq C\Big(\ \big\|u^{k-1}\otimes(b^{k-1}-b^k)+(u^{k-1}-u^k)\otimes b^k\big\|_{\tilde{L}^2_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{7}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\qquad+\big\|b^{k-1}\otimes(u^{k-1}-u^k)+(b^{k-1}-b^k)\otimes u^k\big\|_{\tilde{L}^2_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{7}{2}-3\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\ \Big)\\
&\quad\leq C\Big(\ \|u^{k-1}\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\|b^{k-1}-b^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\qquad+\|u^{k-1}-u^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\|b^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\qquad+\|b^{k-1}\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\|u^{k-1}-u^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\\
&\qquad+\|b^{k-1}-b^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\|u^k\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}\ \Big)\\
&\quad\leq 2CR\|(u^{k-1}-u^k,b^{k-1}-b^k)\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}.
\end{aligned}\tag{27}
$$

Consequently

$$\begin{aligned} &\|(u^{k+1}-u^k, b^{k+1}-b^k)(\omega,.)\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \\ &\leq 4CR\|(u^{k-1}-u^k, b^{k-1}-b^k)(\omega,.)\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \end{aligned} \tag{28}$$

for all $\omega \in \bar{\Omega}$ and $n \in \mathbb{N}$. Now, choosing R given in (25) such that $R < \frac{1}{4C}$ (minimizing C_0, if needed), we deduce that $\|(u^{k+1}-u^k, b^{k+1}-b^k)\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}$ is contractive and therefore the sequence $\left(u^k(\omega,.), b^k(\omega,.)\right)$ is a Cauchy in the space $\tilde{L}^4_\tau\big(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\big)$, for all $\omega \in \bar{\Omega}$. As a consequence of the completeness of $\tilde{L}^4_\tau\big(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\big)$, we may then extract a subsequence (u^k, b^k) denoted by the same symbol, converges to a limit $(u,b) \in \tilde{L}^4_\tau\big(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\big)$ which is a mild solution in $\tilde{L}^4_\tau\big(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\big)$ for (1), for all $\omega \in \bar{\Omega}$.

Finally, we demonstrate the uniqueness of the solution. Suppose that $(\tilde{u}, \tilde{b})$ and (u,b) are two solutions of system (1) with the same initial data (u_0, b_0). Therefore the difference of these two solutions satisfies the following equations:

$$\tilde{u} - u = \int_0^t G_{S,\alpha}(t-s')\mathbb{P}\nabla \cdot (u \otimes u) - \tilde{u} \otimes \tilde{u} - (b \otimes b - \tilde{b} \otimes \tilde{b})(s', \cdot)\, ds'$$

$$\tilde{b} - b = \int_0^t Q_{\nu,\alpha}(t-s')\mathbb{P}\nabla \cdot (u \otimes b - \tilde{u} \otimes \tilde{b} - (b \otimes u - \tilde{b} \otimes \tilde{u}))(s', \cdot)\, ds'$$

Using precisely the same idea of estimates (26)–(28), we get

$$\begin{aligned} &\|(\tilde{u}-u, \tilde{b}-b)(\omega,.)\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)} \\ &\leq \tilde{C}\|(\tilde{u}-u, \tilde{b}-b)(\omega,.)\|_{\tilde{L}^4_\tau\left(\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-\frac{3}{2}\alpha+\frac{\lambda}{2}}_{2,\lambda,q}\right)}, \end{aligned}$$

which allows us to obtain the fact that $\tilde{u} = u$ and $\tilde{b} = b$ due to the condition $0 < \tilde{C} < 1$, i.e., the solution (u,b) is unique. □

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