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Research Article

On duality in convex optimization of second-order differential inclusions with periodic boundary conditions

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Abstract

The present paper is devoted to the duality theory for the convex optimal control problem of second-order differential inclusions with periodic boundary conditions. First, we use an auxiliary problem with second-order discrete-approximate inclusions and focus on formulating sufficient conditions of optimality for the differential problem. Then, we concentrate on the duality that exists in periodic boundary conditions to establish a dual problem for the differential problem and prove that Euler-Lagrange inclusions are duality relations for both primal and dual problems. Finally, we consider an example of the duality for the second-order linear optimal control problem.

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1. Introduction

It is well known that optimization problems for differential inclusions are one of the most actively researched areas of optimal control theory [1, 4–8, 12, 13, 16, 20, 23]. The reputation of duality in optimization theory problems with differential inclusions stems primarily from its importance in formulating necessary and sufficient optimality conditions and, as a result, in generating various algorithmic approaches for solving mathematical programming problems. The investigations conducted in this work demonstrate the importance of duality theory beyond these aspects and highlight its strong connections with various topics in convex analysis, nonlinear analysis, functional analysis, and variational analysis theory.

Convex analysis and, in particular, duality theory, have surprisingly been discovered in recent years' applications in rediscovering classical results as well as providing new powerful ones in the field of optimization theory. In mathematical economics, for example, duality theory is interpreted as prices; in mechanics, potential energy and complementary energy are mutually dual, and the displacement field and stress field are solutions to

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the primal and dual problems, respectively. In economics, there are two major practical applications of duality theory. The first main application of duality theory is that it allows us to derive systems of demand equations that are consistent with maximizing or minimizing behavior on the part of an economic agent, consumer, or producer by simply differentiating a function, rather than explicitly solving a constrained maximization or minimization problem. The second principal advantage of duality theory is that it enables us to derive in an effortless way the "comparative statics" theorems originally deduced from maximizing behavior. Besides the indicated applications, duality often makes it possible to simplify the computational procedure and to construct a generalized solution of variational problems that do not have classical solutions.

Convexity and duality first appear in the classical calculus of variations in the correspondence between Lagrangian and Hamiltonian functions, and how this is related to necessary conditions and the existence of solutions. The paper [14] is concerned with the various sets of sufficient conditions for the existence and nonexistence of solutions to the Dirichlet boundary value problem. The paper [22] investigates the use of convex functions to demonstrate the existence of optimal solutions to an over-determined system of linear equations. The duality theorems demonstrated that a sufficient condition for an extremum is an extremal relation for both the primal and dual problems. That means that if a pair of admissible solutions satisfy this relation, then each of them is a solution to the corresponding (primal and dual) problem. We note that the establishment of extremal relations is a significant part of Ekeland and Temam's investigations for simple variational problems, and there are similar results for differential inclusions in their works [11]. Burachik and Jeyakumar were the first to introduce closedness-type conditions for the optimization problem of minimizing the sum of two functions and its Fenchel dual problem [3]. In contrast to the Lagrange and Fenchel duality, Mahmudov successfully constructed the duality for problems with differential inclusions using the concept of dual operations of addition and infimal convolution of convex functions. The difficulties that have arisen in this case are related to the fact that this approach necessarily requires the construction of a duality of coupled discrete and discrete approximate problems [9, 10, 17–19].

This paper deals with the problem of duality for convex optimal control of second-order differential inclusions with periodic boundary conditions. In the first part of the paper, we formulate the optimality conditions for the Lagrange problem with periodic boundary conditions applying the discrete-approximate method. The second part of the paper focuses on establishing the dual problem using the dual operations of addition and infimal convolution of convex functions. Moreover, we prove that Euler-Lagrange type inclusion is a dual relation and the optimal values in the primal convex and dual concave problems are the same. The posed problem, as well as its duality, are novel. The structure of the paper is as follows.

Section 2 introduces the necessary facts and crucial notions of set-valued mappings, including the hamiltonian function, conjugate function, infimal convolution, and locally adjoint mapping, as well as supplementary results from Mahmudov and Mordukhovich's book [15, 21], and the problem with second-order differential inclusions with periodic boundary conditions.

In Section 3, using difference operators of the first and second-order and an auxiliary set-valued mapping, the differential problem with periodic boundary conditions is approximated with the help of the coupled problem of discrete-approximate problem. Then, we obtain sufficient optimality conditions for the Lagrange problem in terms of the Euler-Lagrange inclusion and transversality conditions proceeding to the limit procedure in Theorem 3.1.

We construct the dual problem to convex problem for differential inclusion with periodic boundary conditions in Section 4. Here the dual problem to the convex problem obtained by using the infimal convolution of convex functions is the starting point of the construction of duality theory. Thus the duality problem for corresponding discrete and discrete approximate problems allows us to formulate this problem for a posed differential problem. However, to avoid lengthy calculations, we omit it and instead formulate only a dual problem for second-order differential inclusions with periodic boundary conditions. Then, we establish the duality relationship between a pair of optimization problems in the duality Theorem 4.1. We show that the Euler-Lagrange type adjoint differential inclusions are a dual relation, which is satisfied by a pair of solutions to primal and dual problems. Finally, the considered duality for the second-order linear optimal control problem demonstrates that maximization in the dual problems occurs over the set of adjoint equation solutions.

2. Preliminaries and problem statement

The first part of this Section includes some notions and results from convex analysis and set-valued analysis theory that can be found in numerous published reports and books for the reader's convenience [15, 21]. Such auxiliary concepts and definitions, it is hoped, are here presented in such a way as to be more readily available.

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space, $\langle x, v\rangle$ be an inner product of elements $x, v \in \mathbb{R}^n, (x, v)$ be a pair of x, v. We say that a set-valued mapping $F : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is convex if its graph $gphF = \{(x, v_1, v_2) : v_2 \in F(x, v_1)\}$ is a convex subset of $\mathbb{R}^{3n}$. It is convex-valued if $F(x, v_1)$ is a convex set for each $(x, v_1) \in domF$, where $domF = \{(x, v_1) : F(x, v_1) \neq \emptyset\}$. For such mappings, we introduce the Hamiltonian function and argmaximum set as

$$H_F(x, v_1, v_2^*) = \sup_{v_2} \left\{ \langle v_2, v_2^*\rangle : v_2 \in F(x, v_1) \right\},$$

$$F_A(x, v_1; v_2^*) = \left\{ v_2 \in F(x, v_1) : \langle v_2, v_2^*\rangle = H_F(x, v_1, v_2^*) \right\}$$

$v_2^* \in \mathbb{R}^n$, respectively. For convex set-valued mappings F, we set $H_F(x, v_1, v_2^*) = -\infty$ if $F(x, v_1) = \emptyset$. For the convex set-valued mapping F, the Hamiltonian function $H_F(\cdot, \cdot, v_2^*)$ is concave.

A subset of $\mathbb{R}^n$ is called a cone if it contains the zero vector and contains with each of its vectors all positive multiples of that vector. Geometrically, this means that a cone, unless it consists of 0 alone, is a "bundle of rays". The convex cone $K_A(z_0)$, $z_0 = (x_0, u_0, v_0)$ is called the cone of tangent directions at a point $z_0 \in A$ to the set A if from $\overline{z} = (\overline{x}, \overline{u}, \overline{v}) \in K_A(z_0)$ it follows that $\overline{z}$ is a tangent vector to the set A at point $z_0 \in A$, i.e., there exists such function $\gamma(\lambda) \in \mathbb{R}^{3n}$ such that $z_0 + \lambda\overline{z} + \gamma(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\gamma(\lambda) \to 0$ as $\lambda \downarrow 0$. We have already seen that the cone of tangent directions involve directions for each of which there exists its own function $\gamma(\lambda)$. For a convex mapping F at a point $(x^0, v_1^0, v_2^0) \in gphF$ setting $\gamma(\lambda) \equiv 0$, we have

$$K_{gphF}(x^0, v_1^0, v_2^0) = cone[gphF - (x^0, v_1^0, v_2^0)] = \Big\{ (\overline{x}, \overline{v}_1, \overline{v}_2) : \ \overline{x} = \lambda(x - x^0), \ \overline{v}_1 = \lambda(v_1 - v_1^0),$$

$$\overline{v}_2 = \lambda(v_2 - v_2^0) \Big\}, \ \forall \, (x, v_1, v_2) \in gphF.$$

A function f is called a proper function if it does not assume the value $-\infty$ and is not identically equal to $+\infty$. Clearly, f is proper if and only is $domf \neq \emptyset$ and $f(x)$ is finite for $x \in domf = \{x : f(x) < +\infty\}$. We say a vector $x^* \in \mathbb{R}^n$ is a subgradient of f at $x_0 \in domf$ if for all $x \in domf$, $f(x) - f(x_0) \geq \langle x^*, x - x_0\rangle$. If f is convex and differentiable, then its gradient at x_0 is a subgradient. But a subgradient can exist even when f is not differentiable at x_0 and there can be more than one subgradient of a function f at a point x_0. There are several ways to interpret a subgradient. A vector x^* is a subgradient of f at x_0 if the affine function of x, $f(x_0) + \langle x^*, x - x_0\rangle$ is a global underestimator of f. Geometrically, x^* is a subgradient of f at x_0 if $(x^*, -1)$ supports epi$f$ at $(x_0, f(x_0))$. A function f is called subdifferentiable at x_0 if there exists at least one subgradient at x_0. The set of subgradients of f at the point x_0 is called the subdifferential of f at x_0, and is denoted $\partial f(x_0)$. A function f is called subdifferentiable if it is subdifferentiable at all $x_0 \in dom f$. The subdifferential $\partial f(x_0)$ is always a closed convex set, even if f is not convex.

Theorem 2.1. (Moreau-Rockafellar)[15] *Let f_1, f_2 be a proper convex function and $f = f_1 + f_2$, $x_0 \in dom f_1 \cap dom f_2$. Suppose that either* (1) *there is a point $x_1 \in dom f_1 \cap dom f_2$ where f_1 is continuous or* (2) *$ri dom f_1 \cap ri dom f_2 \neq \emptyset$. Then*

$$\partial f(x_0) = \partial f_1(x_0) + \partial f_2(x_0).$$

Local properties of differentiable functions are well described by the concept of the derivative and the gradient function. For convex functions, instead of the gradient, we use subdifferentials. In the case of set-valued mappings, a similar role is played by an important concept of this paper: the locally adjoint mappings (LAM). The definition of LAM to F was introduced by Pshenichnyi and applied in works of Mahmudov [17, 18]. Note that a similar notion is given by Mordukhovich [21] too, and is called co-derivative of multifunctions at a given point. For a convex mapping F, a set-valued function defined by

$$F^*(v_2^*; (x^0, v_1^0, v_2^0)) := \Big\{(x^*, v_1^*) : (x^*, v_1^*, -v_2^*) \in K^*_{gphF}(x^0, v_1^0, v_2^0)\Big\}$$

is a locally adjoint set-valued mapping to F at a point $(x^0, v_1^0, v_2^0) \in gphF$, where $K^*_{gphF}(x^0, v_1^0, v_2^0)$ is the dual to the cone of tangent vectors $K_{gphF}(x^0, v_1^0, v_2^0)$.

Theorem 2.2 ([15])**.** *Let F be a convex set-valued mapping. Then*

$$F^*(v_2^*; (x, v_1, v_2)) = \begin{cases} \partial_x H_F(x, v_1, v_2^*), & v_2 \in F_A(x, v_1; v_2^*), \\ \emptyset, & v_2 \notin F_A(x, v_1; v_2^*), \end{cases}$$

where $\partial_x H_F(x, v_1, v_2^*) = -\partial_x(-H_F(x, v_1, v_2^*))$.

The conjugate of f is the function defined by $f^*(x^*) = \sup_x \{\langle x, x^*\rangle - f(x)\}$. It is clear that f^* is convex even if f is not convex. This can be easily verified using that fact that the supremum of a set of convex functions (in our case, for a fixed x, the difference $\langle x, x^*\rangle - f(x)$ is a linear function of x^* and hence the difference is convex) is convex function. This operation play an important role in many applications such as duality. We present an interesting property of conjugate function, in particular, that Young's inequality $f(x) + f^*(x^*) \geq \langle x, x^*\rangle$ holds for any function. If here f is a proper convex function, then we refer to this relation as Fenchel's inequality.

We now associate with each nonempty convex set a convex function known as its support function. The support function W_M of a nonempty set $M \in \mathbb{R}^n$ is defined by $W_M(x^*) = \sup_x \{\langle x, x^*\rangle : x \in M\}$. Moreover, $\delta_N(y) = \left\{ \begin{array}{cl} 0 & , y \in N \\ +\infty & , y \notin N \end{array} \right.$ is called the indicator function of the set N and the operation of infimal convolution $\oplus$ of functions f_1^* and f_2^* is defined as follows:

$$(f_1^* \oplus f_2^*)(y) = \inf \Big\{f_1^*(y_1) + f_2^*(y_2) : \ y_1, y_2 \in \mathbb{R}^n, \ y_1 + y_2 = y\Big\}.$$

Let us denote

$$M_F(x^*, y^*, z^*) := \inf_{x,y,z} \Big\{\langle x, x^*\rangle + \langle y, y^*\rangle - \langle z, z^*\rangle : \ (x, y, z) \in gphF\Big\}.$$

It is clear that for every $x, y \in \mathbb{R}^n$

$$M_F(x^*, y^*, z^*) \leq \langle x, x^*\rangle + \langle y, y^*\rangle - H_F(x, y, z^*).$$

Here the Hamiltonian function $H_F(x, y, z^*) = \sup_z\{\langle z, z^*\rangle : z \in F(x, y)\}, z^* \in \mathbb{R}^n$ and for convex set-valued mapping F, we put $H_F(x, y, z^*) = -\infty$, if $F(x, y) = \emptyset$. Moreover it is easy to see that the function

$$M_F(x^*, y^*, z^*) = \inf_{x,y}\{\langle x, x^*\rangle + \langle y, y^*\rangle - H_F(x, y, z^*)\}$$

is a support function of the set $gphF$ taken with a minus sign. It follows that for a fixed z^*, $M_F(x^*, y^*, z^*) = -\big(-H_F(\cdot, \cdot, z^*)\big)^*(x^*, y^*)$, that is, M_F is the conjugate function for $-H_F(\cdot, \cdot, z^*)$ taken with a minus sign.

The present paper deals with the duality results for the following convex optimal control problem of second-order differential inclusions with periodic boundary conditions, labelled by (PB)

$$\text{infimum } J(x(\cdot)) = \int_0^T f(x(t), t)dt \tag{2.1}$$

$$(PB) \qquad x''(t) \in F(x(t), x'(t), t)\ ,\ \text{a.e. } t \in [0, T], \tag{2.2}$$

$$x(0) = x(T)\ ,\ x'(0) = x'(T), \tag{2.3}$$

where $F(\cdot, \cdot, t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $f(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is proper convex function. The problem is to find an arc $\widetilde{x}(\cdot)$ of the Lagrange problem satisfying (2.2) almost everywhere (a.e.) on a time interval $[0, T]$ and the boundary value conditions (2.3) that minimizes the cost functional $J\big(x(\cdot)\big)$. Here a feasible trajectory $x(\cdot)$ is understood to be an absolutely continuous function on a time interval $[0, T]$ together with the first-order derivatives for which $x''(\cdot) \in L_1^n([0, T])$. Clearly, such a class of functions is a Banach space, endowed with the different equivalent norms.

Our investigations begin with a general approach to constructing a dual optimization problem to a primal problem (PB) based on conjugate function theory. The importance of duality in optimization theory stems primarily from its role in formulating optimality conditions for periodic boundary problems (PB). As a result, we begin by providing sufficient optimality conditions for the problem (PB), which will play a significant role in the subsequent duality investigations.

We introduce the first and second-order difference operators to give an idea of how to construct the discrete-approximate problem for the problem (PB) with second-order differential inclusions:

$$\Delta x(t) = \frac{1}{h}(x(t+h) - x(t)), \quad \Delta^2 x(t) = \frac{1}{h}(\Delta x(t+h) - \Delta x(t)), t = 0, h, \ldots, 1-h,$$

where h is a step on the t-axis and $x(t) \equiv x_h(t)$ is a grid functions on a uniform grid on $[0, T]$. The discrete-approximate problem associated with the continuous problem (PB) is defined as follows

$$\begin{aligned} &\text{minimize} \sum_{t=2,\ldots,T-2h} hf(x(t), t), \\ &\Delta^2 x(t) \in F(x(t), \Delta x(t), t),\ t = 0, h, 2h, \ldots, T-2h, \\ &x(0) = x(T), \quad \Delta x(0) = \Delta x(T). \end{aligned} \tag{2.4}$$

By presenting the auxiliary set-valued mapping

$$G(x, v_1, t) := 2v_1 - x + h^2 F(x, \frac{v_1 - x}{h}, t), \tag{2.5}$$

we rewrite the discrete-approximate inclusion (2.4) as $x(t+2h) \in G(x(t), x(t+h), t)$. Then some equivalence theorems are required for any progress in problems with differential inclusions to express the LAM G^* in terms of LAM F^*. Let us first give two propositions concerning the Hamiltonian functions of the set-valued mappings F and G, and the sets of subdifferential of the Hamiltonian functions H_G and H_F.

Lemma 2.3 ([15]). *Let F and G be formula-specified convex set-valued mappings (2.5). Then there is the following relation between the Hamiltonian H_G and H_F functions:*

$$H_G(x, v_1, v_2^*) = \langle 2v_1 - x\ ,\ v_2^*\rangle + h^2 H_F(x, \frac{v_1 - x}{h}, v_2^*).$$

Lemma 2.4 ([15]). *The following relation holds for subdifferentials of the Hamiltonian functions H_G and H_F:*

$$\partial H_G(x, v_1, v_2^*) = \{-v_2^*\} \times \{2v_2^*\} + h^2\Lambda^*\partial H_F(x, \frac{v_1 - x}{h}, v_2^*),$$

where $\Lambda = \begin{pmatrix} I & 0 \\ \frac{-I}{h} & \frac{I}{h} \end{pmatrix}$ is a $2n \times 2n$ matrix partitioned into submatrices, $I, \frac{-I}{h}, \frac{-I}{h}$ and $n \times n$ zero matrix, where I is an $n \times n$ identity matrix and Λ^* is transposes of Λ.

The following theorem plays a critical role in the construction of LAM for the original discrete approximate problem (2.4) associated with the continuous problem (PB).

Theorem 2.5 ([15]). *If G is a convex set-valued mapping defined by (2.5), then the following statements for the LAMs are equivalent:*

$$(a) \quad (x^*, v_1^*) \in G^*(v_2^*; (x, v_1, v_2)),\ v_2 \in G_A(x, v_1; v_2^*),\ v_2^* \in \mathbb{R}^n$$

$$(b) \quad \Big(\frac{x^* + v_1^* - v_2^*}{h^2}, \frac{v_1^* - 2v_2^*}{h}\Big) \in F^*\Big(v_2^*; (x, \frac{v_1 - x}{h}, \frac{x - 2v_1 + v_2}{h^2})\Big),$$
$$\frac{x - 2v_1 + v_2}{h^2} \in F_A(x, \frac{v_1 - x}{h}; v_2^*),$$

where $G_A(x, v_1; v_2^*)$ is the argmaximum set for mapping G.

Thus by using the discrete-approximate method [15] and Theorem 2.5, we can construct the optimality conditions for the discrete-approximate problem as follows:

Theorem 2.6. *Let F be a convex set-valued mapping and f be proper convex functional and continuous at the points of some feasible trajectory. Then for optimality of the trajectory $\{\tilde{x}(t)\}$ in the discrete approximate problem, it is necessary and sufficient that there exist vectors $x^*(t), u^*(t)$ which are not all equal zero, satisfying the approximate Euler-Lagrange and transversality inclusions:*

$$(a) \quad \Big(\Delta^2 x^*(t) + \Delta u^*(t)\ ,\ u^*(t+h)\Big) \in F^*(x^*(t+2h); (\tilde{x}(t), \Delta\tilde{x}(t), \Delta^2\tilde{x}(t), t))$$
$$-\partial f(\tilde{x}(t), t) \times \{0\}, \quad t = 0, h, \ldots, T - 2h,$$

$$(b) \quad \Delta^2\tilde{x}(t) \in F_A(\tilde{x}(t); x^*(t), t)\ ,$$

$$(c) \quad x^*(0) = x^*(T - h)\ , \quad \Delta x^*(0) = \Delta x^*(T - h).$$

The derivation of these conditions is omitted in this paper to avoid long calculations because the proof is a simple modification of the proof of Theorem 4.4. [16]. We derive sufficient optimality conditions for second-order differential inclusions with periodic boundary conditions using the results obtained in this section. The necessity of these conditions for optimality can be justified by using the functional analysis approach in convex problems-Arzela-Ascoli type theorem for compactness in corresponding function spaces, uniformly convergence, and another functional analysis approach, substantiate passing limit. However, proving necessary conditions is difficult and is a separate topic of discussion that has been omitted.

3. Optimality conditions of periodic boundary problem

The formal limiting procedure in the optimality conditions for second-order discrete approximate problems is used to formulate the optimality conditions of periodic boundary problems. By formally passing to the limit as $h \to 0$ in the conditions of Theorem 2.6, sufficient conditions of optimality for the continuous problem (PB) can be formulated.

Theorem 3.1. *Let F be a convex set-valued mapping and f be proper convex functional and continuous at the points of some feasible trajectory. Then for optimality of the trajectory $\tilde{x}(t)$ in the problem (PB), it is sufficient that there exist vectors $x^*(t), u^*(t)$ which are not all equal zero, satisfying the second-order Euler-Lagrange type adjoint and the transversality conditions*

$$(i) \quad \Big(x^{*''}(t) + u^{*'}(t), u^*(t)\Big) \in F^*(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), t)) - \partial f(\widetilde{x}(t), t) \times \{0\},\ t \in [0, T],$$

$$(ii) \quad \tilde{x}''(t) \in F_A(x(t), x'(t); x^*(t), t)\ , a.e.\ t \in [0, T],$$

$$(iii) \quad x^*(0) = x^*(T)\ , \quad x^{*'}(0) = x^{*'}(T).$$

Proof. For a convex set-valued mapping F, using the Moreau-Rockafellar Theorem 2.1 and Theorem 2.2 and the condition (i) of the theorem, we obtain the following inclusion

$$\Big(x^{*''}(t) + u^{*'}(t), u^*(t)\Big) \in \partial_x \Big[H_F(\tilde{x}(t), \tilde{x}'(t), x^*(t)) - f(\widetilde{x}(t), t)\Big],\ t \in [0, T].$$

By definition of subdifferential, we have

$$\begin{aligned} &H_F(x(t), x'(t), x^*(t)) - H_F(\tilde{x}(t), \tilde{x}'(t), x^*(t)) - f(x(t), t) + f(\widetilde{x}(t), t) \\ &\le \Big\langle x^{*''}(t) + u^{*'}(t)\ ,\ x(t) - \widetilde{x}(t)\Big\rangle + \Big\langle u^*(t)\ ,\ x'(t) - \widetilde{x}'(t)\Big\rangle, \end{aligned}$$

and so, considering the Hamiltonian function properties, we get

$$\begin{aligned} &\Big\langle x''(t)\ ,\ x^*(t)\Big\rangle - \Big\langle \widetilde{x}''(t)\ ,\ x^*(t)\Big\rangle - \Big\langle x^{*''}(t) + u^{*'}(t)\ ,\ x(t) - \widetilde{x}(t)\Big\rangle \\ &-\Big\langle u^*(t)\ ,\ x'(t) - \tilde{x}'(t)\Big\rangle \le f(x(t), t) - f(\widetilde{x}(t), t). \end{aligned} \tag{3.1}$$

By integrating the inequality (3.1) over the interval $[0, T]$, it holds

$$\begin{aligned} \int_0^T \Big[f(x(t), t) - f(\widetilde{x}(t), t)\Big] dt &\ge \int_0^T \Big[\Big\langle x^*(t)\ ,\ x''(t) - \widetilde{x}''(t)\Big\rangle - \Big\langle x^{*''}(t)\ , x(t) - \widetilde{x}(t)\Big\rangle\Big] dt \\ &- \int_0^T \Big[\Big\langle u^{*'}(t)\ ,\ x(t) - \widetilde{x}(t)\Big\rangle + \Big\langle u^*(t)\ , x'(t) - \widetilde{x}'(t)\Big\rangle\Big] dt. \end{aligned} \tag{3.2}$$

Then it is clear that

$$\begin{aligned} &\Big\langle x^*(t)\ ,\ x''(t) - \widetilde{x}''(t)\Big\rangle - \Big\langle x^{*''}(t)\ , x(t) - \widetilde{x}(t)\Big\rangle \\ &= \frac{d}{dt}\Big\langle x^*(t)\ ,\ x'(t) - \widetilde{x}'(t)\Big\rangle - \frac{d}{dt}\Big\langle x^{*'}(t)\ , x(t) - \widetilde{x}(t)\Big\rangle, \end{aligned}$$

and

$$\Big\langle u^{*'}(t)\ ,\ x(t) - \widetilde{x}(t)\Big\rangle + \Big\langle u^*(t)\ , x'(t) - \widetilde{x}'(t)\Big\rangle = \frac{d}{dt}\Big\langle u^*(t)\ ,\ x(t) - \widetilde{x}(t)\Big\rangle.$$

Thus, from here it follows that

$$\begin{aligned} \int_0^T \Big[f(x(t), t) - f(\widetilde{x}(t), t)\Big] dt &\ge \Big\langle x^*(T)\ ,\ x'(T) - \widetilde{x}'(T)\Big\rangle - \Big\langle x^*(0)\ ,\ x'(0) - \widetilde{x}'(0)\Big\rangle \\ &-\Big\langle x^{*'}(T)\ , x(T) - \widetilde{x}(T)\Big\rangle + \Big\langle x^{*'}(0)\ , x(0) - \widetilde{x}(0)\Big\rangle - \Big\langle u^*(T)\ ,\ x(T) - \widetilde{x}(T)\Big\rangle \\ &+\Big\langle u^*(0)\ ,\ x(0) - \widetilde{x}(0)\Big\rangle. \end{aligned} \tag{3.3}$$

Taking into account the condition (iii) of the theorem and the boundary conditions $x(0) = x(T)$, $x'(0) = x'(T)$ of the differential problem (PB), we derive that

$$\int_0^T \Big[f(x(t),t) - f(\widetilde{x}(t),t)\Big]dt \geq 0, \tag{3.4}$$

i.e. for all feasible solutions $x(t)$, we have $J(x(t)) \geq J(\tilde{x}(t))$, so $\tilde{x}(t)$, $t \in [0,T]$ is optimal. The desired result is proved completely. □

Remark 3.2. Here, adjoint differential inclusion is constructed using the discrete method of the continuous problem. It is noteworthy that condition (i) of Theorem 3.1 is immediately deduced from the adjoint Mahmudov's differential inclusions [19].

4. Duality results

We present conjugate dual problem formulations using a general perturbation approach for various classes of primal problems encountered in convex programming. Considering a so-called perturbation function $\phi : X \times Y \to \mathbb{R}^n$, where X and Y are supposed to be separated locally convex spaces, one can attach to the optimization problem

$$\inf_{x\in X} \phi(x,0) \tag{4.1}$$

the following dual problem

$$\sup_{y^*\in Y^*} \{-\phi^*(0,y^*)\}$$

where $\phi^* : X^* \times Y^* \to \mathbb{R}^n$ is the conjugate function of ϕ, while X^* and Y^* are the topological dual spaces of X and Y, respectively. Some facts related to this approach, which has been well-described in monographs by Bot, Ekeland and Temam, are remarked in the books [2, 11], such as the existence of weak and strong duality for the primal and dual pair of optimization problems. We call strong duality the situation when the optimal objective values of the primal and dual coincide and the dual problem has an optimal solution. To establish our duality results, we use the idea of the previously mentioned general perturbation approach in the problem (4.1), consider the following problem

$$\inf_{x\in N} \psi(x) \qquad (P)$$

where ψ is a closed, proper convex function and that N is a convex closed set. It is well known from convex analysis that the operations of addition and infimal convolution of convex functions are dual to each other. Here, if there exists a point $y \in N$, where ψ is continuous on $ri dom\psi$, the optimal value of problem (P) is

$$\inf_{y\in N} \psi(y) = -\sup_{y\in N} \big\{ -\psi(y) - \delta_N(y)\big\} = -(\psi^* \oplus \delta_N^*)(0) = \sup \big\{ -\psi^*(y^*) - \delta_N^*(-y^*)\big\},$$

where $\delta_N(y)$ is the indicator function of the set N. Moreover, since the operations $+$ and $\oplus$ are dual to each other with respect to taking conjugates, it can be noted that $\inf_{y\in N} \psi(y) \geq \sup \big\{ -\psi^*(y^*) - \delta_N^*(-y^*)\big\}$. Then the dual problem to the primal problem (P) can be formulated as being

$$\sup \big\{ -\psi^*(y^*) - \delta_N^*(-y^*)\big\}. \qquad (P^*)$$

The problem (P^*) is called the dual problem to the primal problem (P).

To establish a dual problem to the primal problem (PB), we have to obtain the dual problem for the discrete-approximate problem (2.4). For this, as above, we compute the conjugate functions of the function ψ and indicator function δ_N, wherein the discrete-approximate problem $\psi = \sum_{t=2,\dots,T-2h} hf(x(t),t)$ and N is the set of the intersection set of boundary conditions. By using the well-known fact that the conjugate function of the indicator function of a convex set is the support function of this set, Theorem 1.25 [15], the converse assertion is true if the considered set is closed, we establish the dual problem for the discrete-approximate problem (2.4). Then, we used a limiting process in the dual problem for the discrete-approximate problem by passing to the formal limit as a discrete step tends to zero. Consequently, the construction of the duality problem would deviate too far from the main themes of this paper and is thus omitted.

Therefore, we give the final result that the following dual problem, labeled $(PB)^*$ to the continuous primal problem (PB):

$$\sup_{x^*(\cdot),u^*(\cdot),v^*(\cdot)} \Big\{ -\int_0^T f^*(v^*(t)\ ,\ t)dt + \int_0^T M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big)dt$$
$$+\Big\langle x(0), u^*(0)\Big\rangle - \Big\langle x(T), u^*(T)\Big\rangle\Big\}. \tag{4.2}$$

Now we are in a position to prove the main result of this paper.

Theorem 4.1. *Let $f(\cdot,t)$ be a continuous and proper convex function and F be a convex set-valued mapping. Moreover, let $\widetilde{x}(t)$ be an optimal solution of the primal problem (PB) with periodic boundary conditions. Then, for the optimality of a triple functions $\{x^*(\cdot), u^*(\cdot), v^*(\cdot)\}$ in the dual problem $(PB)^*$, it is necessary and sufficient that the conditions $(i)-(iii)$ of Theorem 3.1 are satisfied. Besides, the optimal values in the primal (PB) and dual $(PB)^*$ problems are equal.*

Proof. By Young's inequality and definition of the function M_F, we have

$$-f^*(v^*(t),t) \leq f(x(t),t) - \langle x(t), v^*(t)\rangle \tag{4.3}$$

and

$$M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big) \leq \Big\langle x(t)\ ,\ x^{*''}(t) + u^{*'}(t) + v^*(t)\Big\rangle$$
$$+\langle x'(t)\ ,\ u^*(t)\rangle - \langle x''(t)\ ,\ x^*(t)\rangle. \tag{4.4}$$

Now, integrating both sides of the inequalities (4.3) and (4.4) on the interval $t \in [0,T]$, and adding them it yields

$$-\int_0^T f^*(v^*(t),t)dt + \int_0^T M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big)dt$$
$$\leq \int_0^T f(x(t),t)dt + \int_0^T \Big[\Big\langle x(t)\ ,\ x^{*''}(t)\Big\rangle - \Big\langle x''(t)\ ,\ x^*(t)\Big\rangle\Big]dt$$
$$+\int_0^T \Big\langle x(t)\ ,\ u^{*'}(t)\Big\rangle + \Big\langle x'(t)\ ,\ u^*(t)\Big\rangle dt. \tag{4.5}$$

Then, in more convenient form

$$-\int_0^T f^*(v^*(t),t)dt + \int_0^T M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big)dt \leq \int_0^T f(x(t),t)dt$$
$$+\int_0^T \frac{d}{dt}\Big[\Big\langle x(t)\ ,\ x^{*'}(t)\Big\rangle - \Big\langle x'(t)\ ,\ x^*(t)\Big\rangle\Big]dt + \int_0^T \frac{d}{dt}\Big\langle x(t)\ ,\ u^*(t)\Big\rangle dt$$
$$= \int_0^T f(x(t),t)dt + \Big\langle x(T)\ ,\ x^{*'}(T)\Big\rangle - \Big\langle x'(T)\ ,\ x^*(T)\Big\rangle - \Big\langle x(0)\ ,\ x^{*'}(0)\Big\rangle$$
$$+\Big\langle x'(0)\ ,\ x^*(0)\Big\rangle + \Big\langle x(T)\ ,\ u^*(T)\Big\rangle - \Big\langle x(0)\ ,\ u^*(0)\Big\rangle. \tag{4.6}$$

From the transversality conditions of the theorem $x^*(0) = x^*(T)$ and $x^{*'}(0) = x^{*'}(T)$, the relation (4.6) becomes as follows

$$-\int_0^T f^*(v^*(t),t)dt + \int_0^T M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big)dt$$
$$\le \int_0^T f(x(t),t)dt + \Big\langle x^{*'}(0)\ ,\ x(T) - x(0)\Big\rangle + \Big\langle x^*(0)\ ,\ x'(0) - x'(T)\Big\rangle$$
$$+\Big\langle x(T)\ ,\ u^*(T)\Big\rangle - \Big\langle x(0)\ ,\ u^*(0)\Big\rangle. \tag{4.7}$$

From the boundary conditions of problem (PB), we prove that for all feasible solutions $x(\cdot)$ and dual variables $x^*(\cdot), u^*(\cdot), v^*(\cdot)$ of the primal (PB) and dual $(PB)^*$ problems, respectively, the following inequality holds:

$$-\int_0^T f^*(v^*(t),t)dt + \int_0^T M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big)dt$$
$$+\Big\langle x(0), u^*(0)\Big\rangle - \Big\langle x(T), u^*(T)\Big\rangle \le \int_0^T f(x(t),t)dt. \tag{4.8}$$

Furthermore, suppose that the collection $\{\widetilde{x}^*(\cdot), \widetilde{u}^*(\cdot), \widetilde{v}^*(\cdot)\}$ satisfies the conditions $(i)-(iii)$ of Theorem 3.1. Then, by using the definition of LAM, the Euler-Lagrange type inclusion (i) and the condition (ii) imply that

$$H_F(x(t), x'(t), x^*(t)) - H_F(\tilde{x}(t), \tilde{x}'(t), x^*(t))$$
$$\le \Big\langle x^{*''}(t) + u^{*'}(t) + v^*(t)\ ,\ x(t) - \widetilde{x}(t)\Big\rangle + \Big\langle u^*(t)\ ,\ x'(t) - \widetilde{x}'(t)\Big\rangle. \tag{4.9}$$

Then by definition of M_F, we have

$$\Big\langle \widetilde{x}^{*''}(t) + \widetilde{u}^{*'}(t) + \widetilde{v}^*(t)\ ,\ \widetilde{x}(t)\Big\rangle + \Big\langle \widetilde{u}^*(t)\ ,\ \widetilde{x}'(t)\Big\rangle - H_F(\tilde{x}(t), \tilde{x}'(t), \widetilde{x}^*(t))$$
$$= M_F\Big(\widetilde{x}^{*''}(t) + \widetilde{u}^{*'}(t) + \widetilde{v}^*(t)\ ,\ \widetilde{u}^*(t)\ ,\ \widetilde{x}^*(t)\Big). \tag{4.10}$$

And $\widetilde{v}^*(t) \in \partial f(\widetilde{x}(t), t)$ means that

$$f^*(\widetilde{v}^*(t), t) = \langle \widetilde{x}(t), \widetilde{v}^*(t)\rangle - f(\widetilde{x}(t), t). \tag{4.11}$$

Then according to relations (4.10) and (4.11), inequality (4.4) and (4.8) are fulfilled as equalities, ensuring the equality of values of the primal and dual problems for $\widetilde{x}(\cdot)$ and $\widetilde{v}^*(t) \in \partial f(\widetilde{x}(t), t)$ and the collection $\{\widetilde{x}^*(\cdot), \widetilde{u}^*(\cdot), \widetilde{v}^*(\cdot)\}$, respectively. Moreover, $\widetilde{x}(\cdot)$ and the collection $\{\widetilde{x}^*(\cdot), \widetilde{u}^*(\cdot), \widetilde{v}^*(\cdot)\}$ satisfy the conditions $(i)-(iii)$ of Theorem 3.1 and the collection $(i)-(iii)$ is a dual relation for the primal (PB) and dual $(PB)^*$ problems. That leads to the desired result. □

Example 4.2. (Second-order Linear Optimal Control Problem)
Here we construct the dual problem for the following second-order linear optimal control problem:

$$(PL) \qquad \begin{aligned} &\text{infimum} \int_0^T f(x(t),t)dt \\ &x''(t) = Ax(t) + Bx'(t) + Cu(t)\ ,\ u(\cdot) \in U,\ t \in [0,T], \\ &x(0) = x(T)\ ,\ x'(0) = x'(T), \end{aligned} \tag{4.12}$$

where A, B are $n \times n$ dimensional matrices, C is $n \times r$ dimensional matrix, $C \subset \mathbb{R}^r$ is convex closed set, f is continuously differentiable function. It is required to find controlling parameters $\widetilde{u}(t) \in U$, $t \in [0,T]$ such that the corresponding trajectory $\widetilde{x}(t)$ minimizes f.

We observe that in this case $F(x,y) = \{z : z = Ax + By + Cu \ , \ u \in U\}$. Then it is easy to see that

$$\begin{aligned} M_F(x^*, y^*, z^*) &= \inf_{x,y,z} \Big\{ \langle x, x^* \rangle + \langle y, y^* \rangle - \langle z, z^* \rangle : \ (x,y,z) \in gphF \Big\} \\ &= \inf_{x,y} \Big\{ \langle x, x^* - A^* z^* \rangle + \langle y, y^* - B^* z^* \rangle \Big\} - \sup_{u \in U} \Big\{ \langle u, C^* z^* \rangle \Big\} \\ &= \begin{cases} -W_U(C^* z^*) \ , & \text{if } x^* = A^* z^*, \ y^* = B^* z^*, \\ -\infty \ , & \text{otherwise.} \end{cases} \end{aligned} \tag{4.13}$$

Then using the formula M_F, in view of the dual problem $(PB)^*$, we write

$$M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big) = \begin{cases} -W_U(C^* x^*(t)), & \text{if } u^*(t) = B^* x^*(t), \\ & x^{*''}(t) + u^{*'}(t) \\ & +v^*(t) = A^* x^*(t) \ , \\ -\infty \ , & \text{otherwise,} \end{cases}$$

or more convenient form

$$M_F\Big(x^{*''}(t) + u^{*'}(t) + v^*(t), u^*(t), x^*(t)\Big) = \begin{cases} -W_U(C^* x^*(t)) \ , & \text{if } v^*(t) = A^* x^*(t) \\ & -x^{*''}(t) - B^* x^{*'}(t), \\ -\infty \ , & \text{otherwise.} \end{cases}$$

Then the dual problem of the problem (4.12) is

$$\sup_{x^*(\cdot)} \Big\{ -\int_0^T f^*(A^* x^*(t) - x^{*''}(t) - B^* x^{*'}(t) \ , \ t) dt - \int_0^T W_U(C^* x^*(t)) dt \\ + \Big\langle x(0), B^* x^*(0) \Big\rangle - \Big\langle x(T), B^* x^*(T) \Big\rangle \Big\}, \tag{4.14}$$

where $x^*(\cdot)$ is a solution of the adjoint Euler-Lagrange inclusion (equation) $x^{*''}(t) = A^* x^*(t) - B^* x^{*'}(t) - v^*(t)$. Consequently, maximization in this dual problem to primal problem (4.12) is realized over the set of solutions of the adjoint equation.

5. Conclusions

The present paper studies the duality theory for the convex optimal control problem of second-order differential inclusions with periodic boundary conditions. Using the discretization method and locally adjoint mappings, we obtain sufficient conditions of optimality for differential problems in the form of Euler-Lagrange inclusion and transversality conditions. Then, we construct the dual problem for second-order differential inclusions with periodic boundary conditions. We prove that if β and β^* are the optimal values of primal and dual problems, respectively, then $\beta \geq \beta^*$ for all feasible solutions. Furthermore, if the standard convex analysis condition of the existence of an interior point is satisfied, both problems have the solution and $\beta = \beta^*$. Finally, we consider second-order linear optimal control problems to obtain duality results that demonstrate this approach. In the future, similar duality results to optimal control problems with any higher-order differential inclusions can be obtained using the method described in this paper. For clarity, the work focuses on convex problems, but the results are easily generalizable to the non-convex case. Of course, the convergence of the limiting procedure and the establishment of the necessary optimality conditions for the primal differential problem are also points of conclusion.

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