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1-generator two-dimensional quasi-cyclic codes over $\mathbb{Z}_4[u]/\langle u^2-1\rangle$

Research Article

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Abstract: In this paper, we obtain generating set of polynomials of two-dimensional cyclic codes over the ring $R=\mathbb{Z}_4[u]/\langle u^2-1\rangle$, where $u^2=1$. Moreover, we find generator polynomials for two-dimensional quasi-cyclic codes and two-dimensional generalized quasi-cyclic codes over R and specify a lower bound on minimum distance of free 1-generator two-dimensional quasi-cyclic codes and two-dimensional generalized quasi-cyclic codes over R.

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1. Introduction

There are many generalizations of cyclic codes. One of them is two-dimensional cyclic codes. A lot of works on two-dimensional cyclic codes has done. Ikai et al. first introduced the concept of common zeros for characterizing two-dimensional codes [6], and showed the existence of two-dimensional codes that can be characterized by the common zeros. After that the researchers have studied with different concepts in these codes. The reader can find some of such studies in the papers [11–13]. Moreover, Lalason et al. [7] construct a basis of an s-dimensional cyclic code over a finite field. On the other hand, quasi-cyclic codes are another natural generalizations of cyclic codes. The study of quasi-cyclic codes over finite rings has provided useful information in coding theory. We shall use the phrase 'QC code' as an abbreviation for 'quasi-cyclic code' and 'GQC code' for 'generalized quasi-cyclic codes'. QC codes form an important class of linear codes which also include cyclic codes (when we consider the case $\ell=1$). Ling and Solé studied the algebraic structure of QC codes over finite fields and provided a new algebraic approach to QC codes (see also [8]). There have been a lot of investigations of QC codes and GQC codes over

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the rings, for example [1–3, 5, 9, 14]. In [10], the authors studied the constacyclic codes over the finite non-chain ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ with $u^2 = 1$ and obtained some new $\mathbb{Z}_4$-linear codes. Finally, Gao et al. [4] have generalized QC codes and GQC codes over the finite non-chain ring $R = \mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ with $u^2 = 1$. They have determined the structure of the generators and the minimal generating sets of 1-generator QC codes and GQC codes. They also have given a lower bound on minimum distance of free 1-generator QC codes and GQC codes over R. Furthermore, in [4], some new $\mathbb{Z}_4$-linear codes were constructed by 1-generated QC codes and GQC codes over R. Hence, there are many examples of cyclic codes and QC codes over R. There exist many researches of two-dimensional cyclic codes over finite fields. However, the research of two-dimensional cyclic codes over R has not been considered by any coding scientist. Moreover, quasi-cyclic codes perform very well on the codes have great lengths. Therefore, these codes are the important and most intensively studies classes of linear codes. The ring $R = \mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ with $u^2 = 1$ is a Frobenius non-chain ring with 16 elements. There are some examples of cyclic codes over R whose $\mathbb{Z}_4$ Gray images have better parameters than previous best-known $\mathbb{Z}_4$-linear codes were presented (see for example [4] and [10]). The main purpose of this paper is to obtain sets of generator polynomials of two-dimensional cyclic codes over R. We also determine the structure of the generators and the minimal generating sets of 1-generator two-dimensional QC codes and two-dimensional GQC codes. This method probably helps to decode two-dimensional cyclic codes and two-dimensional QC codes as it has done for cyclic codes and QC codes. This paper is organized as follows: at first, we find the generator polynomials corresponding to two-dimensional cyclic codes over R. Then, by using these polynomials, we obtain generator polynomials for two-dimensional QC codes over R. Moreover, we study the structure of generators two-dimensional QC codes. The last part of the paper is devoted to obtain 1-generator polynomial two-dimensional GQC codes and determine a lower bound for the minimum distance of free 1-generator GQC codes.

2. Generator polynomials

As was mentioned in the Introduction, the purpose of this section is to obtain a generating set of polynomials for two-dimensional QC codes over the ring $R = \mathbb{Z}_4[u]/ < u^2 - 1 >$ with $u^2 = 1$. Assume that $S := \mathbb{Z}_4[x]/ < x^m - 1 >$, $R' := R[x, y]/ < x^m - 1, y^3 - 1 >$, where $y^3 = 1$, $x^m = 1$ and m is an odd positive integer. Suppose that $n = 3m\ell$ and $\mathcal{R} := R'^{\ell}$. As before, F_q denotes a finite field with q elements. Recall that a linear code C' of length ms over a finite field F is a two-dimensional cyclic code, if it is closed under row shift and column shift of codewords, whose codewords are viewed as ms arrays. This means that for every codeword c of the form

$$c = \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,s-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,s-1} \\ \vdots & \vdots & & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,s-1} \end{pmatrix}$$

in C', the codewords

$$\begin{pmatrix} c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,s-1} \\ c_{0,0} & c_{0,1} & \cdots & c_{0,s-1} \\ \vdots & \vdots & & \vdots \\ c_{m-2,0} & c_{m-2,1} & \cdots & c_{m-2,s-1} \end{pmatrix}$$

and

$$\begin{pmatrix} c_{0,s-1} & c_{0,0} & \cdots & c_{0,s-2} \\ c_{1,s-1} & c_{1,0} & \cdots & c_{1,s-2} \\ \vdots & \vdots & & \vdots \\ c_{m-1,s-1} & c_{m-1,01} & \cdots & c_{m-1,s-2} \end{pmatrix}$$

also belong to C'.

It is well known that these codes are the ideals of the quotient ring $F[x,y]/<x^m-1, y^s-1>$. Similarly, we consider the above definition for a two-dimensional cyclic code C' of length ms over the ring R. So we define a two-dimensional QC codes over R as follows.

Definition 2.1. *Let C be a linear code of length n. If there exists a least positive integer ℓ such that C is closed the ℓth composition under the row shift and the column shift, then we call C is a two-dimensional QC code over R.*

Clearly, ℓ is a divisor of n. If $\ell = 1$, then C is a two-dimensional cyclic code over R. An r-generator two-dimensional QC code is an ideal of C with r generators. In the rest of this section, we shall focus on 1-generator two-dimensional QC code over R. According to Gao et al.[3], 1-generator two-dimensional QC code C over R can be generated by element $(b_1(x,y),\dots,b_\ell(x,y)) \in \mathcal{R}$, and so

$$\begin{aligned} C &= \{f(x,y)(b_1(x,y),\dots,b_\ell(x,y))|f(x,y)\in R[x,y]\} \\ &= \{(f(x,y)b_1(x,y),\dots,f(x,y)b_\ell(x,y))|f(x,y)\in R[x,y]\}. \end{aligned}$$

Özen et al.[10] have studied cyclic codes over R. In fact, they determined a generators of the cyclic codes over R. In [10], it was proved that if m is odd, then S is a principal ideal ring. Now, by using a method similar that used for two-dimensional cyclic codes over a field in [13], we obtain a generator polynomials for two-dimensional cyclic codes over R. Our generating set has an important role in determining generator polynomials two-dimensional QC codes over R.

Note that R is isomorphic to $\mathbb{Z}_4 + u\mathbb{Z}_4$. We begin with the following lemma.

Lemma 2.2. *Suppose that C' is a two-dimensional cyclic code of length $3m$ over R. Then $\{p_i(x,y) \mid i = 1,\dots,6\ \}$ is a generating set of C', where*

$$\begin{aligned} p_1(x,y) =& \alpha_{01}(x) + (u+1)\alpha_{11}(x) + (\beta_{01}(x) + (u+1)\beta_{11}(x))y \\ &+ (\gamma_{01}(x) + (u+1)\gamma_{11}(x))y^2, \\ p_2(x,y) =& (u+1)\alpha_{12}(x) + (\beta_{02}(x) + (u+1)\beta_{12}(x))y + (\gamma_{02}(x) + (u+1)\gamma_{12}(x))y^2, \\ p_3(x,y) =& (\beta_{03}(x) + (u+1)\beta_{13}(x))y + (\gamma_{03}(x) + (u+1)\gamma_{13}(x))y^2, \\ p_4(x,y) =& (u+1)\beta_{14}(x)y + (\gamma_{04}(x) + (u+1)\gamma_{14}(x))y^2, \\ p_5(x,y) =& (\gamma_{05}(x) + (u+1)\gamma_{15}(x))y^2, \\ p_6(x,y) =& (u+1)\gamma_{16}(x)y^2, \end{aligned}$$

and $\alpha_{0i}(x)$, $\alpha_{1i}(x)$, $\beta_{0i}(x)$, $\beta_{1i}(x)$,$\gamma_{0i}(x)$ and $\gamma_{1i}(x)$ are generator polynomials of cyclic codes over $\mathbb{Z}_4$ for each $i = 1,\dots,6$.

Proof. Suppose that I is an ideal of R' and that $f(x,y)$ is an arbitrary element of I. So it can be written uniquely as the following form

$$f(x,y) = f_0(x) + (u+1)f_1(x) + (f_0'(x) + (u+1)f_1'(x))y + (f_0''(x) + (u+1)f_1''(x))y^2,$$

where $f_0(x)$, $f_0'(x)$, $f_0''(x)$, $f_1(x)$, $f_1'(x)$ and $f_1''(x)$ are polynomials in S. The main strategy employed in our proof is to introduce six auxiliary ideals in S. To achieve this, we break our proof into six steps as follows:

Step 1: Set $I_0 := \{g_0(x) \in S :$ there exists $g(x,y) \in I$ such that

$$\begin{aligned} g(x,y) =& g_0(x) + (u+1)g_1(x) + (g_0'(x) + (u+1)g_1'(x))y + (g_0''(x) + (u+1)g_1''(x))y^2, \\ & \text{where } g_1(x), g_0'(x), g_1'(x), g_0''(x), g_1''(x) \in S\}. \end{aligned}$$

It is not hard to see that the set I_0 is an ideal of S. Since m is odd, S is a principal ideal ring. Thus, there exists a polynomial $\alpha_{01}(x)$ in S such that $I_0 = \langle \alpha_{01}(x) \rangle$. Since α_{01} is an element of I_0, according to the definition of I_0, there exists $p_1(x, y) \in I$ with

$$\begin{aligned} p_1(x,y) = \alpha_{01}(x) + (u+1)\alpha_{11}(x) + (\beta_{01}(x) + (u+1)\beta_{11}(x))y \\ + (\gamma_{01}(x) + (u+1)\gamma_{11}(x))y^2, \end{aligned}$$

where $\alpha_{01}(x)$, $\alpha_{11}(x)$, $\beta_{01}(x)$, $\beta_{11}(x)$, $\gamma_{01}(x)$, $\gamma_{11}(x) \in S$. It is clear that $f_0(x) \in I_0$. Hence there exists $t_0(x) \in \mathbb{Z}_4[x]$ such that

$$f_0(x) = \alpha_{01}(x)t_0(x).$$

Set

$$\begin{aligned} h_1(x,y) :=& f(x,y) - p_1(x,y)t_0(x) \\ =& (u+1)h_{01}(x) + (h'_{01}(x) + (u+1)h'_{11}(x))y + (h''_{01}(x) + (u+1)h''_{11}(x))y^2, \end{aligned}$$

where $h_{01}(x)$, $h'_{01}(x)$, $h'_{11}(x)$, $h''_{01}(x)$, $h''_{11}(x) \in S$. Since $f(x, y)$ and $p_1(x, y)$ are in I and I is an ideal of R, $h_1(x, y)$ is again in I.

Step 2: Put $I'_0 := \{g_1(x) \in S :$ there exists $g(x, y) \in I$ such that

$$\begin{aligned} g(x,y) =& (u+1)g_1(x) + (g'_0(x) + (u+1)g'_1(x))y + (g''_0(x) + (u+1)g''_1(x))y^2, \\ & \text{where } g'_0(x), g'_1(x), g''_0(x), g''_1(x) \in S\}. \end{aligned}$$

Clearly, I'_0 is an ideal of S. Thus, there exists a polynomial $\alpha_{12}(x) \in S$ such that $I'_0 = \langle \alpha_{12}(x) \rangle$. There exists $p_2(x, y) \in I$ such that

$$p_2(x,y) = (u+1)\alpha_{12}(x) + (\beta_{02}(x) + (u+1)\beta_{12}(x))y + (\gamma_{02}(x) + (u+1)\gamma_{12}(x))y^2,$$

where $\beta_{02}(x)$, $\beta_{12}(x)$, $\gamma_{02}(x)$, $\gamma_{12}(x) \in S$. According to the definition of I'_0, $h_{01}(x) \in I'_0$, and so $h_{01}(x) = \alpha_{12}(x)t_1(x)$ for some $t_1(x) \in \mathbb{Z}_4[x]$. Set

$$\begin{aligned} h_2(x,y) :=& h_1(x,y) - p_2(x,y)t_1(x) \\ =& (h'_{02}(x) + (u+1)h'_{12}(x))y + (h''_{02}(x) + (u+1)h''_{12}(x))y^2, \end{aligned}$$

where $h'_{02}(x)$, $h'_{12}(x)$, $h''_{02}(x)$, $h''_{12}(x) \in S$. Since $h_1(x, y)$ and $p_2(x, y)$ are polynomials in I we have that $h_2(x, y) \in I$.

Step 3: Set $I_1 := \{g'_0(x) \in S :$ there exists $g(x, y) \in I$ such that

$$\begin{aligned} g(x,y) =& (g'_0(x) + (u+1)g'_1(x))y + (g''_0(x) + (u+1)g''_1(x))y^2, \text{where} \\ & g'_1(x), g''_0(x), g''_1(x) \in S\}. \end{aligned}$$

Obviously, I_1 is an ideal of S, and so there exists a polynomial $\beta_{03}(x)$ in S such that $I_1 = \langle \beta_{03}(x) \rangle$. There exists a polynomial $p_3(x, y) \in I$ such that

$$p_3(x,y) = (\beta_{03}(x) + (u+1)\beta_{13}(x))y + (\gamma_{03}(x) + (u+1)\gamma_{13}(x))y^2,$$

where $\beta_{13}(x)$, $\gamma_{03}(x)$, $\gamma_{13}(x) \in S$. According to the definition of I_1, $h'_{02}(x)$ in I_1. Hence $h'_{02}(x) = \beta_{03}(x)t_2(x)$ for some $t_2(x) \in \mathbb{Z}_4[x]$. Put

$$\begin{aligned} h_3(x,y) :=& h_2(x,y) - p_3(x,y)t_2(x) \\ =& (u+1)h'_{13}(x)y + (h''_{03}(x) + (u+1)h''_{13}(x))y^2, \end{aligned}$$

where $h'_{13}(x)$, $h''_{03}(x)$, $h''_{13}(x) \in S$. Similar to the previous discussion $h_3(x, y) \in I$.

Step 4: Set $I_1' := \{g_1'(x) \in S$: there exists $g(x,y) \in I$ such that

$$g(x,y) = (u+1)g_1'(x)y + (g_0''(x) + (u+1)g_1''(x))y^2, \text{ where } g_0''(x), g_1''(x) \in S\}.$$

It is clear that I_1' is an ideal of S. Thus, there exists a polynomial $\beta_{13}(x) \in S$ such that $I_1' = \langle \beta_{13} \rangle$. Since $\beta_{13}(x)$ in I_1', according to definition I_1', we have a polynomial $p_4(x,y) \in I$, where

$$p_4(x,y) = (u+1)\beta_{13}(x)y + (\gamma_{03}(x) + (u+1)\gamma_{13}(x))y^2,$$

and $\gamma_{03}(x)$, $\gamma_{13}(x) \in S$. Obviously, $h_{13}'(x) \in I_1'$. So, we get $h_{13}'(x) = \beta_{13}(x)t_3(x)$ for some $t_3(x) \in \mathbb{Z}_4[x]$. Put

$$h_4(x,y) := h_3(x,y) - p_4(x,y)t_3(x) = (h_{04}(x) + (u+1)h_{14}(x))y^2,$$

where $h_{04}(x)$, $h_{14}(x) \in S$. It is clear that $h_4(x,y) \in I$.

Step 5: Set

$$\begin{aligned} I_2 := \{g_0''(x) \in S &: \text{there exists } g(x,y) \in I \text{ such that} \\ & g(x,y) = (g_0''(x) + (u+1)g_1''(x))y^2, \text{where } g_1''(x) \in S\}. \end{aligned}$$

Clearly, I_2 is an ideal of S. Therefore, there exists $\gamma_{05}(x) \in S$ such that

$$I_2 = \langle \gamma_{05}(x) \rangle.$$

Besides, $\gamma_{05}(x)$ in I_2, and so we have a polynomial $p_5(x,y) \in I$, where

$$p_5(x,y) = (\gamma_{05}(x) + (u+1)\gamma_{15}(x))y^2,$$

where $\gamma_{15}(x) \in S$. Obviously, $h_{04}(x) \in I_2$, and so we obtain that $h_{04}(x) = \gamma_{05}(x)t_4(x)$ for some $t_4(x) \in \mathbb{Z}_4[x]$. Put $h_5(x,y) := h_4(x,y) - p_5(x,y)t_4(x) = (u+1)h_{05}(x)y^2$, where $h_{05}(x) \in S$. Similarly, $h_5(x,y)$ in I.

Step 6: Put

$$I_2' := \{g_1''(x) \in S : \text{there exists } g(x,y) \in I \text{ such that } g(x,y) = (u+1)g_1''(x)y^2\}.$$

It is clear that I_2' is an ideal of S. Thus there exists $\gamma_{15}(x) \in S$ such that $I_2' =< \gamma_{15}(x) >$. Also there exists a polynomial $p_6(x,y) \in I$ such that $p_6(x,y) = (u+1)\gamma_{15}(x)y^2$. Now, since $h_{05} \in I_2'$, there exists $t_5(x) \in S$ such that $h_{05}(x) = \gamma_{15}(x)t_5(x)$. Therefore, $h_5(x,y) = (u+1)\gamma_{15}(x)t_5(x)y^2 = p_6(x,y)t_5(x)$. Now, we get

$$\begin{aligned} f(x,y) &= h_1(x,y) + p_1(x,y)t_0(x), \\ h_1(x,y) &= h_2(x,y) + p_2(x,y)t_1(x), \\ h_2(x,y) &= h_3(x,y) + p_3(x,y)t_2(x), \\ h_3(x,y) &= h_4(x,y) + p_4(x,y)t_3(x), \\ h_4(x,y) &= h_5(x,y) + p_5(x,y)t_4(x), \\ h_5(x,y) &= p_6(x,y)t_5(x). \end{aligned}$$

These equality imply that

$$\begin{aligned} f(x,y) = p_1(x,y)t_0(x) + p_2(x,y)t_1(x) + p_3(x,y)t_2(x) + p_4(x,y)t_3(x) \\ + p_5(x,y)t_4(x) + p_6(x,y)t_5(x). \end{aligned}$$

Thus

$$I = \langle p_1(x,y), p_2(x,y), p_3(x,y), p_4(x,y), p_5(x,y), p_6(x,y) \rangle.$$

Now, we state an important lemma.

Lemma 2.3. *Let C be a 1-generator two-dimensional QC code of length $n = 3m\ell$ which is generated by $G(x,y) = (G_1(x,y), G_2(x,y), \ldots, G_\ell(x,y)) \in \mathcal{R}$, where $G_i(x,y) \in R'$ for all i with $1 \le i \le \ell$. Then $G_i(x,y) \in C_i$, where C_i is a two-dimensional cyclic code of length m over R. Furthermore, if m is odd, then $G_i(x,y)$ can be selected as the form*

$$G_i(x,y) = \varphi_{0i}(x) + (u+1)\varphi_{1i}(x) + (\psi_{0i}(x) + (u+1)\psi_{1i}(x))y + (\theta_{0i}(x) + (u+1)\theta_{1i}(x))y^2,$$

where $\varphi_{0i}(x)$, $\varphi_{1i}(x)$, $\psi_{0i}(x)$, $\psi_{1i}(x)$, $\theta_{0i}(x)$ and $\theta_{1i}(x)$ are polynomials in $R[x]$ for all i with $1 \le i \le \ell$ and moreover, $\varphi_{0i}(x)$, $\psi_{0i}(x)$ and $\theta_{0i}(x)$ are monic polynomials for all $i = 1, \cdots, \ell$.

Proof. Consider the projection map $\psi_i : \mathcal{R} \to R'$ given by

$$\psi_i(k_1(x,y), \ldots, k_\ell(x,y)) = k_i(x,y),$$

where $k_i(x,y) \in R'$ for all $i = 1, \cdots, \ell$. It is clear that the set $\psi_i(C)$ is a two-dimensional cyclic code over R for all i with $1 \le i \le \ell$. Now, in view of Lemma 2.2, for all $1 \le i \le \ell$, one can obtain a generator for $\psi_i(C)$ as follows

$$\psi_i(C) = \langle p_{1i}(x,y), p_{2i}(x,y), p_{3i}(x,y), p_{4i}(x,y), p_{5i}(x,y), p_{6i}(x,y)\rangle,$$

where, for each $j = 1, \ldots, 6$, $p_{ji}(x,y)$ are polynomials as described in Lemma 2.2 . Since $G_i(x,y) \in \psi_i(C)$ for all $1 \le i \le \ell$, there exists a polynomial $f_i(x,y) \in R[x,y]$ such that

$$\begin{aligned} G_i(x,y) &= f_i(x,y)(\alpha^i_{01}(x) + (u+1)\alpha^i_{11}(x) + (\beta^i_{01}(x) + (u+1)\beta^i_{11}(x))y \\ &\quad + (\gamma^i_{01}(x) + (u+1)\gamma^i_{11}(x))y^2) \\ &= \varphi_{0i}(x) + (u+1)\varphi_{1i}(x) + (\psi_{0i}(x) + (u+1)\psi_{1i}(x))y \\ &\quad + (\theta_{0i}(x) + (u+1)\theta_{1i}(x))y^2, \end{aligned}$$

where $\alpha^i_{01}(x)$, $\alpha^i_{11}(x)$, $\beta^i_{01}(x)$, $\beta^i_{11}(x)$, $\gamma^i_{01}(x)$ and $\gamma^i_{11}(x)$ are generator polynomials of cyclic codes over $\mathbb{Z}_4$ and, for all $1 \le i \le \ell$, $\varphi_{0i}(x)$, $\varphi_{1i}(x)$, $\psi_{0i}(x)$, $\psi_{1i}(x)$, $\theta_{0i}(x)$ and $\theta_{1i}(x)$ are polynomials in $R[x]$. Moreover, $\varphi_{0i}(x)$, $\psi_{0i}(x)$ and $\theta_{0i}(x)$ are monic polynomials for all $i = 1, \cdots, \ell$.

In the light of the above two lemmas, we will obtain the minimal generating sets for 1-generator two-dimensional QC codes.

Theorem 2.4. *Let C be a 1-generator two-dimensional QC code of length $n = 3m\ell$ over R which is generated by $G = (G_1(x,y), G_2(x,y), \ldots, G_\ell(x,y))$, with m is odd and*

$$\begin{aligned} G_i(x,y) &= \varphi_{0i}(x) + (u+1)\varphi_{1i}(x) + (\psi_{0i}(x) + (u+1)\psi_{1i}(x))y \\ &\quad + (\theta_{0i}(x) + (u+1)\theta_{1i}(x))y^2, \end{aligned}$$

where $\varphi_{0i}(x)$, $\varphi_{1i}(x)$, $\psi_{0i}(x)$, $\psi_{1i}(x)$, $\theta_{0i}(x)$ and $\theta_{1i}(x)$ are polynomials in $R[x]$ for all i with $1 \le i \le \ell$ and moreover, $\varphi_{0i}(x)$, $\psi_{0i}(x)$ and $\theta_{0i}(x)$ are monic polynomials for all $1 \le i \le \ell$.

Assume that

$$\begin{aligned} &\deg(\varphi_{0i}(x)) > \deg(\varphi_{1i}(x)), \\ &\deg(\psi_{0i}(x)) > \deg(\psi_{1i}(x)) \text{ and} \\ &\deg(\theta_{0i}(x)) > \deg(\theta_{1i}(x)), \text{ for all } 1 \le i \le \ell, \end{aligned}$$

and that the polynomials $\varphi_{0i}(x) + (u+1)\varphi_{1i}(x)$, $(\psi_{0i}(x) + (u+1)\psi_{1i}(x))y$ and $(\theta_{0i}(x) + (u+1)\theta_{1i}(x))y^2$ are not zero divisor in R'. Assume that

$$\begin{aligned} g_0(x) &= \gcd\{\varphi_{01}(x), \varphi_{02}(x), \ldots, \varphi_{0\ell}(x)\}, & q_0(x) &= \gcd\{\varphi_{11}(x), \varphi_{12}(x), \ldots, \varphi_{1\ell}(x)\}, \\ g_1(x) &= \gcd\{\psi_{01}(x), \psi_{02}(x), \ldots, \psi_{0\ell}(x)\}, & q_1(x) &= \gcd\{\psi_{11}(x), \psi_{12}(x), \ldots, \psi_{1\ell}(x)\}, \\ g_2(x) &= \gcd\{\theta_{01}(x), \theta_{02}(x), \ldots, \theta_{0\ell}(x)\}, & q_2(x) &= \gcd\{\theta_{11}(x), \theta_{12}(x), \ldots, \theta_{1\ell}(x)\} \end{aligned}$$

and, for $k = 0, 1, 2$, $g_k(x)|x^m - 1$ and $q_k(x)|x^m - 1$. Let

$$S_1 = \bigcup_{j=0}^{r_0-1} \{x^j(\varphi_{01}(x) + (u+1)\varphi_{11}(x), \dots, \varphi_{0\ell}(x) + (u+1)\varphi_{1\ell}(x))\},$$

$$S_2 = \bigcup_{j=0}^{r_1-1} \{x^j((\psi_{01}(x) + (u+1)\psi_{11}(x))y, \dots, (\psi_{0\ell}(x) + (u+1)\psi_{1\ell}(x))y)\},$$

$$S_3 = \bigcup_{j=0}^{r_2-1} \{x^j((\theta_{01}(x) + (u+1)\theta_{11}(x))y^2, \dots, (\theta_{0\ell}(x) + (u+1)\theta_{1\ell}(x))y^2)\},$$

$$S_4 = \bigcup_{j=0}^{t_0-1} \{x^j((u+1)h_0\varphi_{11}(x), \dots, (u+1)h_0\varphi_{1\ell}(x))\},$$

$$S_5 = \bigcup_{j=0}^{t_1-1} \{x^j((u+1)h_1\psi_{11}(x)y, \dots, (u+1)h_1\psi_{1\ell}(x)y)\},$$

$$S_6 = \bigcup_{j=0}^{t_2-1} \{x^j((u+1)h_2\theta_{11}(x)y^2, \dots, (u+1)h_2\theta_{1\ell}(x)y^2)\},$$

where, for all $k = 0, 1, 2$, we set $r_k := \deg(\frac{x^m-1}{g_k(x)})$ and $t_k := \deg(\frac{x^m-1}{q_k(x)})$. Then $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$ is a minimal generating set for C. Moreover, $\mid C \mid = 16^{r_0+r_1+r_2}4^{t_0+t_1+t_2}$ for all $1 \leq i \leq \ell$.

Proof. For $k = 0, 1, 2$, put $h_k(x) := \frac{x^m-1}{g_k(x)}$ and $\delta_k(x) := \frac{x^m-1}{q_k(x)}$, and let $c(x, y) = f(x, y)G$ be a codeword in C, where $f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2$, with $f_i(x) \in R[x]$ for all $1 \leq i \leq 3$. For simplicity of presentation, in our proof, we will use the notion f instead of $f(x)$. By the division algorithm, we get the unique polynomials $Q_0(x)$, $Q_1(x)$, $Q_2(x)$, $R_0(x)$, $R_1(x)$, $R_2(x)$ in $R[x]$ such that

$$\begin{aligned}
f_0 &= h_0Q_0 + R_0, \text{ where } R_0 = 0 \text{ or } \deg(R_0) < r_0,\\
f_1 &= h_1Q_1 + R_1, \text{ where } R_1 = 0 \text{ or } \deg(R_1) < r_1,\\
f_2 &= h_2Q_2 + R_2, \text{ where } R_2 = 0 \text{ or } \deg(R_2) < r_2.
\end{aligned}$$

There exist polynomials a_i, a_i', $a_i'' \in \mathbb{Z}_4[x]$ such that $h_0\varphi_{0i} = h_0g_0a_i = 0$,
$h_1\psi_{0i} = h_1g_1a_i' = 0$, $h_2\theta_{0i} = h_2g_2a_i'' = 0$ for all $1 \leq i \leq \ell$. We have

$$\begin{aligned}
c(x, y) = f(x, y)G &= (h_0Q_0 + R_0)(\varphi_{01} + (u+1)\varphi_{11}, \dots, \varphi_{1\ell} + (u+1)\varphi_{1\ell})\\
&+ (h_1Q_1 + R_1)((\psi_{01} + (u+1)\psi_{11})y, \dots, (\psi_{0\ell} + (u+1)\psi_{1\ell})y)\\
&+ (h_2Q_2 + R_2)((\theta_{01} + (u+1)\theta_{11})y^2, \dots, (\theta_{0\ell} + (u+1)\theta_{1\ell})y^2)\\
&= Q_0h_0((u+1)\varphi_{11}, \dots, (u+1)\varphi_{1\ell})\\
&+ R_0(\varphi_{01} + (u+1)\varphi_{11}, \dots, \varphi_{0\ell} + (u+1)\varphi_{1\ell})\\
&+ Q_1h_1((u+1)\psi_{11}y, \dots, (u+1)\psi_{1\ell}y)\\
&+ R_1((\psi_{01} + (u+1)\psi_{11})y, \dots, (\psi_{0\ell} + (u+1)\psi_{1\ell})y)\\
&+ Q_2h_2((u+1)\theta_{11}y^2, \dots, (u+1)\theta_{1\ell}y^2)\\
&+ R_2((\theta_{01} + (u+1)\theta_{11})y^2, \dots, (\theta_{0\ell} + (u+1)\theta_{1\ell})y^2).
\end{aligned}$$

They are not difficult to verify that

$$\begin{gathered}
R_0(\varphi_{01} + (u+1)\varphi_{11}, \dots, \varphi_{0\ell} + (u+1)\varphi_{1\ell}) \in\text{Span}(S_0),\\
R_1((\psi_{01} + (u+1)\psi_{11})y, \dots, (\psi_{0\ell} + (u+1)\psi_{1\ell})y) \in\text{Span}(S_1), \text{ and}\\
R_2((\theta_{01} + (u+1)\theta_{11})y^2, \dots, (\theta_{0\ell} + (u+1)\theta_{1\ell})y^2) \in\text{Span}(S_2).
\end{gathered}$$

Again, using the division algorithm, we get the unique polynomials $Q'_0(x), Q'_1(x)$, $Q'_2(x), R'_0(x), R'_1(x)$, $R'_2(x) \in R[x]$ such that

$$\begin{aligned} Q_0 &= \delta_0 Q'_0 + R'_0, \text{ where } R'_0 = 0 \text{ or } \deg(R'_0) < t_0, \\ Q_1 &= \delta_1 Q'_1 + R'_1, \text{ where } R'_1 = 0 \text{ or } \deg(R'_1) < t_1, \text{ and} \\ Q_2 &= \delta_2 Q'_2 + R'_2, \text{ where } R'_2 = 0 \text{ or } \deg(R'_2) < t_2. \end{aligned}$$

There exist polynomials b_i, b'_i, $b''_i \in \mathbb{Z}_4[x]$ such that

$$\begin{aligned} \delta_0 Q'_0 (u+1) h_0 \varphi_{1i} &= (u+1) Q'_0 q_0 \delta_0 b_i = 0, \\ \delta_1 Q'_1 (u+1) h_1 \psi_{1i} &= (u+1) Q'_1 q_1 \delta_1 b'_i = 0, \text{ and} \\ \delta_2 Q'_2 (u+1) h_2 \theta_{1i} &= (u+1) Q'_2 q_2 \delta_2 b''_i = 0 \end{aligned}$$

in R' for all $1 \leq i \leq \ell$. It is not hard to see that

$$Q_0((u+1)h_0\varphi_{11}, \ldots, (u+1)h_0\varphi_{1\ell}) = R'_0((u+1)h_0\varphi_{11}, \ldots, (u+1)h_0\varphi_{1\ell}) \in \text{Span}(S_4),$$

$Q_1((u+1)h_1\psi_{11}y, \ldots, (u+1)h_1\psi_{1\ell}y) = R'_1((u+1)h_1\psi_{11}y, \ldots, (u+1)h_1\psi_{1\ell}y) \in \text{Span}(S_5)$, and
$Q_2((u+1)h_2\theta_{11}y^2, \ldots, (u+1)h_2\theta_{1\ell}y^2) = R'_2((u+1)h_2\theta_{11}y^2, \ldots, (u+1)h_2\theta_{1\ell}y^2) \in \text{Span}(S_6)$.
Thus $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$ is a Spanning set for C. Also, it is clear $S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6 = \{0\}$.

With the aid of the above theorem, we obtain the following corollary.

Corollary 2.5. *If ℓ is a positive integer and $\varphi_{1i}(x) = x^m - 1$, $\psi_{1i}(x) = x^m - 1$ and $\theta_{1i}(x) = x^m - 1$ are polynomials over R, for all i with $1 \leq i \leq \ell$, then C is a free two-dimensional QC code of rank $r_0 + r_1 + r_2$ over R and its minimal generating set is $S_1 \cup S_2 \cup S_3$ such that*

$$\begin{aligned} S_1 &= \bigcup_{j=0}^{r_0-1} \{x^j(\varphi_{01}(x), \ldots, \varphi_{0\ell}(x))\}, \\ S_2 &= \bigcup_{j=0}^{r_1-1} \{x^j(\psi_{01}(x)y, \ldots, \psi_{0\ell}(x)y)\}, \\ S_3 &= \bigcup_{j=0}^{r_2-1} \{x^j(\theta_{01}(x)y^2, \ldots, \theta_{0\ell}(x)y^2)\}, \end{aligned}$$

where, for all $k = 0, 1, 2$, we set $r_k := \deg(\frac{x^m-1}{g_k(x)})$. Furthermore, $\mid C \mid = 16^{r_0+r_1+r_2}$.

Proof. By Theorem 2.4, if $\varphi_{1i}(x) = x^m - 1$, $\psi_{1i}(x) = x^m - 1$ and $\theta_{1i}(x) = x^m - 1$ are polynomials over R, for all i with $1 \leq i \leq \ell$, then

$$\begin{aligned} q_0 =& \gcd\{\varphi_{11}(x), \ldots, \varphi_{1\ell}(x), x^m - 1\} = x^m - 1, \\ q_1 =& \gcd\{\psi_{11}(x), \ldots, \psi_{1\ell}(x), x^m - 1\} = x^m - 1, \text{ and} \\ q_2 =& \gcd\{\theta_{11}(x), \ldots, \theta_{1\ell}(x), x^m - 1\} = x^m - 1. \end{aligned}$$

Hence $\delta_0 = 1$, $\delta_1 = 1$ and $\delta_2 = 1$. Clearly, $S_1 \cap S_2 \cap S_3 = \{0\}$. Therefore, its minimal generating set is $S_1 \cup S_2 \cup S_3$. This means that C is a free two-dimensional QC code of rank $r_0 + r_1 + r_2$. Thus $\mid C \mid = 16^{r_0+r_1+r_2}$.

In the next theorem, we provide a lower bound on minimum distance of the free 1-generator two-dimensional QC codes over R.

Theorem 2.6. *Let C be a free 1-generator two-dimensional QC code of length $n = 3m\ell$ over R as in Corollary 2.5. Suppose that*

$$\begin{aligned} h_{0i} &= (x^m - 1)/\varphi_{0i}(x), & h_{1i} &= (x^m - 1)/\psi_{0i}(x), \\ h_{2i} &= (x^m - 1)/\theta_{0i}(x), & h_0 &= \mathrm{lcm}\{h_{01}, \ldots, h_{0\ell}\}, \\ h_1 &= \mathrm{lcm}\{h_{11}, \ldots, h_{1\ell}\} \text{ and} & h_2 &= \mathrm{lcm}\{h_{21}, \ldots, h_{2\ell}\} \end{aligned}$$

for all i with $1 \le i \le \ell$. Then we have the following statements.

(i) $d_{min}(C) \ge \sum_{i \notin A} d_{0i} + \sum_{j \notin B} d_{1j} + \sum_{t \notin D} d_{2t}$ for all $1 \le i, j, t \le \ell$, where $A, B, C \subseteq \{1, 2, \ldots, \ell\}$ are sets from maximum size for which

$$\begin{aligned} &\mathrm{lcm}\{h_{0i}, i \in A\} \ne h_0, & &\mathrm{lcm}\{h_{1j}, j \in B\} \ne h_1 \text{ and} \\ &\mathrm{lcm}\{h_{2t}, t \in D\} \ne h_2. \end{aligned}$$

(ii) If $h_{01} = h_{02} = \ldots = h_{0\ell}$, $h_{11} = h_{12} = \ldots = h_{1\ell}$ and $h_{21} = h_{22} = \ldots = h_{2\ell}$, then

$$d_{min}(C) \ge \sum_{i=1}^{\ell} d_{0i} + \sum_{i=1}^{\ell} d_{1i} + \sum_{i=1}^{\ell} d_{2i}.$$

Proof. (i) Consider the projection map $\psi_i : \mathcal{R} \to R'$ given by

$$\psi_i(k_1(x, y), \ldots, k_\ell(x, y)) = k_i(x, y),$$

where $k_i(x, y) \in R'$ for all i with $1 \le i \le \ell$. It is easy to show that $\psi_i(C)$ is a two-dimensional code over R. Let $c(x, y) = f(x, y)G$ be a nonzero codeword in C, where $f(x, y) \in R[x, y]$. Since C is a free 1-generator two-dimensional QC code, we have that $\varphi_{1i}(x) = x^m - 1$, $\psi_{1i}(x) = x^m - 1$, $\theta_{1i}(x) = x^m - 1$ for all $1 \le i \le \ell$. So, the i-th component is zero if and only if $(x^m - 1) \mid f(x, y)G$. This means that $(x^m - 1) \mid f_0(x)\varphi_{0i}(x)$, $(x^m - 1) \mid f_1(x)\psi_{0i}(x)$ and $(x^m - 1) \mid f_2(x)\theta_{0i}(x)$, that is, if and only if $h_{0i} \mid f_0(x)$, $h_{1i} \mid f_1(x)$, $h_{2i} \mid f_2(x)$ for all $1 \le i \le \ell$. Thus $c(x, y) = 0$ if and only if $h_0 \mid f_0(x)$, $h_1 \mid f_1(x)$ and $h_2 \mid f_2(x)$. Therefore, $c(x, y) \ne 0$ if and only if $h_0 \nmid f_0(x)$ or $h_1 \nmid f_1(x)$ or $h_2 \nmid f_2(x)$. Thus, $c(x, y) \ne 0$ have the most number of zero blocks whenever

$$\begin{aligned} h_0 &\ne \mathrm{lcm}\{h_{0i}, i \in A\}, \text{ where } \mathrm{lcm}\{h_{0i}, i \in A\} \mid f_0(x), \\ h_1 &\ne \mathrm{lcm}\{h_{1j}, j \in B\}, \text{ where } \mathrm{lcm}\{h_{1j}, j \in B\} \mid f_1(x), \\ h_2 &\ne \mathrm{lcm}\{h_{2t}, t \in D\}, \text{ where } \mathrm{lcm}\{h_{2t}, t \in D\} \mid f_2(x), \end{aligned}$$

where A, B and D are a maximal subset of $\{1, 2, \ldots, \ell\}$ having this property. Thus

$$d_{min}(C) \ge \sum_{i \notin A} d_{0i} + \sum_{i \notin B} d_{1i} + \sum_{i \notin D} d_{2i}.$$

(ii) Now, we know that $A = \varnothing$ if and only if $h_{01} = h_{02} = \ldots = h_{0\ell}$ and also, $B = \varnothing$ if and only if $h_{11} = h_{12} = \ldots = h_{1\ell}$. Moreover, $D = \varnothing$ if and only if $h_{21} = h_{22} = \ldots = h_{2\ell}$. Thus, $d_{min}(C) \ge \sum_{i=1}^{\ell} d_{0i} + \sum_{i=1}^{\ell} d_{1i} + \sum_{i=1}^{\ell} d_{2i}$.

Corollary 2.7. *Let C be a 1-generator two-dimensional QC code of length $n = 3m\ell$ over R which is generated by*

$$\begin{aligned} G = (&\varphi_{01}(x) + (u+1)\varphi_{11}(x) + (\psi_{01}(x) + (u+1)\psi_{11}(x))y + \\ &(\theta_{01}(x) + (u+1)\theta_{11}(x))y^2, \ldots, \varphi_{0\ell}(x) + (u+1)\varphi_{1\ell}(x) + \\ &(\psi_{0\ell}(x) + (u+1)\psi_{1\ell})y + (\theta_{0\ell}(x) + (u+1)\theta_{1\ell}(x))y^2), \end{aligned}$$

where m is odd. Assume that $\varphi_{1i}(x) = x^m - 1$, $\psi_{1i}(x) = x^m - 1$ and $\theta_{1i}(x) = x^m - 1$ for each $i = 1, 2, \ldots, \ell$. Let $h_{0i} = (x^m - 1)/\varphi_{0i}(x)$, $h_{1i} = (x^m - 1)/\psi_{0i}(x)$ and $h_{2i} = (x^m - 1)/\theta_{0i}(x)$, for all i with $1 \leq i \leq \ell$, and that

$$\begin{aligned} h_0 &= \operatorname{lcm}\{h_{01}, h_{02}, \ldots, h_{0\ell}\}, \\ h_1 &= \operatorname{lcm}\{h_{11}, h_{12}, \ldots, h_{1\ell}\}, \text{ and} \\ h_2 &= \operatorname{lcm}\{h_{21}, h_{22}, \ldots, h_{2\ell}\}. \end{aligned}$$

Then

(i) C is a free two-dimensional QC code from $\deg(h_0) + \deg(h_1) + \deg(h_2)$. Moreover, $\mid C \mid = 16^{\deg(h_0)+\deg(h_1)+\deg(h_2)}$.

(ii) $d_{min}(C) \geq \sum_{i \notin A} d_{0i} + \sum_{i \notin B} d_{1i} + \sum_{i \notin D} d_{2i}$, where $A, B, D \subseteq \{1, 2, \ldots, \ell\}$ are set from maximum size for which

$$\begin{aligned} &\operatorname{lcm}\{h_{0i}, i \in A\} \neq h_0, \qquad \operatorname{lcm}\{h_{1j}, j \in B\} \neq h_1 \text{ and} \\ &\operatorname{lcm}\{h_{2t}, t \in D\} \neq h_2. \end{aligned}$$

(iii) If $h_{01} = h_{02} = \ldots = h_{0\ell}$, $h_{11} = h_{12} = \ldots = h_{1\ell}$ and $h_{21} = h_{22} = \ldots = h_{2\ell}$, then

$$d_{min}(C) \geq \sum_{i=1}^{\ell} d_{0i} + \sum_{i=1}^{\ell} d_{1i} + \sum_{i=1}^{\ell} d_{2i}.$$

Proof. Let $c(x, y) = f(x, y)G$ be a codeword in C such that $f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2$, where $f_i(x) \in R[x]$ for $i = 0, 1, 2$. By the division algorithm, we can find unique polynomials $Q_1(x), Q_2(x), Q_3(x), R_1(x), R_2(x), R_3(x) \in R[x]$ such that

$$\begin{aligned} f_0(x) &= h_0 Q_1(x) + R_1(x), \text{ where } R_1(x) = 0, \text{ or } \deg R_1(x) < \deg(h_0), \\ f_1(x) &= h_1 Q_2(x) + R_2(x), \text{ where } R_2(x) = 0, \text{ or } \deg R_2(x) < \deg(h_1), \\ f_2(x) &= h_2 Q_2(x) + R_3(x), \text{ where } R_3(x) = 0, \text{ or } \deg R_3(x) < \deg(h_2). \end{aligned}$$

Now, we have

$$\begin{aligned} c(x, y) &= f(x, y)G \\ &= (h_0 Q_1(x) + R_1(x))(\varphi_{01}(x), \ldots, \varphi_{0\ell}(x)) \\ &+ (h_1 Q_2(x) + R_2(x))(\psi_{01}(x)y, \ldots, \psi_{0\ell}(x)y) \\ &+ (h_2 Q_3(x) + R_3(x))(\theta_{01}(x)y^2, \ldots, \theta_{0\ell}(x)y^2). \end{aligned}$$

We know that $h_0\varphi_{0i}(x) = h_1\psi_{0i}(x) = h_2\theta_{0i}(x) = x^m - 1$. Therefore, we obtain

$$\begin{aligned} &R_1(x)(\varphi_{01}(x), \ldots, \varphi_{0\ell}(x)) \in \operatorname{Span}(S_1), \\ &R_2(x)(\psi_{01}(x)y, \ldots, \psi_{0\ell}(x)y) \in \operatorname{Span}(S_2) \text{ and} \\ &R_3(x)(\theta_{01}(x)y^2, \ldots, \theta_{0\ell}(x)y^2 \in \operatorname{Span}(S_3). \end{aligned}$$

Thus $t_0 = 0$, $t_1 = 0$ and $t_2 = 0$ which implies that $S_4 = S_5 = S_6 = \varnothing$. Using the definition of free module, we obtain C is a free two-dimensional QC code of rank $\deg(h_0) + \deg(h_1) + \deg(h_2)$. Therefore, $\mid C \mid = 16^{\deg(h_0)+\deg(h_1)+\deg(h_2)}$.

The statements (ii) and (iii) follow from Theorem 2.6.

3. 1-generator two-dimensional GQC codes

In this section, we study two-dimensional GQC codes over R. At first, we recall the definition of 1-generator two-dimensional GQC codes over R.

Definition 3.1. *Let $m_1, m_2, \ldots, m_\ell$ be positive integers and*

$$R_i = R[x,y]/\langle x^{m_i}-1, y^3-1\rangle$$

for all i with $1 \le i \le \ell$. Any ideal of $\mathcal{R} = R_1 \times R_2 \times \ldots \times R_\ell$ is called a two-dimensional GQC code of length $(m_1, m_2, \ldots, m_\ell)$ with index ℓ over R.

If C is a two-dimensional GQC code of length $(m_1, m_2, \ldots, m_\ell)$ with $m = m_1 = \ldots = m_\ell$, then C is a two-dimensional QC code with length $n = 3m\ell$.

Lemma 3.2. *Let C be a 1-generator two-dimensional GQC code of length $(m_1, \ldots, m_\ell)$ and $G'(x,y) = (G'_1(x,y), G'_2(x,y), \ldots, G'_\ell(x,y)) \in \mathcal{R}$ be a generator of C, where $G'_i(x,y) \in R_i$ for all i with $1 \le i \le \ell$. Then $G'_i(x,y) \in C_i$, where C_i is a two-dimensional cyclic code of length m_i over R for $i = 1, \cdots, \ell$.*

Also, if m_i is odd, then $G'_i(x,y)$ can be selected to be of the form $G'_i(x,y) = (\varphi_{0i}(x) + (u+1)\varphi_{1i}(x) + (\psi_{0i}(x) + (u+1)\psi_{1i}(x))y + (\theta_{0i}(x) + (u+1)\theta_{1i}(x))y^2)$ where $\varphi_{0i}(x)$, $\varphi_{1i}(x)$, $\psi_{0i}(x)$, $\psi_{1i}(x)$, $\theta_{0i}(x)$ and $\theta_{1i}(x)$ are polynomials in $R[x]$ for all i with $1 \le i \le \ell$. Furthermore, $\varphi_{0i}(x)$, $\psi_{0i}(x)$ and $\theta_{0i}(x)$ are monic polynomials for all $1 \le i \le \ell$.

By using a method similar to that we used in the proof of Theorem 2.4, one can obtain the next theorem which gives the minimal generating set of 1-generator two-dimensional GQC codes over R.

Theorem 3.3. *Let C be a 1-generator two-dimensional GQC code of length $(m_1, m_2, \ldots, m_\ell)$ over R which is generated by $G'(x,y) = (G'_1(x,y), G'_2(x,y), \ldots, G'_\ell(x,y))$, where m_i is odd for all i with $1 \le i \le \ell$. Then $G'_i(x,y) = \varphi_{0i}(x) + (u+1)\varphi_{1i}(x) + (\psi_{0i}(x) + (u+1)\psi_{1i})y + (\theta_{0i}(x) + (u+1)\theta_{1i}(x)y^2)$, where $\varphi_{0i}(x)$, $\varphi_{1i}(x)$, $\psi_{0i}(x)$, $\psi_{1i}(x)$, $\theta_{0i}(x)$ and $\theta_{1i}(x)$ are polynomials in $R[x]$ for all i with $1 \le i \le \ell$. Furthermore, $\varphi_{0i}(x)$, $\psi_{0i}(x)$ and $\theta_{1i}(x)$ are monic polynomials for all $1 \le i \le \ell$.*

Assume that $\deg(\varphi_{0i}(x)) \ge \deg(\varphi_{1i}(x))$, $\deg(\psi_{0i}(x)) \ge \deg(\psi_{1i}(x))$ and that $\deg(\theta_{0i}(x)) \ge \deg(\theta_{1i}(x))$. Suppose that polynomials $\varphi_{0i}(x) + (u+1)\varphi_{1i}(x)$, $(\psi_{0i}(x) + (u+1)\psi_{1i})y$, $(\theta_{0i}(x) + (u+1)\theta_{1i}(x))y^2$ are not zero-divisor of R_i. Let

$$\begin{aligned} &h_{0i} = (x^{m_i}-1)/\gcd(\varphi_{0i}(x), x^{m_i}-1), && \\ &h_0 = \mathrm{lcm}(h_{01}, \ldots, h_{0l}), && \deg(h_0) = r_0, \\ &\delta_{0,i} = (x^{m_i}-1)/\gcd(h_0\varphi_{1i}(x), x^{m_i}-1), && \\ &\delta_0 = \mathrm{lcm}(\delta_{01}, \ldots, \delta_{0\ell}) \text{ and} && \deg(\delta_0) = t_0. \end{aligned}$$

Let

$$\begin{aligned} &h_{1i} = (x^{m_i}-1)/\gcd(\psi_{0i}(x), x^{m_i}-1), && \\ &h_1 = \mathrm{lcm}(h_{11}, h_{12}, \ldots, h_{1\ell}), && \deg(h_1) = r_1, \\ &\delta_{1i} = (x^{m_i}-1)/\gcd(h_1\psi_{1i}(x), x^{m_i}-1), && \\ &\delta_1 = \mathrm{lcm}(\delta_{11}, \delta_{12}, \ldots, \delta_{1\ell}) \text{ and} && \deg(\delta_1) = t_1. \end{aligned}$$

Suppose that

$$\begin{aligned} &h_{2i} = (x^{m_i}-1)/\gcd(\theta_{0i}(x), x^{m_i}-1), && \\ &h_2 = \mathrm{lcm}(h_{21}, \ldots, h_{2\ell}), && \deg(h_2) = r_2, \\ &\delta_{2i} = (x^{m_i}-1)/\gcd(h_2\theta_{1i}(x), x^{m_i}-1), && \\ &\delta_2 = \mathrm{lcm}(\delta_{21}, \delta_{22}, \ldots, \delta_{2\ell}) \text{ and} && \deg(\delta_2) = t_2. \end{aligned}$$

Then the minimal generating set of C is $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$, where

$$S_1 = \bigcup_{j=0}^{r_0-1} \{x^j(\varphi_{01}(x) + (u+1)\varphi_{11}(x), \ldots, \varphi_{0\ell}(x) + (u+1)\varphi_{1\ell}(x))\},$$

$$S_2 = \bigcup_{j=0}^{r_1-1} \{x^j((\psi_{01}(x) + (u+1)\psi_{1i}(x))y, \ldots, (\psi_{0\ell}(x) + (u+1)\psi_{1\ell}(x))y)\},$$

$$S_3 = \bigcup_{j=0}^{r_2-1} \{x^j((\theta_{01}(x) + (u+1)\theta_{11}(x))y^2, \ldots, (\theta_{0\ell}(x) + (u+1)\theta_{1\ell}(x))y^2)\},$$

$$S_4 = \bigcup_{j=0}^{t_0-1} \{x^j((u+1)h_0\varphi_{11}(x), \ldots, (u+1)h_0\varphi_{1\ell}(x))\},$$

$$S_5 = \bigcup_{j=0}^{t_1-1} \{x^j((u+1)h_1\psi_{11}(x)y, \ldots, (u+1)h_1\psi_{1\ell}(x)y)\},$$

$$S_6 = \bigcup_{j=0}^{t_2-1} \{x^j((u+1)h_2\theta_{11}(x)y^2, \ldots, (u+1)h_2\theta_{1\ell}(x)y^2)\}.$$

Thus $\mid C \mid = 16^{r_0+r_1+r_2}4^{t_0+t_1+t_2}$.

According to Theorem 3.3, we have the following corollary.

Corollary 3.4. *If ℓ is a positive integer and $\varphi_{1i}(x) = x^{m_i} - 1$, $\psi_{1i}(x) = x^{m_i} - 1$ and $\theta_{1i}(x) = x^{m_i} - 1$ are polynomials over R for all i with $1 \leq i \leq \ell$, then C is a free two-dimensional GQC code of rank $r_0+r_1+r_2$ over R and its minimal generating set is $S_1 \cup S_2 \cup S_3$. Furthermore, C has $16^{r_0+r_1+r_2}$ codewords.*

In the following theorem, we give a lower bound on the minimum distance of free 1-generator two-dimensional GQC codes over R. Its proof is exactly the same as the proof of Theorem 2.6, so we delete it.

Theorem 3.5. *Let C be a free 1-generator two-dimensional GQC code of length $(m_1, m_2, \ldots, m_\ell)$ over R as in Corollary 3.4. Let*

$$\begin{aligned} h_{0i} &= (x^{m_i} - 1)/\varphi_{0i}(x), & h_0 &= \mathrm{lcm}\{h_{01}, \ldots, h_{0\ell}\}, \\ h_{1i} &= (x^{m_i} - 1)/\psi_{0i}(x), & h_1 &= \mathrm{lcm}\{h_{11}, \ldots, h_{1\ell}\}, \\ h_{2i} &= (x^{m_i} - 1)/\theta_{0i}(x) \text{ and} & h_2 &= \mathrm{lcm}\{h_{21}, \ldots, h_{2\ell}\}. \end{aligned}$$

Then

(i) $d_{min}(C) \geq \sum_{i \notin A} d_{0i} + \sum_{i \notin B} d_{1i} + \sum_{i \notin D} d_{2i}$, *where $A, B, D \subseteq \{1, 2, \ldots, \ell\}$ are sets of maximum size for which*

$$\begin{aligned} &\mathrm{lcm}\{h_{0i}, i \in A\} \neq h_0, && \mathrm{lcm}\{h_{1j}, j \in B\} \neq h_1 \text{ and} \\ &\mathrm{lcm}\{h_{2t}, t \in D\} \neq h_2. \end{aligned}$$

(ii) If $h_{01} = h_{02} = \ldots = h_{0\ell}$, $h_{11} = h_{12} = \ldots = h_{1\ell}$ and $h_{21} = h_{22} = \ldots = h_{2\ell}$, then

$$d_{min}(C) \geq \sum_{i=1}^{\ell} d_{0i} + \sum_{i=1}^{\ell} d_{1i} + \sum_{i=1}^{\ell} d_{2i}.$$

According to Corollary 3.4 and Theorem 3.5 we obtain the following corollary.

Corollary 3.6. *Assume that ℓ is a positive integer and $\varphi_{1i}(x) = x^{m_i} - 1$, $\psi_{1i}(x) = x^{m_i} - 1$ and $\theta_{1i}(x) = x^{m_i} - 1$ are polynomials over R for all i with $1 \leq i \leq \ell$. Let*

$$\begin{aligned} h_{0i} &= (x^{m_i} - 1)/\varphi_{0i}(x), & h_0 &= \mathrm{lcm}\{h_{oi}, \dots, h_{0\ell}\}, \\ h_{1i} &= (x^{m_i} - 1)/\psi_{0i}(x), & h_1 &= \mathrm{lcm}\{h_{11}, \dots, h_{1\ell}\}, \\ h_{2i} &= (x^{m_i} - 1)/\theta_{0i}(x) \text{ and} & h_2 &= \mathrm{lcm}\{h_{21}, \dots, h_{2\ell}\} \end{aligned}$$

for all $1 \leq i \leq \ell$. Then we have the following statements.

(i) C is a free two-dimensional code of rank $\deg(h_0) + \deg(h_1) + \deg(h_2)$. Moreover, $| C |= 16^{\deg(h_0)+\deg(h_1)+\deg(h_2)}$,

(ii) $d_{min}(C) \geq \sum_{i\notin A} d_{0i} + \sum_{j\notin B} d_{1j} + \sum_{k\notin D} d_{2k}$, where A, B, $D \subseteq \{1, \dots, \ell\}$,

(iii) Let $h_{01} = \dots = h_{0\ell} = h_0$, $h_{11} = \dots = h_{1\ell} = h_1$ and $h_{21} = \dots = h_{2\ell} = h_2$. Then we have $d_{min}(C) \geq \sum_{i=1}^{\ell} d_{0i} + \sum_{i=1}^{\ell} d_{1i} + \sum_{i=1}^{\ell} d_{2i}$.

4. Conclusion

This paper is devoted to the study of two-dimensional quasi-cyclic codes and two-dimensional generalized quasi-cyclic codes of length $3m\ell$ which are a natural generalization of quasi-cyclic codes and generalized quasi-cyclic codes over the ring $R = \mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ with $u^2 = 1$. We first determine the generator polynomials of two-dimensional cyclic codes over R. Then we find the generator polynomials of two-dimensional quasi-cyclic codes and two-dimensional generalized quasi-cyclic codes over R and give their minimal generating sets. Moreover, we study the minimum distances of the family of the free 1-generator two-dimensional quasi-cyclic codes and two-dimensional generalized quasi-cyclic codes.

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